Finite Element Method for Solving Elliptic 2D Problem on Linear Triangular Element

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ABSTRACT

This study investigates the finite element method for second order elliptic issues in a two-dimensional polygonal region. In view of the practical regularity assumptions of the real solution, optimal order error estimates in L^2 and H^1-norms are found for a finite element based on uniform mesh for $2D$ elliptic problem in Square domain .On the other side of the research several examples have been written to illustrate numerical solutions.

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1.Introduction

In this article, we show how to use the finite element method (FEM) to solve the following problem:

$$-\Delta u + \boldsymbol{b} \cdot \nabla u = f \qquad \text{in } \Omega, \qquad (1)$$
$$u = 0 \qquad \text{on } \partial\Omega. \qquad (2)$$

Where $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ and $\nabla u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$. Suppose that Ω has a polygonal region in $\mathbb{R}^2$ and bounded $\partial\Omega$ and $\boldsymbol{b} \in \boldsymbol{W}^1_\infty(\Omega)$.

Finite difference algorithms are notoriously bad at handling shapes with irregular domains. The FEM can get around this drawback. Therefore, it is undoubtedly the most reliable and popular method to solve differential equations, which Courant initially proposed [1], who conducted research using a set of triangle elements. Then, in the early 1950s, engineers separately reinvented the process. The early authors include Argyris [2], Turner and others [3], among others.

Clough coined the phrase "finite element" [4]. As indicated by the widespread adoption of numerous cutting-edge commercial packages, FEMs have currently replaced other numerical approaches as the standard for solving all types of PDEs.

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Numerous numerical techniques for second order elliptic problems have been developed in recent years [5,6,7,8,9,10,11,12,13,14].

The following is the paper outline: In Section 2, FEM equations construction for Elliptic 2D Problem is described. In Section 3, Priori Estimates of error for finite element method are proved. Numerical examples are given to show the effectiveness of the suggested strategy in Section 4. In the last part, we provide a brief summary of the methodology and findings.

1.1 FEM equations construction

First of all, we explore the weak form of the problem in order to build an approximation of the finite element. We multiply the first equation by an arbitrary function (test function) $v \in H_0^1(\Omega)$, integrate the result and then use the Green formula.

$$\int_\Omega fv\,dx = \int_\Omega -\Delta u\, v\,dx + \int_\Omega \boldsymbol{b}\cdot\nabla u$$

$$\int_\Omega fv\,dx = \int_\Omega \nabla u\cdot\nabla v\,dx - \underbrace{\int_{\partial\Omega} (n\cdot\nabla u)\,v\,ds}_{=0} + \int_\Omega \boldsymbol{b}\cdot\nabla u\,vdx$$

$$\int_\Omega fv\,dx = \int_\Omega \nabla u\cdot\nabla v\,dx + \int_\Omega \boldsymbol{b}\cdot\nabla u\,vdx$$

The weak formulation of $(1)-(2)$ and by inner product form is: Find $u \in H^1(\Omega)$ where

$$(\nabla u,\nabla v) + (\boldsymbol{b}\cdot\nabla u, v) = (f,v), \qquad \forall\, v \in H_0^1(\Omega) \tag{3}$$

The bilinear define $a(\cdot,\cdot) = H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}$ by

and $$a(u,v) = (\nabla u,\nabla v) + (\boldsymbol{b}\cdot\nabla u, v) \tag{4}$$

Then ,The FEM is: Find $u_h \in V_h \subset H^1(\Omega)$ such that

$$(\nabla u_h,\nabla v_h) + (\boldsymbol{b}\cdot\nabla u_h, v_h) = (f,v_h), \qquad \forall\, v_h \in V_h, \tag{5}$$

and $$a_h(u_h,v_h) = (f,v_h) \qquad \forall\, v_h \in V_h, \tag{6}$$

where the space of finite elements

$$V_h = \{v : v \text{ is continuous on } \Omega\,;\ v|_K \in P_1(K),\ K \in T_h\}.$$

Assume that u is approximated over a finite element triangle K by

$$u(x,y) \approx u_h(x,y) = \sum_{j=1}^{3} u_j^K \varphi_j^K(x,y), \tag{7}$$

where u_j^K is the value of u_h at the jth node of the element, and φ_j^K is the Lagrange interpolation function, such that

$$\varphi_j^K(x_i, y_i) = \delta_{ij}.$$

We must compute the following element matrices over each element K.

Putting (7) into (5) and test function $v_h = \varphi_i^K$, $i = 1,2,3$, respectively,

and the source function f is

$$f(x,y) \approx \sum_{j=1}^{3} f_j\, \varphi_j^K(x,y), \qquad f_j = f\big(x_j, y_j\big),$$

The element diffusion matrix is obtained (stiffness matrix)

$$A_{ij} \equiv \int_K \nabla\varphi_j^K \cdot \nabla\varphi_i^K \; dx\, dy, \quad i,j = 1,2,3, \tag{8}$$

the element convection matrix

$$B_{ij} \equiv \int_K \big(\boldsymbol{b} \cdot \nabla\varphi_j^K\big)\varphi_i^K \; dx\, dy, \quad i,j = 1,2,3, \tag{9}$$

and the element mass matrix

$$M_{ij} \equiv \int_K \varphi_j^K \varphi_i^K \; dx\, dy, \quad i,j = 1,2,3, \tag{10}$$

we collect all the elements K_n, $1 \le n \le N_K$, of the grid T_h, We find a set of linear equations for the numerical solution u_j at each node:

$$\sum_{n=1}^{N_K} \big(A_{ij} + B_{ij}\big) u_j = \sum_{n=1}^{N_K} M_{ij} f_j \, \cdot \tag{11}$$

For a unique of the solution, a(·,·) must be coercive provided that

$(-\frac{1}{2}\nabla \cdot \boldsymbol{b} \ge 0)$. Indeed,

$$a(v,v) = (\nabla v, \nabla v) + (\boldsymbol{b} \cdot \nabla v, v) = (|\nabla v|^2) + \left(-\frac{1}{2}\nabla \cdot \boldsymbol{b}\right) v^2 \ge C\|v\|_{1,\Omega}^2,$$

Where C is a positive constant with $|\cdot|_{1,\Omega}$ be the norm in $H^1(\Omega)$. Thus, the Lax-Milgram lemma leads to an unique special solvability.

1.2 A Priori Estimates of Error

We begin this section with the following theorem:

Theorem 1: In accordance with (5), the finite element approximation u_h satisfies the Galerkin orthogonally.

$$(\nabla(u-u_h),\nabla v_h)+(\boldsymbol{b}\cdot\nabla(u-u_h),v_h)=0,\quad \forall\, v_h\in V_h, \tag{12}$$

Proof. From the equation (3)

$$(\nabla u,\nabla v)+(\boldsymbol{b}\cdot\nabla u,v)=(f,v),\qquad \forall\, v\in H_0^1(\Omega) \tag{13}$$

and from the equation (5)

$$(\nabla u_h,\nabla v_h)+(\boldsymbol{b}\cdot\nabla u_h,v_h)=(f,v_h),\qquad \forall\, v_h\in V_h, \tag{14}$$

Subtracting (14) from (13) and using $V_h\subset H_0^1(\Omega)$ proves the proof.

■

The following lemma will prove that we need it.

Lemma 1: There is a constant C, such that

$$|a(u,v)|\le C\|u\|_{H^1}\|v\|_{H^1}\qquad \forall\, u,v\in H_0^1(\Omega) \tag{15}$$

Where C is independent of h.

Proof: The problem $(1)-(2)$ satisfies the variational form

$$a(u,v)=(f,v)\qquad \forall\, v\in H_0^1(\Omega),$$

where

$$a(u,v)=(\nabla u,\nabla v)+(\boldsymbol{b}\cdot\nabla u,v)=\int \nabla u\cdot\nabla v\,dx+\int_\Omega \boldsymbol{b}\cdot\nabla u\, v dx$$

Appling a Cauchy-Schwarz inequality, we have

$$|a(u,v)|=\left|\int_\Omega \nabla u\cdot\nabla v\,dx\right|+\left|\int_\Omega \boldsymbol{b}\cdot\nabla u\, v dx\right|$$

$$\le \|\nabla u\|\|\nabla v\|+|\boldsymbol{b}|_{L^\infty(\Omega)}\|\nabla u\|\|v\|$$

$$\le \|\nabla u\|\|\nabla v\|+C_1\|\nabla u\|\|v\|$$

Where, C_1 depends on $|\boldsymbol{b}|_{L^\infty(\Omega)}$, then using the Poincare inequality, we have

$$\leq \|\nabla u\|\|\nabla v\| + C_1 C_2\|\nabla u\|\|\nabla v\|$$

$$\leq \|\nabla u\|\|\nabla v\| + C_3\|\nabla u\|\|\nabla v\|$$

Here, $C_3 = C_3(C_1, C_2)$

$$\leq (1 + C_3)\|\nabla u\|\|\nabla v\| + \|\nabla u\|\|\nabla v\|$$

$$\leq C\|\nabla u\|\|\nabla v\| + \|\nabla u\|\|\nabla v\|,$$

Such that $C = 1 + C_3$. Since $\|\nabla u\| \leq \|u\|_{H^1}$, we have

$$|a(u, v)| \leq C\|u\|_{H^1}\|v\|_{H^1}.$$

■

The following approximation property [15] is known to be satisfied by V_h.

$$\inf_{v_h \in V_h} \|v - v_h\| + h\|v - v_h\|_{H^1} \leq ch^{m+1}H_{m+1}, \quad v \in H^{m+1}(\Omega)$$

The following projection operators are necessary in order to obtain the error estimates. Let $\Psi_h: H_0^1(\Omega) \to V_h$ the definition of the RTZ projection is by

$$(\nabla(u - \Psi_h u), \nabla v_h) = 0, \quad \forall v \in V_h \tag{16}$$

The following outcomes are well known to exist[16]

$$\|u - \Psi_h u\| + h\|\nabla(u - \Psi_h u)\| \leq ch^{m+1}\|u\|_{m+1} \tag{17}$$

Now, we deconstruct the mistakes as follows to obtain a priori error estimates:

Let $u - u_h = u - \pi_h u + \pi_h u - u_h = \alpha + \beta$

Applying (3), (5) and auxiliary (16), we get .

$(\nabla u, \nabla v) - (\nabla u_h, \nabla v_h) + (\boldsymbol{b} \cdot \nabla u, v) - (\boldsymbol{b} \cdot \nabla u_h, v_h) = 0$

$(\nabla(u - u_h), \nabla v_h) + (\boldsymbol{b} \cdot \nabla(u - u_h), v_h) = 0$

$(\nabla(u - \pi_h u + \pi_h u - u_h), \nabla v_h) + (\boldsymbol{b} \cdot \nabla(u - \pi_h u + \pi_h u - u_h), v_h) = 0$

$$(\nabla(\pi_h u - u_h), \nabla v_h) + \underbrace{(\nabla(u - \pi_h u), \nabla v_h)}_{=0} + (\boldsymbol{b} \cdot \nabla(u - \pi_h u), v_h)$$

$$+(\boldsymbol{b} \cdot \nabla(\pi_h u - u_h), v_h) = 0$$

$$(\nabla\beta, \nabla v_h) = -(\boldsymbol{b} \cdot \nabla\alpha, v_h) - (\boldsymbol{b} \cdot \nabla\beta, v_h) \qquad \forall v \in V_h \qquad (18)$$

The above equation is called the error equation in α and β.

Here, we demonstrate the error estimates for $u - u_h$ in L^2 and H^1-norms.

Theorem 2: Suppose that u and u_h be the solution of (3) and (5), respectively. Then , the following optimal order error estimate hold.

$$\|u - u_h\|_{H^1} \leq Ch^{r+1}\|u\|_{r+1}$$

Where $r \geq 1$, for $d = 2,3$. The constant C depends on c_0, C_4, C_5.

Proof: From the triangle inequality , we have

$$\|u - u_h\|_{H^1} = \|u - \pi_h u + \pi_h u - u_h\|_{H^1}$$

$$\leq \|u - \pi_h u\|_{H^1} + \|\pi_h u - u_h\|_{H^1}$$

$$\leq \alpha + \beta \qquad (19)$$

Since the estimate of α, we can take it from (17), as for the estimate β, we find it in the following way: Put $v_h = \beta$ in (18) we obtain

$$(\nabla\beta, \nabla\beta) = -(\boldsymbol{b} \cdot \nabla\alpha, \beta) - (\boldsymbol{b} \cdot \nabla\beta, \beta). \qquad (20)$$

Appling a Cauchy-Schwarz inequality to every term, we obtain

$$\|\nabla\beta\|^2 \leq |-\boldsymbol{b}|_{L^\infty(\Omega)}\|\nabla\alpha\|\|\nabla\beta\| + |-\boldsymbol{b}|_{L^\infty(\Omega)}\|\nabla\beta\|^2$$

$$\|\nabla\beta\|^2 \leq C_4\|\nabla\alpha\|\|\nabla\beta\| + C_5\|\nabla\beta\|^2$$

$$\|\nabla\beta\|^2 - C_5\|\nabla\beta\|^2 \leq C_4\|\nabla\alpha\|\|\nabla\beta\|$$

$$\|\nabla\beta\|^2 \leq C_4\|\nabla\alpha\|\|\nabla\beta\|$$

$$\|\nabla\beta\|^2 \leq \frac{C_4}{(1 - C_5)}\|\nabla\alpha\|\|\nabla\beta\|$$

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$$\|\nabla\beta\| \leq \frac{C_4}{(1 - C_5)}\|\nabla\alpha\|$$

Since α , $\beta \in V_h \subset H_0^1(\Omega)$, then $\|\alpha\| \le c_0\|\nabla\alpha\|$ and $\|\beta\| \le c_0\|\nabla\beta\|$, thus, we get

$$c_0\|\beta\| \le \frac{C_4}{(1-C_5)}c_0\|\alpha\|$$

$$\|\beta\| \le C\|\alpha\|$$

Here, $C = C(c_0, C_4, C_5)$,

From (17) we have

$$\|\beta\| \le Ch^{m+1}\|u\|_{m+1}, \qquad (21)$$

Substituting (17) and (20) into (19). Then complete the proof .

Similarly, we analyse the error estimates for $u - u_h$ in L^2-norm.

3 Illustration Example

Two instances are provided in this part to highlight the numerical findings.

Example (1): Assume that $\Omega = [0,1]^2$ is the domain. The exact solution $u(x,y,)$ and fours function $f(x,t)$ for (1) are selected as:

$$u = x^2y^2,$$

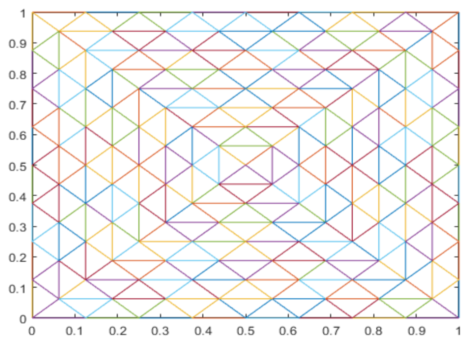
$$f = -2y^2 - 2x^2 + 2xy^2 + 2yx^2,$$

with $\vec{p} = (1,1)$. we use multiple levels of meshes and the linear element to solve this equation.

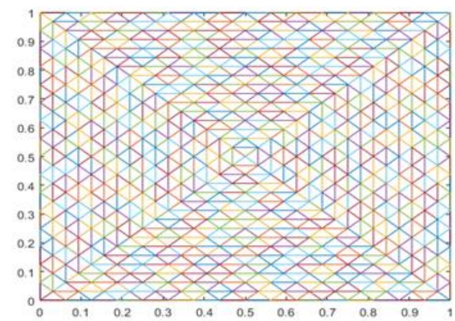
Table 1- The Maximum error and Convergence rate of the GFEM (5).

h	$max\|u-uh\|$	Rate
5.0000e-01	4.1667e-02	1.5779e+00
2.5000e-01	1.3957e-02	1.6617e+00
1.2500e-01	4.4112e-03	1.6935e+00
6.2500e-02	1.3638e-03	1.7329e+00
3.1250e-02	4.1030e-04	1.7698e+00
1.5625e-02	1.2032e-04	1.8000e+00

7.8125e-03	3.4552e-05

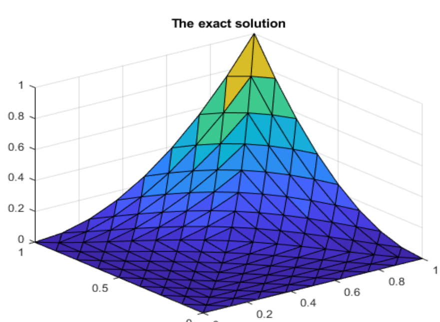


(a)

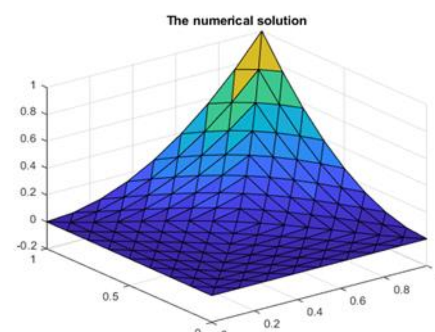


(b)

Figure -1 The levels of grid at (a) $h = \frac{1}{16}$ (b) $h = \frac{1}{32}$

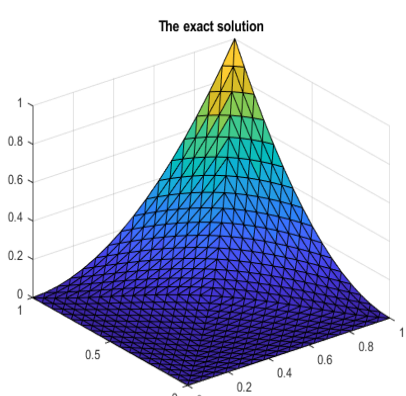


(a)

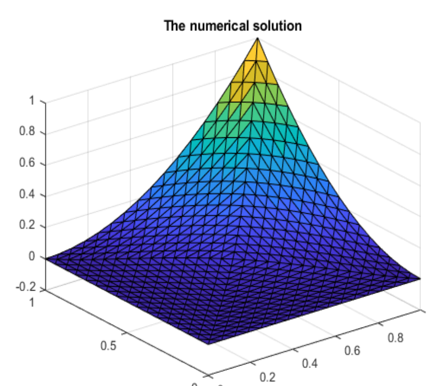


(b)

Figure -2 (a) The exact solution at $h = \frac{1}{16}$ (b) The numerical solution at $h = \frac{1}{16}$



(a)



(b)

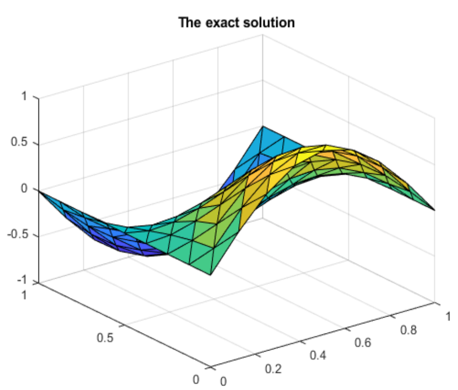
Figure -3 (a) The exact solution at $h = \frac{1}{32}$ (b) The numerical solution at $h = \frac{1}{32}$

Example (2): Assume that $\Omega = [0, 1]^2$ is the domain. The exact solution $u(x, y,)$ for (1) is selected as:

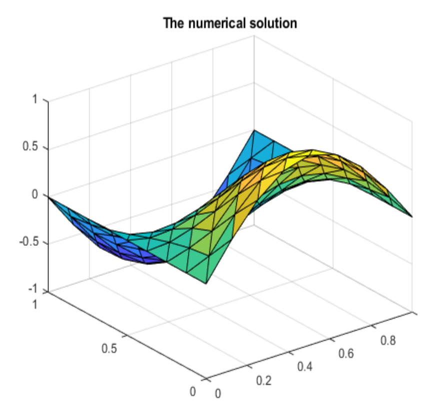
$$u = \sin(\pi x)\cos(\pi y), \text{ with } \vec{p} = (1,1).$$

Table 2 The Maximum error and Convergence rate of the GFEM (5).

h	$max\|u - uh\|$	Rate
5.0000e-01	1.3090e-01	1.3080e+00
2.5000e-01	5.2869e-02	1.2486e+00
1.2500e-01	2.2250e-02	1.9372e+00
6.2500e-02	5.8099e-03	1.9800e+00
3.1250e-02	1.4728e-03	1.9927e+00
1.5625e-02	3.7008e-04	1.9957e+00
7.8125e-03	9.2794e-05	

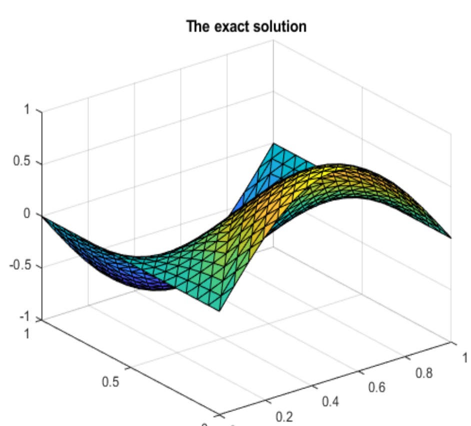


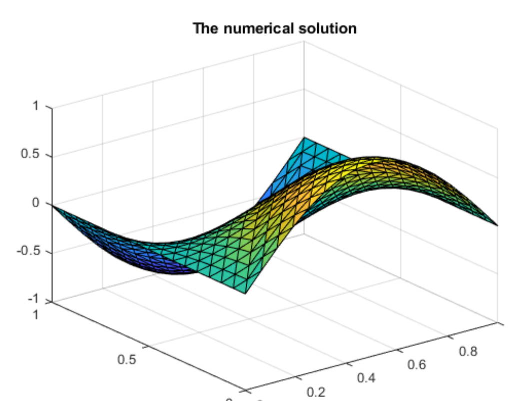
(a)



(b)

Figure - 4 (a) The exact solution at $h = \frac{1}{16}$ (b) The numerical solution at $h = \frac{1}{16}$





(a) (b)

Figure -5 (a) The exact solution at $h = \frac{1}{32}$ (b) The numerical solution at $h = \frac{1}{32}$

4. Conclusion

In this article, GFEMs are applied for 2D Problems on Linear Triangular in square domain. We can see that for u the convergence rate is equal to 2.

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