Numerical solution of Boundary value problems of fractional order using Cubic-spline Interpolation combined with method function

الحل العددي لمسائل القيم الحدوديه ذات المرتبه الكسريه بأستخدام تقريب Cubic-spline مركباً مع داله طريقه shooting

Mohamed Saleh Mehdi
Department of Mathematics / College of Education for Science
Karbala University
Moh.saleh81@yahoo.com

Abstract

In this paper we shall use the Cubic spline method combined with shooting method for solving fractional boundary value problems. In this approach the fractional order differential equation will be transformed into a system of ordinary differential equations used for approximating the fractional term. Numerical comparisons between the solution using this new method and the methods introduced in [17, 29] are presented. The obtained numerical results show that the proposed method maintains a remarkable high accuracy.

الخلاصة

نقدم في هدا البحث نظاما جديد لحل مسائل كسور القيمة الحدية باستخدام طريقة ال Cubic spline جنبا الى جنب مع طريقة الshooting method . واستخدمت تحويل الرتبة الكسرية الى نظام من المعادلات التفاضلية العادية لتقريب الحد الكسري . واخيرا عمل مقارنة بين الحلول العددية التي حصلنا عليها مع حلول استخدام طرق ال exact لحل المعادلات التفاضيلة الكسرية [17,29] وكان هناك نسبة خطأ قليلة جدا مما يثبت دقة الطريقة الجديدة.

Introduction

Fractional calculus attracted the attention of many researchers because it has recently gained popularity in the investigation of dynamical systems. There are many applications of fractional derivative and fractional integration in several complex systems such as physics, chemistry, fluid mechanics, viscoelasticity, signal processing, mathematical biology, and bioengineering, and various applications in many branches of science and engineering [3].

One of the applications where the fractional differential equation appears is the equation describing the motion of fluids, which are encountered down hole during the process of oil well logging, through a device that has been designed to measure fluids viscosity.

The fluid flow is governed by the Navies-Stokes equations:

$$q_t + (q.\nabla)q = -\frac{1}{\rho}\nabla \mathrm{p} + \sigma\nabla^2 \mathrm{q} \,, \qquad (1)$$

$$\nabla.q = 0 \,,$$

Where q denotes the fluid velocity, p denotes pressure, t denotes time, and ρ and σ are the fluid density and kinematic viscosity, respectively. Then, it was found that the equation governing the motion of the fluid through the instrument is

$$y''(x) + k\sqrt{\pi}\, D^{1.5}\, y(x) + \alpha y'(x) = 0, \qquad y(0) = 1, \quad y'(0) = 0 \qquad (2)$$

The above fractional deferential equation is well known as Bagley-Troika equation when $\alpha = 0$ .Which appears in modeling the motion of a rigid plate immersed in a Newtonian fluid [12, 17]

Several methods have been proposed to obtain the analytical solution of fractional deferential equations (FDEs) such as Laplace and Fourier transforms, eigenvector expansion, method based on Laguerre integral formula, direct solution based on Grunwald Letnikov approximation, truncated Taylor series expansion, and power series method [9,18-23]. There are also several methods have recently been proposed to solve FDEs numerically such as fractional Adams-Moulton methods, explicit Adams multistep methods, fractional deference method, decomposition method, variation iteration method, least squares finite element solution, extrapolation method, and the Kansa method which is mesh less, easy-to-use, and has been used to handle a broad range of partial differential equation models [24–31]. Also, I considered the numerical solution of the fractional boundary value problem method.

(FBVP) $y^{2-\propto}y(x) + p(x)y = g(x)$, $0 \le \alpha < 1,\ x \in [a,b],$ with Dirichlet boundary conditions using quadratic polynomial spline, [32].

The existence of at least one solution of fractional problems can be seen in
[3, 11, 14, 16, 31].

We consider the numerical solution of the following fractional boundary value
Problem [FBVPs]:

$$y''(x) + \theta D^{\alpha} y + \beta y = f(x), \ \ m-1 \le \alpha < m, \ \ x \in [a,b] \qquad (3)$$

Subject to boundary conditions:

$$y(a) = y_a, \quad y(b) = y_{b,} \qquad (4)$$

Where the function f(x) is continuous on the interval [a,b] and the operator D^{α} represents
The Caputo fractional derivative. Where, the Caputo fractional derivative is [22]

$$D^{\alpha} y(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} y^{(m)}(s)ds, \quad \propto > 0, \ \ m-1 < \propto < m, \qquad (5)$$

When $\alpha = 0$, (3) is reduced to the classical second order boundary value problem.

2. Method of Solution

The following is a brief derivation of the algorithm used to solve problem (3)-(4). The method of solution presented in the following section is based on cubic spline approach combined with shooting method.

2.1. Cubic Spline Solution for FDEs

In order to develop cubic spline approximation for the fractional differential equation (3)-(4), we would discuss the solution of (3) as initial value problem of the form:

$$y''(x) + \theta D^{\alpha} y + \beta y = f(x), \ \ 0 \le \propto < 1, \ \ x \in [a,b] \qquad (6)$$

$$y(a) = y_a, \qquad y'(a) = y'_a \qquad (7)$$

Let

$$\Delta x : \ x_i = a + ih, \ x_0 = a, \ x_n = b, \ h = \frac{(b-a)}{n}, \ \ i = 0,1,2,\ldots,n-1 \qquad (8)$$

Be a partition of [a ,b] which divides the interval into n-equal parts.

Cubic spline approximation will be built in each subinterval $[a+ih, a+(i+1)h]$ to Approximate the solution of (6)-(7). Starting with the first interval [a, a + h], consider that the cubic polynomial spline segment $s_{0(x)}$ has the form:

$$s_0(x) = a_0 + b_o(x-a) + \frac{c_0}{2}(x-a)^2 + \frac{d_0}{6}(x-a)^3, \tag{9}$$

where $a_0\ , b_0, c_0, and, d_0$ are constants to be determined. It is straightforward to check:

$$s_0(a) = a_0 = y_a\ ,\ s_0{}'(a) = b_0 = y_a{}'\ ,\ s_0{}''(a) = c_0 = y''(a) = f(a) - \beta y_a - \theta D^{\alpha} y_a$$
(10)

By construction, (4) satisfies (6) for $x = a$. Then, for complete determination of the spline In the first interval, we have to find d_0.From (9), we have

$$s_0{}''(x) = c_0 + d_0(x-a). \tag{11}$$

We will impose that the spline be a solution of the problem (6) at the point $x = a + h$ hence, we obtain

$$s_0{}''(a+h) = y''(a+h) = f(a+h) - \beta y(a+h) - \theta D^{\alpha} y(a+h) \tag{12}$$

From (11), (12) and using (9) we obtain:

$$\left(h + \frac{\beta h^3}{6}\right) d_0 = f(a+h) - \beta\left(y_a + y_a{}'h + \frac{h^2}{2} y''(a)\right) - y''(a) - \theta D^{\alpha} y,\ at\ x = a+h \tag{13}$$

Then the spline is fully determined in the first subinterval. In the next subinterval [a+h, a+2h] the cubic spline segment $s_1(x)$ has the form:

$$s_1(x) = s_0(a+h) + s_0{}'(a+h)\big(x-(a+h)\big) + \frac{s_0{}''(a+h)}{2}(x-(a+h))^2 + \frac{d_1}{6}(x-(a+h))^3 \tag{14}$$

From which we get

$$s_1{}''(x) = s_0{}''(a+h) + d_1(x-(a+h)) \tag{15}$$

Taking into consideration that this cubic spline is of class $c^2([a, a+h] \cup [a+h, a+2h])$, and again all of the coefficients of $s_1(x)$ are determined with exception of d_1.It is easy to check that the spline $s_1(x)$ be a solution of the problem (6) at the point x = a + h, then for determining d_1 we will impose that the spline be a solution of the problem (6) at the point x = a + 2h . Hence, by repeating the previous procedure we obtain

$$s_1{}''(a+2h) = y''(a+2h) = f(a+2h) - \beta y(a+2h) - \theta D^{\alpha} y,\ at\ x = a+2h$$
(16)

Substituting by $x = a + 2h$ into (15) and equating the result by (16), we get

$$\left(h + \frac{\beta h^3}{6}\right) d_1 = f(a+2h) - \beta\left(s_0(a+h) + h s_0'(a+h) + \frac{h^2}{2} s_0''(a+h)\right) - s_0''(a+h) - \theta D^{\alpha} y \quad at\ \ x = a+2h \tag{17}$$

By this way the spline is totally determined in the subinterval $[a+h, a+2h]$ .Iterating this process, let us consider that the cubic spline is constructed until the subinterval$[a+(i-1)h, a+ih]$, then we can define it in the next the subinterval $[a+ih, a+(i+1)h]$ as:

$$s_i(x) = \psi_i + \frac{d_i}{6}(x-(a+ih))^3 \, , \tag{18}$$

Where

$$\psi_i = \sum_{k=0}^{2} \frac{1}{k!}(s_{k-1})^{(k)}(a+ih)(x-(a+ih))^k \tag{19}$$

Then the cubic spline S(x) $\in c^2(\bigcup_{j=0}^{i}[a+j, a+(j+1)])$ and easy to check that (18) verifies the differential equation (6) at the point $x = a+ih$. The constant d_i can be determined by imposing that the spline be a solution of the problem (6) at the point $x = a+(i+1)h$ Hence, we obtain

$$\left(h+\frac{\beta h^3}{6}\right)d_i = f(a+(i+1)h) - \beta\psi_i(a+(i+1)h) - \Psi_i''(a+(i+1)h) - \theta D^{\alpha}y$$
$$\text{at } x = a+(i+1)h \tag{20}$$

From (19)-(20), the spline approximation for the solutions of (3) and (6) at $x_i = a+ih$, i= 1,2,...,n can be written in the following form:

$$s_i(x_{i+1}) = \sum_{k=0}^{2} \frac{1}{k!} h^k \, {s_{i-1}}^{(k)}(a+ih) + \frac{h^3}{6} d_i \, , \tag{21}$$

where $d_i = \frac{1}{h}\left[s''_{i(a+(i+1)} \, h\right] - s''_{i-1}(a+ih)]$ $, i = 0,1,2,\ldots$

Lemma 2.1. Let $y \in c^4[a,b]$then the error bound associated with (21) is $|e(X)| = 0(h^2)$

Proof. For each subinterval [a + ih,a +(i + 1)h], the error terms are

$$e_{i+1} = y(x_{i+1}) - s_i(x_{i+1}) \ , \ \ i = 0,1,2,\ldots,n-1 \tag{22}$$

Using, Taylor expansion for $y(x_{i+1})$,in the general form of Taylor we get,

$$y(a+(i+1)h) = y(a+ih) + hy'(a+ih) + \frac{h^2}{2}y''(a+ih) + \frac{h^3}{6}y'''(a+ih) + 0(h^2) \tag{23}$$

Then (21) & (23) led to

$$e_{i+1} = y(x_{i+1}) - s_i(x_{i+1})$$
$$= e_i + e'_i + \left(\frac{h^2}{2}\right)e''_i + \left(\frac{h^3}{6}\right)e'''_i + 0(h^4) \ , \ i = 1,2,\ldots,n-1 \tag{24}$$

For the subinterval $[a, a+h]$:

$$e_1 = y(a+h) - s_0(a+h) = \frac{h^3[y'''(a) - d_0]}{6} + 0(h^4) = 0(h^3),$$
$$e'_1 = y'(a+h) - s'_0(a+h) = 0(h^2) \, , \tag{25}$$
$$e''_1 = y''(a+h) - s''_0(a+h) = 0(h),$$

Then, for $i = 1$ in (24) we get:

$$e_2 = y(a+2h) - s_1(a+2h) = e_1 + he'_1 + \left(\frac{h^2}{2}\right)e''_1 + \left(\frac{h^3}{6}\right)e'''_1 + 0(h^4)$$
$$= e_1 + 0(h^3) = 0(h^3) \, . \tag{26}$$

In general, it can be written as $e_{i+1} = e_i + 0(h^3)$. Then, it can be proved that

$$|e(x)| = n \, 0(h^3) = 0(h^2).$$

2.2. Numerical Approximation of Fractional Term

The algorithm used for solving fractional differential equation is based on transforming the fractional derivative into a system of ordinary differential equation. Firstly, the Caputo fractional derivative for y(x) can be written as:

$$D^{\alpha}y(x)=\frac{x^{m-\alpha-1}}{\Gamma(m-\alpha)}\int_0^x\left(1-\frac{s}{x}\right)^{m-\alpha-1}y^{(m)}(s)ds\,,\quad \alpha>0\;\; m-1<\alpha<m$$

We now use the binomial formula [9]:

$$(1+z)^{\lambda}=\sum_{p=0}^{\infty}\binom{\lambda}{p}z^{p}=\sum_{p=0}^{\infty}\frac{(-1)^{p}\,\Gamma(p-\lambda)}{\Gamma(-\lambda)p!}(z)^{p},\;\;|z|<1 \qquad (27)$$

With (27) the expression for $D^{\alpha}y(x)$ can be written as follows with $\lambda=\;m\;-\;\alpha-1$:

$$D^{\alpha}y(x)=\frac{x^{\lambda}}{\Gamma(\lambda+1)}\int_0^x y^{(m)}(s)\left[\sum_{p=0}^{\infty}\frac{\Gamma(p-\lambda)}{\Gamma(-\lambda)p!}\left(\frac{s}{x}\right)^{p}\right]ds \qquad , \qquad \alpha>0,m-1<\alpha<m$$

(28)

The integral:

$$\sigma_p=\int_0^x s^{p}\,y^{(m)}(s)ds\,,\,p=0,1,2,\ldots \qquad (29)$$

are solutions to the following system of differential equations:

$$\sigma'_p=x^{p}y^{(m)}(x)\qquad \sigma_p(0)=0,\;\;p=0,1,2,\ldots \qquad (30)$$

According to (28) – (30) the expression for D^{α}y(x) can be rewritten as:

$$D^{\alpha}y(x)=\frac{x^{m-\alpha-1}}{\Gamma(m-\alpha)}\sum_{p=0}^{\infty}\left(\frac{\Gamma(p-m+\alpha+1)}{\Gamma(-m+\alpha+1)p!x^{p}}\sigma_p\right),\,\alpha>0,m-1<\alpha<m \qquad (31)$$

with σ_p satisfying (30), (31) will represent the fundamental relation used in numerical representation of the fractional term in fractional differential equations. In application, we will use finite number of terms N suitably chosen, so (31) will be

$$D^{\alpha}y(x)\cong\frac{x^{m-\alpha-1}}{\Gamma(m-\alpha)}\sum_{p=0}^{N}\left(\frac{\Gamma(p-m+\alpha+1)}{\Gamma(-m+\alpha+1)p!x^{p}}\sigma_p\right),\,\propto>0,m-1<\propto<m$$

3. Convergence Analysis

Let $s^{\Delta}{}_3$ be the space of cubic splines with respect to Δ and with smoothness $c^2[a,b]$. Also, let us denote by $y_{\Delta}(x)$ the cubic spline approximation to y(x). This implies that $y_{\Delta}\in s^{\Delta}{}_3$ which can be written as $y_{\Delta}=s_i(x)\,,\,i\;=\;0,1,2\,\ldots,n\,-\,1.$

Without loss of generality, we will consider problem (1.3) with homogeneous Dirichlet boundary conditions [33]:

$$y(a)=0\;\;,\;\;y(b)=0 \qquad (32)$$

It will be assumed that y and y_{Δ}satisfy these boundary conditions.

if we assume that the BVP $y''(x) = 0$ along with boundary conditions (32) has a unique solution then there is a Green's function $G(x, s)$ for the problems

$$z = y'' ,\ z_\Delta = y''_\Delta \ , \tag{33}$$

Where

$$y(x) = \int_a^b G(x,s)z(s)ds = Gz(x) \tag{34}$$

$$y_\Delta(x) = \int_a^b G(x,s)z_\Delta(s)ds = Gz_\Delta(x) \tag{35}$$

Where

$$G(x,s) = \begin{Bmatrix} (x-s) - \frac{(x-a)(b-s)}{(b-a)} & ,a\le s\le x\le b \\ -\frac{(x-a)(b-s)}{(b-a)} \ , & a\le x\le s\le b \end{Bmatrix} \tag{36}$$

G is a compact operator, since $G(x,s)$ is continuous in $[a,b] \times [a,b]$, [33].

Lemma 3.1. Consider the following:

$$D^\alpha y(x) = D^\alpha \int_a^b G(x,z)z(s)ds = \int_a^b (D^\alpha G(x,z))z(s)ds = D^\alpha Gz(x) \tag{37}$$

Proof. From the Caputo fractional derivative $D^\alpha y(x)$, we get

$$D^\alpha y(x) = D^\alpha \int_{s=a}^{s=b} G(x,s)z(s)ds$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_{t=a}^{t=x} (x-t)^{m-\alpha-1} \left(\frac{d^m}{dt^m} \left[\int_{s=a}^{s=b} G(t,s)z(s)ds \right] \right) dt \tag{38}$$

Using the principle of differentiation under the integral sign, for the function g(x) with the form:

$$g(x) = \int_{\delta_1(x)}^{\delta_2(x)} \Phi(x,t)dt \tag{39}$$

We have that

$$\frac{dg(x)}{dx} = \int_{\delta_1(x)}^{\delta_2(x)} \frac{\partial}{\partial x} \Phi(x,t)dt + \Phi(x,\delta_2)\frac{d\delta_2}{dx} - \Phi(x,\delta_1)\frac{d\delta_1}{dx} \tag{40}$$

where the functions $\Phi(x,t)$ and $(\partial/\partial x)\Phi(x,t)$ are both continuous in both t and x in some region of the (t,x) plane, including $\delta_1 \le t \le \delta_2$ and $x_0 \le x \le x_1$, then we can deduce that

$$\frac{d^m}{dt^m}\left[\int_a^b G(t,s)z(s)ds\right] = \int_a^b \frac{\partial^m}{\partial t^m} G(t,s)z(s)ds \tag{41}$$

Then we have

$$D^\alpha y(x) = \frac{1}{\Gamma(m-\alpha)} \int_{t=a}^{t=x} (x-t)^{m-\alpha-1} \left[\int_{s=a}^{s=b} \frac{\partial^m}{\partial t^m} G(t,s)z(s)ds \right] dt \tag{42}$$

Changing the order of integration leads to

$$D^\alpha y(x) = \frac{1}{\Gamma(m-\alpha)} \int_{s=a}^{s=b} \left[\int_{t=a}^{t=x} (x-t)^{m-\alpha-1} \frac{\partial^m}{\partial t^m} G(t,s)z(s)dt \right] ds$$

$$D^\alpha y(x) = \int_{s=a}^{s=b} \left[\frac{1}{\Gamma(m-\alpha)} \int_{t=0}^{t=x} (x-t)^{m-\alpha-1} \frac{\partial^m}{\partial t^m} G(t,s)dt \right] z(s)ds \tag{43}$$

$$D^\alpha y(x) = \int_a^b \left(D^\alpha G(x,s)\right) z(s)ds = D^\alpha Gz(x)$$

and this the proof of lemma.

Substituting from (33) – (35) and (37) into (3) leads to

$$z(x) + \theta D^\alpha Gz(x) + \beta Gz(x) = f(x) \tag{44}$$

We will introduce the operator $Ky(x)$ defined by:

$$ky(x) = \theta \int_a^b \left(D^\alpha G(x,s)\right) y(s)ds + \beta \int_a^b G(x,s)y(s)ds \tag{45}$$

which maps $c^2[a,b]$ to $c[a,b]$. We also introduce a linear projection p_Δ that maps $c^2[a,b]$ to ${s^1}_\Delta$ piecewise linear interpolation at the grid points $\{x_i\}_0^n$. Then (44) can be rewritten as:

$$z(x)+Kz(x)=f(x)\ , \tag{46}$$

and we have also:

$$z_\Delta(x)+Kz_\Delta(x)=f(x) \tag{47}$$

By the definition of p_Δ [33], $\|p_\Delta z-z\|_\infty$ converges to zero as h approaches zero for continuous function $z(x)$. This in turn implies that $\|p_\Delta K-K\|_\infty$ converges to zero as h approaches zero.

Theorem 3.2 (see [34]). If there is N_0 large enough, then $\{(I\ +\ p_\Delta K)^{-1}\colon n\geq N_0\}$ exists and consists of a sequence of bounded linear operators. Which means, for a constant δ independent of N_0 and $z\in C[a,b], if\ n\geq N_0, then$

$$\|(I\ +\ p_\Delta K)^{-1}z\|\leq\delta\|z\|$$

Theorem 3.3. Assuming that

(H1) the BVP (3) along with boundary conditions (32) has a unique solution in $C^2[a,b]$,

(H2) the BVP $y''(x)=\ 0$ along with boundary conditions (32) has a unique Solution, then, for some n $\geq N_0$ one has

$$\begin{aligned}&\|y-y_\Delta\|_\infty\leq C_k\|y^{(k+2)}\|h^k\ \forall y\in C^{k+2}[a,b],\ 1\leq k\leq 2\\&\|y-y_\Delta\|_\infty\leq C_0\Psi(y'',h),\forall y\in C^2[a,b],\end{aligned} \tag{48}$$

Where c_k is a constant and independent of y, h and $\Psi(y'',h)$ and y(x) be the solution of (3)-(4). Then, operating on both sides of (46) by the linear projection operator p_Δ gives

$$p_\Delta z(x)+p_\Delta Kz(x)=p_\Delta f(x) \tag{49}$$

Adding z(x) to both sides of (3.18) and subtracting (3.16) from the results lead to

$$(I+p_\Delta K)\big(z(x)-z_\Delta(x)\big)=z(x)-p_\Delta z(x) \tag{50}$$

Operating on both sides of (50) by $(I+p_\Delta K)^{-1}$ leads to

$$z(x)-z_\Delta(x)=(I+p_\Delta K)^{-1}(z(x)-p_\Delta z(x)) \tag{51}$$

Operating on both sides of (51) by the operator G and using (33)–(35), we get

$$y(x)-y_\Delta(x)=G(I+p_\Delta K)^{-1}(y''(x)-p_\Delta y''(x)) \tag{52}$$

Since the operator G is bounded and from Theorem 3.2 the operator $(I + p_\Delta K)^{-1}$ is also bounded, then

$$\|y(x) - y_\Delta(x)\| \le \|G\| \|(I + p_\Delta K)^{-1}\| \|y''(x) - p_\Delta y''(x)\| \qquad (53)$$

From [33], we have that $\|y''(x) - p_\Delta y''(x)\| \le C_k \left\|y^{(k+2)}\right\| h^k,\ \forall y \in C^{k+1}[a,b]$
Where $1 \le k \le 2$ (54)

$$\|y - y_\Delta\|_\infty \le C_0 \Psi(y'', h),\ \ \forall y \in C^2[a,b] \qquad (55)$$

Where , $\Psi(y'', h) = \sup\{|y''(\tau + h^{\sim}) - y''| : \tau\, , \tau + h^{\sim} \in [a,b],\ h^{\sim} \le h\}$

4. Numerical Examples

We will consider some numerical examples illustrating the solution using cubic spline methods. and we used implicit Adams-Bashforth three-step method in approximating the fractional term.

Example 4.1. Consider the initial value problem:

$$y''(x) + k\sqrt{\pi}\, D^{1.5}\, y(x) + y(x) = 0, \qquad y(0) = 1, \qquad y'(0) = 0 \qquad (56)$$

The analytical solution of (56), as found in [17] , has the following form:

$$y(x) = 1 - \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r \left(-k\sqrt{\pi}\right)^j (j+r)!\, x^{2+2r+j/2}}{j!\, r!\, (2+2r+j/2)\, \Gamma(\, 2+2r+j/2)} \qquad (57)$$

Then by using MathCAD program we get the following numerical results

Table 1 : Numerical results of Example 4.1.

	K=1		k=1/5		k=0.005	
X	Analytical solution	Approx solution	Analytical solution	Approx solution	Analytical solution	Approx solution
0	1	1	1	1	1	1
0.125	0.99437	0.993126	0.992747	0.992391	0.992212	0.992212
0.250	0.979919	0.974802	0.971922	0.970148	0.968995	0.968983
0.375	0.958424	0.944545	0.938558	0.933609	0.930733	0.930674
0.500	0.930957	0.904813	0.893615	0.883958	0.878038	0.877899
0.625	0.898335	0.857938	0.838087	0.822499	0.811743	0.811497
0.750	0.861241	0.805442	0.773025	0.750552	0.732892	0.732514
0.875	0.820277	0.748795	0.699540	0.669584	0.642719	0.642193
1	0.775989	0.688838	0.618798	0.580978	0.542633	0.541945

Substituting from Theorem 3.2 and (54) into (53) completes the proof.

Note that the difference between the two solution is very small and this error shows that the method is accurate

Example 4.2. Consider the initial value problem:

$$y''(x) + D^{0.5}\, y(x) + y(x) = 8, \qquad y(0) = y'(0) = 0 \tag{58}$$

Table 2 : Numerical results of Example 4.2.

X	Fractional diff. method [29]	Our method
0	0	0
0.1	0.039473	0.039933
0.2	0.157703	0.158981
0.3	0.352402	0.353996
0.4	0.622083	0.619900
0.5	0.957963	0.950455
0.6	1.360551	1.348551
0.7	1.823267	1.796370
0.8	2.340749	2.899808
0.9	2.907324	2.295551
1	3.517013	3.499200

This example had been solved for many methods. Table 2 shows a comparison between the solution of (58) by our method and fractional differential method.

5. Conclusion

New scheme for solving class of fractional boundary value problem is presented using cubic spline method combined with shooting method. Transforming the fractional derivative into a system of ordinary differential equations is used for approximating the fractional term. Convergence analysis of the method is considered and is shown to be second order. Numerical comparisons between the solution using this new method and the methods introduced in [17, 29] are presented. The obtained numerical results show that the proposed method maintains a remarkable high accuracy which makes it encouraging for dealing with the solution of two-point boundary value problem of fractional order.

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