# Solution of Third Order Ordinary BVPs Using Osculatory Interpolation Technique

# حل مسائل القيم الحدودية الاعتيادية من الرتبة الثالثة باستخدام تقنية الاندراج التماسي

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## Abstract

The aim of this paper is to present a method for solving third order ordinary differential equations with two point boundary condition , we propose two-point osculatory interpolation to construct polynomial solution. The original problem is concerned using two-points osculatory interpolation with the fit equal numbers of derivatives at the end points of an interval [0 , 1] .

Also, many examples are presented to demonstrate the applicability, accuracy and efficiency of the method by compared with conventional method .


**Key ward :** ODE , BVP's , Osculator Interpolation **.**


## خلاصة

الهدف من هذا البحث عرض طريقة لحل معادلات تفاضلية اعتيادية من الرتبة الثالثة ذات الشروط الحدودية عند نقطتين حيث أننا نقترح الاندراج التماسي ذو النقطتين للحصول على الحل كمتعددة حدود,أن أصل المسالة يتعلق باستخدام الاندراج التماسي ذو النقطتين والذي يتفق مع الدالة ومشتقاتها عند نقطتي نهاية الفترة [0,1] أيضا ناقشنا بعض الأمثلة لتوضيح الدقة و الكفاءة وسهولة الأداء للطريقة المقترحة من خلال المقارنة مع الطرق التقليدية الأخرى .


## 1. Introduction

In the study of nonlinear phenomena in physics, engineering and other sciences, many mathematical models lead to two-point BVP's associated with non-linear high order ordinary differential equations . In recent decades, many works have been devoted to the analysis of these problem and many different techniques have been used or developed in order to deal with two main questions : existence and uniqueness of solutions [1],[2] and Computation of solutions.

In this paper we use two-point osculatory interpolation ,essentially this is a generalization of interpolation using Taylor polynomials . The idea is to approximate a function y by a polynomial P in which values of y and any number of its derivatives at given points are fitted by the corresponding function values and derivatives of P .

We are particularly concerned with fitting function values and derivatives at the two end points of a finite interval, say [0,1] where a useful and succinct way of writing osculatory interpolant $P_{2n+1}$ of degree 2n + 1 was given for example by Phillips [3] as :

$$P_{2n+1}(x) = \sum_{j=0}^{n} \{ y^{(j)}(0)\, q_j(x) + (-1)^j\, y^{(j)}(1)\, q_j(1-x) \} \quad , \quad (1)$$

$$q_j(x) = (x^j / j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s = Q_j(x) / j! \quad , \quad (2)$$

so that (1) with (2) satisfies :

$$y^{(j)}(0) = P_{2n+1}^{(j)}(0) \ , \quad y^{(j)}(1) = P_{2n+1}^{(j)}(1) \ , \quad j = 0, 1, 2, \ldots, n \ .$$

implying that $P_{2n+1}$ agrees with the appropriately truncated Taylor series for y about x = 0 and x = 1. We observe that (1) can be written directly in terms of the Taylor coefficients $a_i$ and $b_i$ about x = 0 and x = 1 respectively, as :

$$P_{2n+1}(x) = \sum_{j=0}^{n} \{ a_j Q_j(x) + (-1)^j b_j Q_j(1-x) \} \qquad , \qquad (3)$$

## 2. Solution of Two-Point Third Order BVP's for ODE

A general form of 3rd - order ordinary BVP's is **:-**

$$y^{(3)}(x) = f( x, y, y^{(1)}, y^{(2)} ) \quad , \quad 0 \le x \le 1 \quad , \qquad (4)$$

subject to the boundary conditions :

$$y^{(i)}(0) = A_i \ , \quad y^{(j)}(1) = B_j \quad , i= 0,1,\ldots, k-1 \ , j= 0,1 \ , \ldots , 3- k+1 \ , \qquad (5)$$

The simple idea of suggested method is use a two - point polynomial interpolation to replace y in problem (4) and (5) by a $P_{2n+1}$ which enables any unknown derivatives of y to be computed, the first step therefore is to construct the $P_{2n+1}$, to do this we need evaluate Taylor coefficients of y about $x = 0$ **:**

$$y= \sum_{i=0}^{\infty} a_i \ x \qquad \ni a_i = y^{(i)}(0) / i! \qquad , \qquad (6a)$$

Then insert the series form (6a) into (4) and equate the coefficients of powers of x to obtain $a_n$ **.** Also**,** evaluate Taylor coefficients of y(x) about $x$ =1 **:**

$$y = \sum_{i=0}^{\infty} b_i (x-1)^i \quad \ni b_i = y^{(i)}(1) / i! \quad , \qquad (6b)$$

Then insert the series form (6b) into (4) and equate coefficients of powers of ( x-1) **,** to obtain $b_n$ ,then derive equation (4) with respect to x and iterate the above process to obtain $a_{n+1}$and $b_{n+1}$ ,now iterate the above process many times to obtain $a_{n+2}$, $b_{n+2}$,then $a_{n+3}$, $b_{n+3}$and so on, that is ,we can get $a_i$ and $b_i$, for all i ≥ n .

Now, to evaluate $a_i$ , $b_i$, for i < n, we get half number of these unknown coefficients from given boundary condition ,then use all these $a_i$'s and $b_i$'s to construct $P_{2n+1}$ of the form :

$$P_{2n+1}(x) = \sum_{i=0}^{n} \{a_i Q_i(x) + (-1)^i \; b_i Q_i(1-x)\} \qquad , \quad (7a)$$

Where $Q_j(x) / j! = ( x^j / j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s \qquad , \quad (7b)$

we see that (7a) have n unknown coefficients **.**

Now**,** to evaluate the remainder coefficients integrate equation (4) on [0 , x] 3 - times to obtain :

$$y''(x) - 2!\, a_2 = \int_0^x f(s, y, y', y'')\, ds \qquad , \quad (8_1)$$

$$y'(x) - a_1 - 2!\, a_2 x = \int_0^x (1-s) f(s, y, y', y'')\, ds \qquad , \quad (8_2)$$

$$y(x) - a_0 - a_1 x - 2!\, a_2\, x^2 / 2! = \int_0^x f(s, y, y', y'')\, ds \quad , \quad (8_3)$$

use $P_{2n+1}$ as a replacement of y **,**y '**,**y" in (8) and putting $x = 1$ in all above integration **,** then we have system of 3 equations with 3 unknown coefficients which can be solved using the MATLAB package, version 7.9**,** to get the unknown coefficients**,** thus insert it into (7), thus (7) represent the solution of (4) **.**

Now, we introduce many examples of third order TPBVP's for ODE to illustrates suggested method . Accuracy and efficiency of the suggested method is established through comparison with B – Spline [4] .

**Example**

Consider the following linear third order BVP's :

$$y''' - y' = e^x \quad , \quad 0 < x < 1 \quad ,$$

subject to the BC : $y(0) = 0$ , $y(1) = 1$ , $y'(1) = 0$ .

The exact solution for this problem is :

$$y(x) = 4.8618 - 1.4603e^x - 3.4015e^{-x} + 1/2 \, x \, e^x$$

Now, we solve this equation using suggested method from equations (2) and (3) we have :

$P_{17} = 0.000000050x^{17} - 0.000000421x^{16} + 0.000001569x^{15} - 0.000003343x^{14} + 0.000004470x^{13} - 0.000003839x^{12} + 0.000002261x^{11} - 0.000000606x^{10} + 0.000017838x^{9} - 0.000021374x^{8} + 0.001079587x^{7} - 0.002585821x^{6} + 0.037009314x^{5} - 0.119241301x^{4} + 0.573519604x^{3} - 1.930895608x^{2} + 2.441117621x$

For more details ,table (1) give the results for different nodes in the domain, for n = 8 , i.e. $P_{17}$ and errors obtained by comparing it with the exact solution. Table (2) give a comparison between the $P_{17}$ and B – Spline method given in[4] to illustrate the accuracy of suggested method. Also, figure (1) gives comparison between the exact and suggested method $P_{17}$.

### 3. Conditioning of BVP's

In particular, BVP's for which a small change to the ODE or boundary conditions results in a small change to the solution must be considered, a BVP's that has this property is said to be well-conditioned.[5] Otherwise, the BVP's is said to be ill-conditioned. To be useful in applications, a boundary value problems should be well posed. This means that given the input to the problem there exists a unique solution, which depends continuously on the input .Consider the following third order BVP's :

$$y^{(3)}(x) = f(x, y(x), y'(x), y''(x)) \;, x \in [0, 1] \qquad , \qquad (9a)$$

With BC: $y^{(i)}(0) = A_i , y^{(j)}(1) = \beta_j$ , $i= 0,1,\dots,k-1$ , $j= 0,1,\dots, n-k+1$ , (9b)

For a well-posed problem we now make the following assumptions:

1. Equation (9) has an approximate solution $P \in C^n[0, 1]$, with this solution and $\rho > 0$ , we associate the spheres :

$$S\rho(P(x)) := \{ y \in IR^n : | P(x) - y(x) | \le \rho \}$$

2. $f( x, P(x), P'(x), P''(x) )$ is continuously differentiable with respect to P, and $\partial f / \partial P$ is continuous .

The following assumptions are important due to the error associated with approximate solutions to BVP's, depending on the semi-analytic technique, approximate solution $\check{y}(x)$ to the linear nth-order BVP's (9) may exactly satisfy the perturbed ODE :

$$\check{y}^{(n)} = u(x)\, \check{y}^{(n-1)} + \dots + d(x)\check{y}' + q(x)\, \check{y} + r(x) \;;\; 0 < x < 1 \;; \qquad (10a)$$

where $r : R \rightarrow R^m$ , and the linear BC :

$$B_0\, \breve{y}(0) + B_1\, \breve{y}(1) = \beta + \propto \quad ; \qquad (10b)$$

where $\beta + \propto = \sigma$ , $\sigma \in R^m$ and $\{\propto, \beta, \sigma\}$ are constants. If $\breve{y}$ is a reasonably good approximate solution to (9), then $\|r(x)\|$ and $\|\sigma\|$ are small. However, this may not imply that $\breve{y}$ is close to the exact solution y. A measure of conditioning for linear BVP's that relates both $\|r(x)\|$ and $\|\sigma\|$ to the error in the approximate solution can be determined. The following discussion can be extended to nonlinear BVP's by considering the variational problem on small sub domains of the nonlinear BVP's [6].

Letting : $e(x)= |\breve{y}(x) - y(x)|$; then subtracting the original BVP's (9) from the perturbed BVP's (10) results in :

$$e^{(n)}(x) = |\breve{y}^{(n)}(x) - y^{(n)}(x)| \qquad ; \quad (11a)$$

$$e^{(n)}(x)= u(x)\, e^{(n-1)}(x) +\ldots+d(x)\, e'(x)+ q(x)\, e(x) + r(x); \;\; 0 < x < 1 \; ; \quad (11b)$$

with BC : $B_0\, e(0) + B_1\, e(1) = \sigma$ ; (11c) However, the form of the solution can be furthered simplified by letting : $\Theta(x) = Y(x)\, Q^{-1}$ ; where Y is the fundamental solution and Q is defined in (7b) . Then the general solution can be written as :

$$e(x) = \Theta(x)\, \sigma + \int_0^1 G(x, t)\, r(t)\, dt \qquad ; \qquad (12)$$

where G(x, t) is Green's function [7], taking norms of both sides of (12) and using the Cauchy - Schwartz inequality [7] results in :

$$\|e(x)\|_\infty \le k_1 \|\sigma\|_\infty + k_2 \|r(x)\|_\infty \qquad ; \qquad (13)$$

where $k_1 = \|Y(x)Q^{-1}\|_\infty$ ; and $k_2 = \sup\limits_{0\le x\le 1} \int_0^1 \|G(x, t)\|_\infty\, dt$ ,

In (13), the $L_\infty$ norm, sometimes called a maximum norm, is used due to the common use of this norm in numerical BVP's software. For any vector $v \in R^N$, the $L_\infty$ norm is defined as : $\|v\|_\infty = \max\limits_{1\le i\le N} |v_i|$:The measure of conditioning is called the conditioning constant k, and it is given by : $k = \max(k_1, k_2)$ ; (14)

When the conditioning constant is of moderate size, then the BVP's is said to be well-conditioned.

Referring again to (13), the constant k thus provides an upper bound for the norm of the error associated with the perturbed solution,

$$\| e(x) \|_{\infty} \leq k \left[ \| \sigma \|_{\infty} + \| r(x) \|_{\infty} \right] \quad ; \quad (15)$$

It is important to note that the conditioning constant only depends on the original BVP's and not the perturbed BVP's. As a result, the conditioning constant provides a good measure of conditioning that is independent of any numerical technique that may cause such perturbations. The well conditioned nature of a BVP's and the local uniqueness of its desired solution are assumed in order to solve the problem numerically.

## References


1. Prasad K. R.,and Kameswara Rao A., "Multiple positive solutions for nonlinear third order general two-point boundary value problems ", Electronic Journal of Qualitative Theory of Differential Equations , No. 9, pp: 1-17, 2009 .
2. Wang L., and Pei M.," Existence and Uniqueness for Nonlinear Third-Order Two-Point Boundary Value Problems ", Mathematical Analysis J , Vol.14, PP: 321-332, 2007 .
3. Phillips G .M. , "Explicit forms for certain Hermite approximations ", BIT 13 (1973) 177-180.
4. Kashem B.E.," Partition Method for Solving Boundary Value Problem Using B-Spline Functions", Eng.& Tech. Journal ,Vol. 27, No. 11, 2009 .
5. Howell K.B. ," Ordinary Differential Equations" , Spring 2009, USA
6. Shampine L .F. , " Singular Boundary Value Problems for ODEs " , Southern Methodist University J., Vol. 1, 2002 .
7. Boisvert J. J., A Problem-Solving Environment for the Numerical Solution of Boundary Value Problems ,MSC Thesis, in the Department of Computer Science, University of Saskatchewan ,Saskatoon , Canada , January 2011.


**Table 1: The result of the suggested method for $P_{17}$ of example**

| $X_i$ | Exact Solution y(x) | Osculatory interpolation $P_{17}$ | Errors $\lvert y(x)-P_{17} \rvert$ |
|---|---|---|---|
| 0 | 0.000000000000000 | 0.000000000000000 | 0.000000000000000 |
| 0.1 | 0.225372976788598 | 0.225364769121097 | 8.207667501E-6 |
| 0.2 | 0.415413171479965 | 0.415396761919977 | 1.6409559988E-5 |
| 0.3 | 0.573186826784299 | 0.573162139019653 | 2.4687764646E-5 |
| 0.4 | 0.701559696972480 | 0.701526571840217 | 3.3125132263E-5 |
| 0.5 | 0.803238607059112 | 0.803196800952235 | 4.1806106877E-5 |
| 0.6 | 0.880812775733061 | 0.880761958161851 | 5.0817571211E-5 |
| 0.7 | 0.936795458049050 | 0.936735208333540 | 6.0249715510E-5 |
| 0.8 | 0.973666482074707 | 0.973596285135070 | 7.0196939637E-5 |
| 0.9 | 0.993916279190485 | 0.993835520392147 | 8.0758798338E-5 |
| 1 | 1.000092040986118 | 1.000092040986118 | 0.000000000000000 |
| *S.S.E = 2.992474308328639E-008* | | | |

**Table 2 : A comparison between $P_{17}$ and B – Spline method for example.**

| $X_i$ | Exact Solution y(x) | B – Spline | Osculatory interpolation $P_{17}$ |
|---|---|---|---|
| 0 | 0.000000000000000 | 0 | 0.000000000000000 |
| 0.1 | 0.225372976788598 | 0.2254 | 0.225364769121097 |
| 0.2 | 0.415413171479965 | 0.4154 | 0.415396761919977 |
| 0.3 | 0.573186826784299 | 0.5732 | 0.573162139019653 |
| 0.4 | 0.701559696972480 | 0.7015 | 0.701526571840217 |
| 0.5 | 0.803238607059112 | 0.8032 | 0.803196800952235 |
| 0.6 | 0.880812775733061 | 0.8808 | 0.880761958161851 |
| 0.7 | 0.936795458049050 | 0.9367 | 0.936735208333540 |
| 0.8 | 0.973666482074707 | 0.9736 | 0.973596285135070 |
| 0.9 | 0.993916279190485 | 0.9938 | 0.993835520392147 |
| 1 | 1.000092040986118 | 1 | 1.000092040986118 |
| | | *S.S.E* =0.00001 | *S.S.E* = *2.992474308328639E-008* |

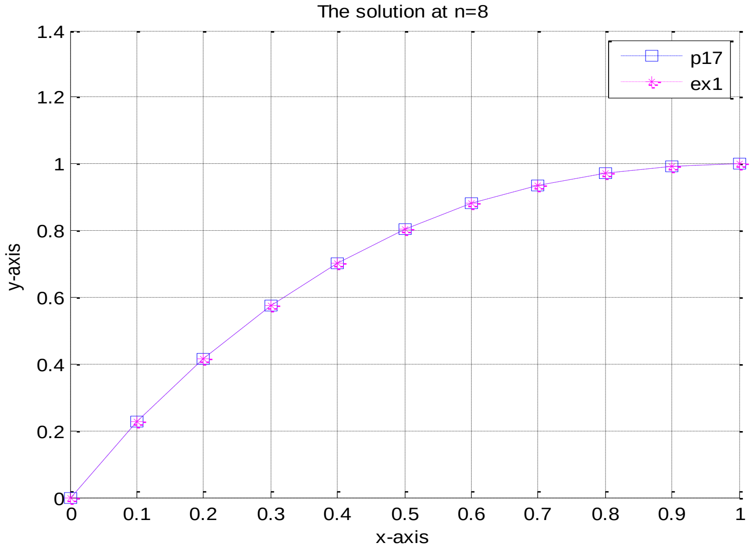


Figure 1: Comparison between the exact and suggested method $P_{17}$ of example