Local and Global Uniqueness Theorems of the N-th Order Partial Differential Equations

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Abstract:

In this paper, we consider inequalities in which the function is an element of n-th partially order space. Local and Global uniqueness theorem of solutions of the n-the order Partial differential equation Obtained which are applications of Gronwall's inequalities.

Key words: Local and global Uniqueness theorem, N-th order partial differential equations.

Introduction:

Consider the differential equation of the type

$$U(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n, U)$$

$$\text{With} \quad \begin{aligned} U(x_0, x_i) &= g(x) \\ U(x_i, x_0) &= h(x) \\ g(x_i) &= h(x_0) \end{aligned}$$

i=0, 1, 2,…., n… (1)

where *J=[0,a)* ,a>0 , $f \in [J \times R^n, R]$ and R^n denotes the real n-dimensional Euclidean space, $x_i \; i = 0,1,\ldots,n-1$ is a real positive constants and both g(x) and h(x) are continuous functions , *Liu* and *Ge[2]* based on the coincidence degree method of Gaines and Mawhin [3]. Proved that (1) has at least one solution ***U. Elias*** [4] proved the existence of global at least one solution to (1).

In this paper, Bihari's inequality is applied to obtain local uniqueness and Gronwall's inequality to obtain global uniqueness of solution to (1).
It is important mentioning that it was shown by Baihov D. and Simeonov [1] that the solution of (1) is of the form:

$$U(x) = \sum_{i=0}^{n-1} \frac{U_{i-1}}{i!} + \int_{x_0}^{x_1}\int_{x_2}^{x_2} \ldots \int_{x_{n-1}}^{x_n} f(v, s, \ldots, t, U)\, dv\, ds \ldots dt$$

$$x \in J \quad i = 0,1,2,\ldots,n$$

… (2)

with $0 < x_i < a$, $a > 0$ and

$$\begin{aligned} U(x_0, x_i) &= g(x) \\ U(x_i, x_0) &= h(x) . \\ g(x_i) &= h(x_0) \end{aligned}$$

are initial constant.

- Local Uniqueness:

In this section, a local uniqueness result is proved by applying Bihari's inequality theorem.

Bihari's Inequality Theorem [1]

Suppose the following conditions holds:

1. $a(t)$ is positive continuous function in $J = [\alpha, \beta)$
2. $K_j(t, s)$, $j = 1,2,3,\ldots,n$,are non negative continuous functions for $\alpha \le s \le t \le \beta$ which are no decreasing in **t** for any fixed **s.**
3. $g_j(u)$, $j = 1,2,\ldots,n$ are non decreasing continuous functions in R_+ ,with $g_j(u) > 0$ for *u>o* and

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$g(au) \le r(a)\,w(u)$,for $a > 0, u \ge 0$ were r(a)is non negative continuous function in R_+,which is positive for *u>0.*

4.u(t) is non negative continuous function in *J* and

$$u(t) \le a(t) + \sum_{j=1}^{n} \int_{\alpha}^{t} k_j(t,s) g_j(u(s))ds, t \in J$$

then

$$r(t) \le a(t)\psi_{n-1}(t) G_n^{-1}\left[G_n(1) + \frac{r_n(a(t))\psi_{n-1}(t)}{a(t)}\right] \int_0^t k_n(t,s)ds$$

, where

$$G_n(u) = \int_{u_n}^{u} \frac{dx}{g_n(x)} \quad , u > 0, (u_n > 0)$$

***Theorem 1**: (Local Uniqueness)*

The initial value problem (1) has a unique solution on the interval *0<u<a,* if the function f is continuous in the region

$$0 < x < a, |(x_1, x_2,...,x_n, u) - (x_1, x_2,...,x_n, v)| \le b$$

and such that

$$|f(x_1, x_2,...,x_n, u) - F(x_1, x_2,...,x_n, v)| \le \sum_{i=0}^{n-1} \phi_i(|U^i - V^i|)$$

where ϕ *(z)*is a continuous non decreasing function on 0<z<A, with ϕ*(0)=0* , b>0 and A is a positive constant .

Proof:

Let *U(x)* and *V(x)* be two solutions to (1) which are defined in neighborhood at the right of x_0 .That is

$$U(x) = \sum_{i=0}^{n-1} \left(g(x_i) + h(x_i) - g(x_0)\right)$$

$$+ \int_{x_0}^{x_1}\int_{x_2}^{x_2} .... \int_{x_{n-1}}^{x_n} f(v, s,..., t, U) dv ds....dt$$

$$V(x) = \sum_{i=0}^{n-1} \left(G(x_i) + H(x_i) - G(x_0)\right)$$

$$+ \int_{x_0}^{x_1}\int_{x_2}^{x_2} .... \int_{x_{n-1}}^{x_n} F(v, s,..., t, V) dv ds....dt$$

This leads easily to

$$|U(x) - V(x)| \le \sum_{i=0}^{n-1} \begin{bmatrix} |g(x_i) - G(x_i)| + \\ |h(x_0) - H(x_0)| + |g(x_0) - G(x_0)| \end{bmatrix} +$$

$$\int_{x_0}^{x_1}\int_{x_1}^{x_2} .... \int_{x_{n-1}}^{x_n} \left\| \begin{matrix} f(v, s,..., t, U) - \\ F(v, s,..., t, V) \end{matrix} \right| dv ds..dt$$

If $\prod_{i=0}^{n-1}(x_{i-1} - x_i) \ge 0$ and $|f(v,s,...,t,U) - F(v,s,...,t,V)| < \in$ and $\sum_{i=0}^{n-1} \begin{bmatrix} |g(x_i) - G(x_i)| + \\ |h(x_0) - H(x_0)| + \\ |g(x_0) - G(x_0)| \end{bmatrix} < \in$

$$|U(x) - V(x)| \le \sum_{i=0}^{n-1} \in_i + \in \prod_{i=0}^{n-1} (x_{i-1} - x_i)$$

$$+ \int_{x_0}^{x_1}\int_{x_1}^{x_2} .... \int_{x_{n-1}}^{x_n} K|U - V| dv ds..dt$$

Let $r(x)$ be the right hand side of the above inequality,

$$r(x) \le \sum_{i=0}^{n-1} \in_i + \in \prod_{i=0}^{n-1} (x_{i-1} - x_i) +$$

$$\int_{x_0}^{x_1}\int_{x_1}^{x_2} .... \int_{x_{n-1}}^{x_n} \begin{matrix} \in_i + (v - x_0)(s - x_1).... \\ (t - x_n)\, Kz(v, s,..., t) \end{matrix} dv ds..dt$$

If only $\prod_{i=0}^{n-1}(x_{i-1} - x_i) \ge 0$ .on a compact space, the last equation is bounded by a constant M. $|U - V| \le M \in$ on this set .consequently , the solution of such boundary value problem equation (1) depends continuously on f and the boundary date .if $\in \to 0, |U - V| \to 0$, on the compact set.

By using Bihari's inequality we have

$$r(x) \le \in \psi_{n-1}(x) G_n^{-1}\left[G_n(1) + \frac{r_n(\in)\psi_{n-1}(x)}{\in}\right]$$

Global Uniqueness

The global uniqueness for the initial value problem (1) will be discussed with the aid of Gronwall's inequality, which seems by the following theorem.

***Gronwall's Inequality Theorem* [6]**

Let *a (t), b (t)* and *u (t)* be continuous functions in $J=[\alpha,\beta]$ and let *b(t)* be a nonnegative in $J=[\alpha,\beta]$ and *a(t)* is nondecreasing in $J=[\alpha,\beta]$ suppose

$$u(t)\le a(t)+\int_{\alpha}^{t} b(s)u(s)\,ds \quad ,t\in J$$

Then

$$u(t)\le a(t)e^{\int_{\alpha}^{t} b(s)ds} \quad ,t\in J$$

Theorem (2) *(Global uniqueness theorem)*

Assume that:

1. f is a continuous function in the region R={(s$(x_1,x_2,\ldots,x_n,U)$): $0<x<a,\left|(x_1,x_2,\ldots,x_n,u)-(x_1,x_2,\ldots,x_n,v)\right|\le b$ }$\subset\Omega$ where Ω is an open $(x_0,x_1,x_2,\ldots,x_n,u)$ in R^{n+1} with a, b>0.
2. *f* satisfy Lipschitz condition with respect to $(x_0,x_1,x_2,\ldots,x_n,u)$,

$$\left|f(x_1,x_2,\ldots,x_n,u)-F(x_1,x_2,\ldots,x_n,v)\right|\le L\sum_{i=0}^{n-1}\left|U^i-V^i\right|$$

For some positive constant L, then the solution of (1) is unique.

Proof

Let U(x) and V(x) be two solutions to (1) then

$$U(x)=\sum_{i=0}^{n-1}\left(g(x_i)+h(x_i)-g(x_0)\right)+\int_{x_0}^{x_1}\int_{x_2}^{x_2}\cdots\int_{x_{n-1}}^{x_n} f(v,s,\ldots,t,U)\,dvds\ldots.dt$$

$$V(x)=\sum_{i=0}^{n-1}\left(G(x_i)+H(x_i)-G(x_0)\right)+\int_{x_0}^{x_1}\int_{x_2}^{x_2}\cdots\int_{x_{n-1}}^{x_n} F(v,s,\ldots,t,V)\,dvds\ldots.dt$$

$x\in J$

From which we get

$$|U(x)-V(x)|\le\sum_{i=0}^{n-1}\left[\begin{matrix}|g(x_i)-G(x_i)|+\\ |h(x_0)-H(x_0)|+|g(x_0)-G(x_0)|\end{matrix}\right]+\int_{x_0}^{x_1}\int_{x_1}^{x_2}\cdots\int_{x_{n-1}}^{x_n}\left|\begin{matrix}f(v,s,\ldots,t,U)-\\ F(v,s,\ldots,t,V)\end{matrix}\right|dvds..dt$$

$$\le\sum_{i=0}^{n-1}\left[\begin{matrix}|g(x_i)-G(x_i)|+\\ |h(x_0)-H(x_0)|+|g(x_0)-G(x_0)|\end{matrix}\right]+\int_{x_0}^{x_1}\int_{x_1}^{x_2}\cdots\int_{x_{n-1}}^{x_n}L\left[\left|\begin{matrix}U(v,s,..t)-\\ V(v,s,\ldots,t)\end{matrix}\right|\right]dvds..dt$$

Let

$$z(v,s,\ldots,t)=|U(v,s,.t)-V(v,s,\ldots,t)|\;L$$

Then

$$|U(x)-V(x)|\le\sum_{i=0}^{n-1}\left[\begin{matrix}|g(x_i)-G(x_i)|+\\ |h(x_0)-H(x_0)|+|g(x_0)-G(x_0)|\end{matrix}\right]+\int_{x_0}^{x_1}\int_{x_1}^{x_2}\cdots\int_{x_{n-1}}^{x_n}z(v,s,\ldots,t)dvds..dt$$

Let r(x) equal to the right hand side of the above inequality, then

$$z(v,s,\ldots,t)\le r(v,s,\ldots,t)$$

$$r(x)\le\sum_{i=0}^{n-1}\left[|g(x_i)-G(x_i)|+|h(x_0)-H(x_0)|+|g(x_0)-G(x_0)|\right]+\int_{x_0}^{x_1}\int_{x_1}^{x_2}\cdots\int_{x_{n-1}}^{x_n}z(v,s,\ldots,t)dvds..dt$$

By the above inequality (Gronwall's inequality)

$$r(x)\le\sum_{i=0}^{n-1}\left[\begin{matrix}|g(x_i)-G(x_i)|+\\ |h(x_0)-H(x_0)|+\\ |g(x_0)-G(x_0)|\end{matrix}\right]e^{\int_{x_0}^{x_1}\int_{x_1}^{x_2}\cdots\int_{x_{n-1}}^{x_n}dvds..dt}$$

Since $g(x_i)=G(x_i)$, $h(x_0)=H(x_0)$ and $g(x_0)=G(x_0)$ then $r(x)\le 0$ since

$$|u(x_0,x_1,\ldots,x_n)-v(x_0,x_1,\ldots,x_n)\le|r(x)\le 0$$

Then

$$|u(x_0,x_1,\ldots,x_n)-v(x_0,x_1,\ldots,x_n)|\le 0$$

and since the absolute value larger than or equal to zero then

U(x)=v(x) $x\in J$

Example: consider the two boundary value problems

$$u_{xy} = f(x,y,u)\ ,\ 0 < x,y < a, a > 0 \ \ldots \ (3)$$

$$u_{(x_0,y)} = g(y) = e^y,$$

$$u_{(x,y_0)} = h(x) = e^x$$

$$g(y_0) = e^{y_0} = h(x_0) = e^{x_0}$$

and

$$U_{xy} = F(x,y,U),$$

$$U(x_0,y) = G(y) = \cos y,$$

$$U(x,y_0) = H(x) = \cos x$$

$$G(y_0) = \cos y_0 = H(x_0) = \cos x_0$$

Where all the functions are continuous and f satisfies a Lipchitz condition with constant K. $K > 0$

If and only if $(x - x_0)(y - y_0) \geq 0$ .on a compact set

Conclusion:

1- It is easy to note that the uniqueness of a special cases solution (n=1 or n=2) can be obtain by using Bihari's and Gronwall's inequality which is give the work more accuracy and easer.

2-The quantity between the brackets in the above example is bounded by the constant M, hence $|u - U| \leq M \in$ on this set.

3-the solution of such boundary value problem eq(3) depends consequently on f and the boundary data .

If $\in \to 0$ then $|u - U| \to 0$ on the compact set.

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نظريتي الوحدانية العامة والمحلية للمعادلات التفاضلية الجزئية من الرتبة النونية

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الخلاصة:

لقد تم الاعتماد على المتراجحات التي تساعد مشتقات الدوال الجزئية ذات الرتبة النونية للحصول على نظرية الوحدانية العامة والمحلية لحل المعادلة التفاضلية الجزئية وهي تطبق لمتراجحات كرانول.