



## Equality of Dedekind sums modulo 72Z

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### Abstract:

According to the wide appearance of Dedekind sums in different applications on different subjects, a new approach for the equivalence of the essential and sufficient condition for  $12s(a_1, b) - 12s(a_2, b)$  in  $24Z$  and  $12s(a_1, b) - 12s(a_2, b)$  in  $72Z$  where  $s(a, b) = \sum_{k \bmod b} \left( \left( \frac{ak}{b} \right) \right) \left( \left( \frac{k}{b} \right) \right)$  and the equality of two Dedekind sums with their connections is given. The conditions for  $12s(a_1, b) - 12s(a_2, b)$  in  $24Z$  which is equivalent to  $12s(a_1, b) - 12s(a_2, b)$  in  $72Z$  were demonstrated with condition that of 9 does not divide  $b$ . Some applications for the important of Dedekind sums were given.

**Keywords:** Congruence, Dedekind sums, Equivalent, Inversion number.

## المساواة بين مجاميع ديديكيند موديول 72Z

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### الخلاصة

انطلاقاً من الظهور الواسع لمجاميع ديديكيند في التطبيقات المختلفة لمختلفة المواضيع، تم اتباع للتكافؤ بين الشروط الضرورية والكافية للمعادلة

$$12s(a_1, b) - 12s(a_2, b) \in 24Z \text{ و } 12s(a_1, b) - 12s(a_2, b) \in 72Z$$

وتم ربطها بتساويهما. وكذلك تم تقديم شرط للتكافؤ بين المجموعين وهوان

$b$  لا تقسم التسعة  $(b \nmid 9)$ . التطبيقات التي وضحت اهمية مجاميع ديديكيند تم عرضها في امثلة مختلفة

### 1. Introduction

The appearance of Dedekind sums in a wide range of geometry and number theory are explored and the hopes of illuminating new connections. Dedekind sums are traditional themes of study that familiarized by Richard Dedekind in the nineteenth century in his investigation of the  $\eta$ -function [1]. Dedekind sums emerge in many fields of mathematic, most unmistakably in combinatorial geometry (lattice point account in polytope) [2], analytic number theory (modular forms) [1], topology (signature deformities of manifolds)[3], algorithmic complexity (pseudo-random number generators) [4, 5] and algebraic number theory (class numbers)[5] were investigated. To express the Dedekind sums, let

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer} \\ 0 & \text{otherwise} \end{cases}$$

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Where  $[x]$  means the greatest integer less than or equal to  $x$ , and it is called sawtooth function. For  $a, b \in \mathbb{N}$  is coprime the Dedekind sum that denoted with  $s(a, b)$  is defined by:

$$s(a, b) = \sum_{k \bmod b} \left( \left( \frac{ak}{b} \right) \right) \left( \left( \frac{k}{b} \right) \right).$$

Newly, Jabuka et al. [6] demonstrated the essential condition for correspondence of two Dedekind sums  $s(a_1, b)$  and  $s(a_2, b)$  with the condition that of  $b \mid (a_1 a_2 - 1)(a_1 - a_2)$ . Girstmair [7] confirmed that this condition is proportionate to  $12s(a_1, b) - 12s(a_2, b)$  in  $\mathbb{Z}$ , also Tuskrman [8] gave the essential and sufficient conditions for  $12s(a_1, b) - 12s(a_2, b)$  belong to  $2\mathbb{Z}$  and  $4\mathbb{Z}$ . Tuskrman [5] introduced a proof for the essential and sufficient condition for  $12s(a_1, b) - 12s(a_2, b)$  in  $8\mathbb{Z}$ . In a current note Girstmair gave an essential and sufficient condition for  $12s(a_1, b) - 12s(a_2, b)$  in  $24\mathbb{Z}$ . The aim of present research is to demonstrate this condition which is equivalent to  $12s(a_1, b) - 12s(a_2, b)$  in  $72\mathbb{Z}$ , with the condition that 9 does not divide  $b$ .

**2. Basic concepts**

Some basic concepts that relates to the Dedekind sums and properties are introduced , which are [9]:

Let  $a$  and  $b$  be coprime, then we have the following properties

1. The Dedekind sums is a periodic function of the variable  $a$ , with period  $b$ , by periodicity of the Sawtooth function. That is,

$$s(a + kb, b) = s(a, b) \quad \text{for all } k \text{ in } \mathbb{Z}.$$

2.  $s(1, b) = \frac{-1}{4} + \frac{1}{6b} + \frac{b}{12}$ .

3.  $6b s(a, b)$  in  $\mathbb{Z}$ .

4. The most remarkable outcome regarding the Dedekind sum is the well-known reciprocity formula:

$$s(a, b) + s(b, a) = \frac{1}{12} \left( \frac{a}{b} + \frac{b}{a} + \frac{1}{ab} \right) - \frac{1}{4}.$$

Consequently, reciprocity formula allows us to calculate  $s(a, b)$  in polynomial time analogous in spirit to the Euclidean algorithm.

**Example 1 [2]:**

Let  $a = 100$  and  $b = 147$ , now alternately using the reciprocity formula and some properties of Dedekind sums:  $s(100, 147) = \frac{577}{882}$ . For more details see [2]

A priori, to calculate  $s(100, 147)$ , it takes 147 steps, where as it can be computed with nine step by means of the reciprocity formula.

**Definition 1 [1]:**

For any integer  $a$  and  $n$  in  $\mathbb{Z}^+$ . The symbol  $\left( \frac{a}{n} \right)$  is characterized as the result of the Legendre symbol  $\left( \frac{a}{p} \right)$ . That is,

$$\left( \frac{a}{n} \right) = \prod_{j=1}^r \left( \frac{a}{p_j} \right)$$

Where  $n = p_1 p_2 \dots p_{1j} \dots p_r$  and each  $p_j$  is prime is called the Jacobi symbol.

**Definition 2[10]:**

Let  $A$  be a finite set  $\{1, 2, \dots, n\}$ , a permutation group is defined as a group, whose elements are a permutation of  $A$ .

**Definition 3 [1]:**

Given a permutation  $\pi$  of the numbers  $\{1, 2, \dots, b\}$ , the inversion number of  $\pi$  is define as  $I(\pi) = \#\{(i, j) \mid 1 \leq i < j \leq b, \pi(i) > \pi(j)\}$ . That is the number of times a bigger passage goes before a littler one, it is called number of inversion. If  $a$  is relatively prime to  $b$ , then the map  $\pi(a, b) : \{1, 2, \dots, b\} \rightarrow \{1, 2, \dots, b\}$  given by  $x \rightarrow ax \pmod{b}$  is a permutation of  $\{1, 2, \dots, b\}$ . In this case  $I(a, b)$  signify the number of inversions of  $\pi(a, b)$ .

The following result demonstrates that, the inversions of  $\pi(a, b)$  and Jacobi symbol are firmly related.

**Theorem 1 [1]:**

For odd  $b$  and the greatest common divisor of integers  $a$  and  $b$  equal to one,  $\text{g.c.d}(a, b)$ , then

$$(-1)^{I(a,b)} = \left(\frac{a}{b}\right)$$

where  $I(a, b)$  and  $\left(\frac{a}{b}\right)$  are the number of inversion and the Jacobi symbol respectively.

The relationship between the number of inversions and Dedekind sums are given in the theorem 2 as follows:

**Theorem 2, [1, 11]:**

Let  $\pi(a, b)$  in  $\text{Aut}(Z/bZ)$ , then the number of inversion  $I(a, b)$  of  $\pi(a, b)$  is:

$$I(a, b) = \frac{1}{4}(b-1)(b-2) - 3b\varsigma(a, b)$$

where  $\text{Aut}(Z/bZ)$  means the group of automorphisms of  $Z/bZ$ .

**3. Equality of Dedekind Sums**

A natural question about Dedekind sums is finding condition on the integers  $a_1, a_2$  and  $b$  such that two Dedekind sums are equal. Jabuka et al. [6] proved that theorem 3 which tell us  $\varsigma(a_1, b)$  and  $\varsigma(a_2, b)$  can only be equal under certain conditions.

**Theorem 3:**

Let  $a_1, a_2$  be relatively prime to  $b$ , where  $b$  in  $Z^+$  and  $a_1, a_2$  in  $Z$ . If  $\varsigma(a_1, b) = \varsigma(a_2, b)$ , then  $b|(1 - a_1a_2)(a_1 - a_2)$ .

We immediately have the following corollary

**Corollary 1 [6]:**

Let  $a_1$  and  $a_2$  in  $N$  and  $p$  be a prime integer number then  $\varsigma(a_1, p) = \varsigma(a_2, p)$ , if and only if,  $p|(1 - a_1a_2)$  or  $p|(a_1 - a_2)$ .

Girstmair[7] extends the preceding theorem 3 with the following result.

**Theorem 4:**

Let  $a_1, a_2$  be relatively prime to  $b$ , where  $b$  in  $Z^+$  and  $a_1, a_2$  in  $Z$ , then  $12\varsigma(a_1, b) - 12\varsigma(a_2, b)$  in  $Z$ , if and only if,  $b|(1 - a_1a_2)(a_1 - a_2)$ .

**Definition 4[12]:**

Let  $a$  and  $b$  be integers and  $m$  be positive integer, is said to be congruent to  $b$  modulo  $m$  denoted by  $(a \equiv b \pmod{m})$  if  $m$  divides  $(a - b)$  the integer  $m$  is called the modulo of the congruencies.

Theorem 5 is used to find two other necessary and sufficient conditions for  $12\varsigma(a_1, b) - 12\varsigma(a_2, b)$  in  $2Z$  and  $4Z$ . It gives new necessary conditions on equality of Dedekind sums.

**Theorem 5 [8]:**

Let  $b$  in  $N$  and  $a_1, a_2$  in  $N$ , where  $\text{g.c.d.}(a_1, b) = 1$  and  $\text{g.c.d.}(a_2, b) = 1$ .

1. The ensuing conditions are equivalents:

i.  $4I(a_1, b) \equiv 4I(a_2, b) \pmod{b}$

ii.  $12\varsigma(a_1, b) - 12\varsigma(a_2, b)$  in  $Z$

iii.  $b|(1 - a_1a_2)(a_1 - a_2)$

2. The ensuing conditions are equivalents :

i.  $2I(a_1, b) \equiv 2I(a_2, b) \pmod{b}$

ii.  $12\varsigma(a_1, b) - 12\varsigma(a_2, b)$  in  $2Z$

iii.  $2b|(1 - a_1a_2)(a_1 - a_2)(b + a_1a_2 - 1)$

3. The ensuing conditions are equivalents :

i.  $I(a_1, b) \equiv I(a_2, b) \pmod{b}$

ii.  $12\varsigma(a_1, b) - 12\varsigma(a_2, b)$  in  $4Z$

iii.  $4b|(1 - a_1a_2)(a_1 - a_2)(b + a_1a_2 - 1)$

By consuming a generalization of Zolotarev's theorem concerning the Jacobi symbol, Tsukerman[8] gave the necessary and sufficient conditions for  $12\varsigma(a_1, b) - 12\varsigma(a_2, b)$  in  $8Z$  in Theorem 6.

**Theorem 6 [5]:**

Let  $a_1, a_2, b$  in  $N$ , where  $N = \{0, 1, 2, 3, \dots\}$  and  $a_1, a_2$  be relatively prime to  $b$ .

The ensuing conditions are equivalents:

i.  $(a_1, b) \equiv I(a_2, b) \pmod{2b}$ .

ii.  $12\varsigma(a_1, b) - 12\varsigma(a_2, b)$  in  $8Z$ ;

$$\text{iii. Define } M(a, b) = \begin{cases} \frac{(1 - (\frac{a}{p}))}{2} & \text{if } b \text{ is odd} \\ \frac{(a-1)(a+b-1)}{4} & \text{if } b \text{ is even} \end{cases}$$

Then  $(a_1 - a_2)(b - 1)(a_1 a_2 + b - 1) \equiv 4b(a_2 M(b, a_1) - a_1 M(b, a_2)) \pmod{8b}$ .  
Some theorems, remarks and definitions are needed to reach of our aim.

**Remark 1 [1]:**

Let  $a, b$  in  $N$  be coprime, the following conditions are hold:

- i.  $6b\varsigma(a, b) \equiv 0, \mp 1, \mp 3 \pmod{9}$ .
- ii. If  $3 \nmid b$ , then  $12b\varsigma(a, b) \equiv 0 \pmod{3}$ .
- iii. If  $3|b$ , then  $12b\varsigma(a, b) \equiv 2\epsilon \pmod{9}$ .
- iv.  $a \equiv \epsilon \pmod{3}$ , where  $\epsilon = \{\mp 1\}$ .

**Definition 5, [13]:**

Let  $a, b$  in  $N$  be coprime, then

$$M(a, b) = \begin{cases} 2 - 2\left(\frac{a}{b}\right) & \text{if } b \text{ is odd} \\ (a - 1)(a + b - 1) & \text{if } b \text{ is even} \end{cases}$$

Where  $\left(\frac{a}{p}\right)$  is a Jacobi symbol, the function  $M$  may swap by the following simpler function

$$H(a, b) = \begin{cases} 2 - 2\left(\frac{a}{b}\right) & \text{if } b \text{ is odd} \\ 4 & \text{if } b \equiv 0 \pmod{4} \text{ and } a \equiv 3 \pmod{4} \\ 0 & \text{otherwise} \end{cases}$$

**Theorem 7 [11]:**

Let  $a_1, a_2$  in  $N$  be relatively prime to  $b$  in  $N$ . Suppose, further that of  $9 \nmid b$ , then  $12\varsigma(a_1, b) - 12\varsigma(a_2, b) \in 24Z$  if and only if,  
 $b(a_2 H(b, a_1) - a_1 H(b, a_2)) \equiv (a_1 - a_2)(b - 1)(a_1 a_2 + b - 1) \pmod{8b}$  (1)

**4. Main results and proofs**

**Theorem 8:**

Let  $a_1, a_2$  be relatively prime to  $b$  in  $N$  and  $9 \nmid b$ . Then  $\varsigma(a_1, b) - \varsigma(a_2, b) \in 72Z$  if and only if,  $\varsigma(a_1, b) - \varsigma(a_2, b) \in 24Z$ .

**Proof:**

By using the relationship ii and iii given in remark 1, and from the definition 4 of congruence, we get

$$12\varsigma(a_1, b) = \frac{3k_1}{b} \text{ and } 12\varsigma(a_2, b) = \frac{3k_2}{b}, \text{ with } k_1, k_2 \text{ in } Z.$$

Now, the congruence (1) in theorem 7 is equivalent to  $12\varsigma(a_1, b) - 12\varsigma(a_2, b) \in 24Z$ . Consequently, it is also equivalent to  $\frac{3(k_1 - k_2)}{b} = 24r, r \text{ in } Z$ . However,  $3 \nmid b$  and so this means  $3|r$ , then  $12\varsigma(a_1, b) - 12\varsigma(a_2, b) \in 72Z$ . If  $3|b$ , then  $12b\varsigma(a, b) \equiv 2\epsilon \pmod{9}$ , where,  $\epsilon \in \{\pm 1\}$  also  $12\varsigma(a_1, b) = \frac{2\epsilon + 9k_1}{b}$  and  $12\varsigma(a_2, b) = \frac{2\epsilon + 9k_2}{b}$ , therefore  $12\varsigma(a_1, b) - 12\varsigma(a_2, b) = \frac{9(k_1 - k_2)}{b}$  which is equal to  $24r, r \in Z$ . If  $9|b$ , this simply means  $12\varsigma(a_1, b) - 12\varsigma(a_2, b) \in 24Z$ , which is the result of Gristmair's that given in theorem 7. However,  $9 \nmid b$  this means that  $3|r$ , then  $12\varsigma(a_1, b) - 12\varsigma(a_2, b) \in 72Z$  and the proof is completed.

**Theorem 9:**

Let  $a_1, a_2$  be relatively prime to  $b$  in  $N$ . If  $12\varsigma(a_1, b) - 12\varsigma(a_2, b) = 2^n$ , for some  $n$  in  $Z^+$ , then  $9|b$ .

**Poof:**

Suppose that is true. By Remark (1) that say if  $3|b$ , then  $12b\varsigma(a, b) \equiv 2\epsilon \pmod{9}$ , where  $\epsilon \in \{\pm 1\}$  and from definition 4 of congruence, we get

$$12\varsigma(a_1, b) = \frac{2\epsilon + 9k_1}{b} \text{ and } 12\varsigma(a_2, b) = \frac{2\epsilon + 9k_2}{b} \text{ with integers } k_1 \text{ and } k_2.$$

$$\text{Now, } 12\varsigma(a_1, b) - 12\varsigma(a_2, b) = \frac{9(k_1 - k_2)}{b}.$$

By using, the assumption of  $12s(a_1, b) - 12s(a_2, b) = 2^n$ , for some  $n$  in  $Z^+$ . Hence,  $\frac{9^{(k_1-k_1)}}{b} = 2^n$ . Since  $9 \nmid 2^n$  for any value of  $n$  then  $9 \mid b$ .

**Example 2:**

Let  $a_1 = 1, a_2 = 25$  and  $b = 72$ , then  $s(a_1, b) = \frac{2485}{432}$  and  $s(a_2, b) = \frac{181}{432}$ ,  
 $12s(a_1, b) - 12s(a_2, b) = 64 = 2^6$ , then  $9 \mid b$ .

The converse of this Theorem 9 is not true. For example

Let  $a_1 = 1, a_2 = 13$  and  $b = 72$  such that  $9 \mid b$ .

$s(a_1, b) = \frac{2485}{432}$  and  $s(a_2, b) = \frac{-35}{432}$ , but

$12s(a_1, b) - 12s(a_2, b) = 70 \neq 2^n$ , for any  $n$  in  $Z^+$ .

**ii. Application of Dedekind sum in geometric combinatorial**

In this section, we will give a formula for Ehrhart polynomial, that used to obtain results about lattice polyhedra and Dedekind sums. These applications include a formula for the number of lattice points in an arbitrary lattice tetrahedron.

**Theorem 10 [2](The Mordell-Pommersheim Tetrahedron):**

Let  $P$  be a tetrahedron with vertices  $(0,0,0), (a, 0,0), (0, b, 0), (0,0, c)$  and  $a, b, c$  are pairwise relatively prime. Then

$$L_P(t) = \frac{abc}{6} t^3 + \frac{ab+ac+bc+1}{4} t^2 + \left( \frac{1}{12} \left( \frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c} + \frac{1}{abc} \right) + \frac{a+b+c}{4} + \frac{3}{4} - s(bc, a) - s(ac, b) - s(ab, c) \right) t + 1.$$

**Example 3:**

Let us consider the tetrahedron with vertices  $(0,0,0), (3,0,0), (0,5,0)$  and  $(0,0,7)$ , to find the Ehrhart polynomial for the tetrahedron, we use the formula given by Theorem 10.

Solution:

$$L_P(t) = \frac{abc}{6} t^3 + \frac{ab+ac+bc+1}{4} t^2 + \left( \frac{1}{12} \left( \frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c} + \frac{1}{abc} \right) + \frac{a+b+c}{4} + \frac{3}{4} - s(bc, a) - s(ac, b) - s(ab, c) \right) t + 1.$$

$$= \frac{105}{6} t^3 + \frac{15+21+35+1}{4} t^2 + \left( \frac{3}{4} + \frac{15}{4} + \frac{1}{12} \left( \frac{35}{3} + \frac{21}{5} + \frac{15}{7} + \frac{1}{105} \right) - s(35, 3) - s(21, 5) - s(15, 7) \right) t + 1,$$

$$s(35,3) = -\frac{1}{18}, s(35,3) = \frac{1}{5} \text{ and } s(35,3) = \frac{5}{14}$$

$$L_P(t) = \frac{35}{2} t^3 + 18t^2 + \frac{11}{2} t + 1.$$

The formula of Example 4 is profitable, because it reduces the number of computation of the Dedekind sums, which can be done efficiently. Indeed, recursively by applying the reciprocity formula and the obvious identity

$$s(a+kb, b) = s(a, b), \quad \text{for all } k \in Z.$$

**5. Conclusions:**

it was found that two Dedekind sums is equal. The equivalence between the essential and sufficient conditions for  $12s(a_1, b) - 12s(a_2, b)$  in  $24Z$ , and  $12s(a_1, b) - 12s(a_2, b)$  in  $72Z$  are investigated and it is connected by the equality of two Dedekind sums. In comparable way it can evidence the equivalence conditions for equality of two Dedekind sums modulo  $216Z, 648Z, \dots$ , if the modular that considered were satisfied the following sequence :  $Z, 2Z, 4Z, 8Z, 24Z, 72Z, 216Z, \dots$ , and so on, where  $Z$  be an integer number.

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