Journal of Kerbala University , Vol. 7 No.4 Scientific . 2009

Theorems on n- dimensional Sumudu transforms and their applications

Lecturer Amal Khalaf Haydar,
Kufa University,College of Education for Girls,Math. Dep.

Abstract

In this paper we prove eight fundamental theorems that include the Sumudu transform of $n-$variables and a table of $n-$Sumudu transform of some familiar functions that is calculated in this paper . In addition , two partial differential equations are solved by using the double Sumudu transform **.**

المستخلص

في هذا البحث أثبتنا ثمانية خصائص أساسية لتحويل سامودو النوني ، تم حساب جدول يبين تحويل سامودو النوني لبعض الدوال المألوفة بالإضافة إلى حل معادلتين تفاضليتين جزئيتين باستخدام تحويل سامودو الثنائي .

1 . Introduction

The single Sumudu transform (or Sumudu transform) was proposed by Watugala in [1] for functions of exponential order as follows : Consider functions in the set A , defined by

$$A=\{f(t)\,|\,\exists M,\tau_1\,,and\,/\,or\,\tau_2>0\,,$$
$$\text{such that}\quad |f(t)|<Me^{|t|/\tau_j},\ if\ \ t\in(-1)^j\times[0,\infty)\}\,. \tag{1.1}$$

For a given function in the set A , the constant M must be finite , while τ_1 and τ_2 need not simultaneously exist , and each may be infinite . For a given function $f(t)$ in A the Sumudu transform is defined by

$$F(u)=S\ [f(t)]=\int_0^\infty f(ut)e^{-t}dt,\quad u\in(-\tau_1,\tau_2)\,. \tag{1.2}$$

Or equivalently

$$F(u)=S\,[f(t)]=\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}f(t)dt\ , \tag{1.3}$$

provided the integral exists for some u . Also properties and applications of the Sumudu transform to ordinary differential equations are described in [1] . In [2] , Weerakoon derived formulas for the single Sumudu transform of partial derivatives and applied them in solving initial value problems . Subsequently exploited by many works such as Weerakoon in [3] and [4] . Watugala in [5] extended the Sumudu transform to functions of two variables with emphasis on solutions to partial differential equations . In [6] , the double Sumudu transform of functions expressible as polynomials or convergent series are derived .

The generalization of the single Sumudu transform to $n-$dimensional for a function $f(\bar{t})$ of the exponential order is given by

$$F(\bar{u})=S_n[f(\bar{t})]=\int_0^\infty\int_0^\infty...\int_0^\infty e^{-\sum_{i=1}^{n}t_i}f(\overline{\overline{ut}})\ \underset{i=1}{\overset{n}{\pi}}dt_i\ \ . \tag{1.4}$$

Or by

$$F(\bar{u})=S_n[f(\bar{t})]=\frac{1}{\underset{i=1}{\overset{n}{\pi}}u_i}\int_0^\infty\int_0^\infty...\int_0^\infty e^{-\sum_{i=1}^{n}\frac{t_i}{u_i}}f(\bar{t})\ \underset{i=1}{\overset{n}{\pi}}dt_i\ , \tag{1.5}$$

provided the integral exists for some $(\bar{u})$, $t_i \in R^+$, $(\bar{t}) = (t_1, t_2, ..., t_n)$, $(\bar{u}) = (u_1, u_2, ..., u_n)$ and $(\overline{ut}) = (u_1 t_1, u_2 t_2, ..., u_n t_n)$. The $n-$dimensional Sumudu transform S_n and the $n-$dimensional Laplace transform L_n of the function $f(\bar{t})$ [7] which is given by

$$L_n[f(\bar{t})] = \int_0^\infty \int_0^\infty ... \int_0^\infty e^{-\sum_{i=1}^{n} s_i t_i} f(\bar{t}) \prod_{i=1}^{n} dt_i \quad , \tag{1.6}$$

are theoretically dual . That is

$$F(\bar{u}) \prod_{i=1}^{n} u_i = L_n[f(\bar{t})]\Big|_{s_i = 1/u_i} . \tag{1.7}$$

Also

$$L_n[f(\bar{t})] \prod_{i=1}^{n} s_i = F(\bar{u})\Big|_{u_i = 1/s_i} . \tag{1.8}$$

2. Properties of the $n-$dimensional Sumudu transform

In this section we prove very important properties of the transform S_n through the following eight theorems

Thoerem 2.1 . Suppose that $F_i(u_i)$ be the Sumudu transform of the function $f_i(t_i)$ for $i = 1, 2, ..., n$. Then

$$S_n[\prod_{i=1}^{n} f_i(t_i)] = \prod_{i=1}^{n} F_i(u_i) . \tag{2.1}$$

Proof . From definition (1.4) and definition (1.2) we have

$$\begin{aligned} S_n[\prod_{i=1}^{n} f_i(t_i)] &= \int_0^\infty \int_0^\infty ... \int_0^\infty e^{-\sum_{i=1}^{n} t_i} \prod_{i=1}^{n} f_i(u_i t_i) \prod_{i=1}^{n} dt_i \\ &= \prod_{i=1}^{n} \int_0^\infty e^{-t_i} f_i(u_i t_i) dt_i = \prod_{i=1}^{n} F_i(u_i) . \end{aligned} \tag{2.2}$$

Remark 2.2. By using a similar proof of theorem 2.1 we can easily deduce that if $f(\bar{t})$ be a product of two independent functions , i .e. if

$$f(\bar{t}) = f_1(t_{i_1}, t_{i_2}, ..., t_{i_k}) \, f_2(t_{i_{k+1}}, t_{i_{k+2}}, ..., t_{i_n}) , \tag{2.3}$$

where $i_k \neq i_l$ for $k \neq l$ and $k, l, i_k, i_l = 1, 2, ..., n$. Then

$$S_n[f(\bar{t})] = S_k[f_1(t_{i_1}, t_{i_2}, ..., t_{i_k})] \, S_{n-k}[f_2(t_{i_{k+1}}, t_{i_{k+2}}, ..., t_{i_n})] \, , \tag{2.4}$$

i.e. the dimension of Sumudu transform of any function is equal to the number of the independent variables of that function . This speech is true if $f(\bar{t})$ is a product of any number doesn't exceed n of the independent functions .

Theorem 2.3. Let $f(\bar{t})$ is a function with $n-$dimensional Sumudu transform $F(\bar{u})$. Then

$$S_n[f(\overline{at})] = F(\overline{au}) \ , \tag{2.5}$$

where $(\overline{at}) = (a_1 t_1, a_2 t_2, ..., a_n t_n)$, $(\overline{au}) = (a_1 u_1, a_2 u_2, ..., a_n u_n)$ and a_i's are positive constants .

Proof . The $n-$dimensional Sumudu transform of $f(\overline{at})$ may be obtained directly from definition (1.4)

$$S_n[f(\overline{at})] = \int_0^\infty \int_0^\infty ... \int_0^\infty e^{-\sum_{i=1}^{n} t_i} f(\overline{aut}) \prod_{i=1}^{n} dt_i = F(\overline{au}) , \tag{2.6}$$

where $(\overline{aut}) = (a_1u_1t_1, a_2u_2t_2, ..., a_nu_nt_n)$.

Theorem 2.4 . Suppose that $F(\bar{u})$ is the $n-$dimensional Sumudu transform of the function $f(\bar{t})$. Then

$$S_n[e^{\sum_{i=1}^{n} a_it_i} f(\bar{t})] = \frac{1}{\pi_{i=1}^{n}(1-a_iu_i)} F(\overline{\frac{u}{1-au}}) \ , \tag{2.7}$$

where $(\overline{\frac{u}{1-au}}) = (\frac{u_1}{1-a_1u_1}, \frac{u_2}{1-a_2u_2}, ..., \frac{u_n}{1-a_nu_n})$.

Proof . From definition (1.4) we get that

$$S_n[e^{\sum_{i=1}^{n} a_it_i} f(\bar{t})] = \int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} e^{-\sum_{i=1}^{n}(1-a_iu_i)t_i} f(\overline{ut}) \pi_{i=1}^{n} dt_i . \tag{2.8}$$

Therefore , by the change of variables $w_i = (1-a_iu_i)t_i$, $i = 1,2,...n$, then

$$\begin{aligned} S_n[e^{\sum_{i=1}^{n} a_it_i} f(\bar{t})] &= \frac{1}{\pi_{i=1}^{n}(1-a_iu_i)} \int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} e^{-\sum_{i=1}^{n} w_i} f(\overline{\frac{uw}{1-au}}) \pi_{i=1}^{n} dw_i \\ &= \frac{1}{\pi_{i=1}^{n}(1-a_iu_i)} F(\overline{\frac{u}{1-au}}) . \end{aligned} \tag{2.9}$$

Note [4] : Recall that , the Heaviside function $H(t-a)$ is defined as

$$H(t-a) = \begin{cases} 0 \ , \ if \quad t < a \ , \\ 1 \ , \ if \quad t > a. \end{cases} \tag{2.10}$$

Theorem 2.5. Let $F(\bar{u})$ be the $n-$dimensional Sumudu transform of the function $f(\bar{t})$. Then

$$S_n[f(\overline{t-a}) \pi_{i=1}^{n} H_i(t_i - a_i)] = e^{-\sum_{i=1}^{n} \frac{a_i}{u_i}} F(\bar{u}) , \tag{2.11}$$

where $f(\overline{t-a}) = (t_1 - a_1, ..., t_n - a_n)$ and H is the Heaviside function .

Proof . From equation (2.10) we conclude that

$$\pi_{i=1}^{n} H_i(t_i - a_i) = \begin{cases} 0 \ , \ if \quad \exists \ i \ \ni t_i < a_i , \\ 1 \ , \ if \quad t_i > a_i \ \ \forall i. \end{cases} \tag{2.12}$$

Thus

$$f(\overline{t-a}) \pi_{i=1}^{n} H_i(t_i - a_i) == \begin{cases} 0 \qquad\qquad , \ if \quad \exists \ i \ \ni t_i < a_i , \\ f(\overline{t-a}) \ , \ if \quad t_i > a_i \ \ \forall i. \end{cases} \tag{2.13}$$

From definition (1.5) and equation (2.13) we get that

$$S_n[f(\overline{t-a})\underset{i=1}{\overset{n}{\pi}}H_i(t_i-a_i)]=\frac{1}{\underset{i=1}{\overset{n}{\pi}}u_i}\int_0^\infty\int_0^\infty\cdots\int_0^\infty e^{-\sum_{i=1}^{n}\frac{t_i}{u_i}}f(\overline{t-a})\underset{i=1}{\overset{n}{\pi}}H_i(t_i-a_i)\underset{i=1}{\overset{n}{\pi}}dt_i$$
$$=\frac{1}{\underset{i=1}{\overset{n}{\pi}}u_i}\int_{a_n}^\infty\int_{a_{n-1}}^\infty\cdots\int_{a_1}^\infty e^{-\sum_{i=1}^{n}\frac{t_i}{u_i}}f(\overline{t-a})\underset{i=1}{\overset{n}{\pi}}dt_i\,. \tag{2.14}$$

By setting $(\overline{t-a})=(\overline{x})$, i.e. $t_i-a_i=x_i$ for $i=1,2,...,n$ then

$$S_n[f(\overline{t-a})\underset{i=1}{\overset{n}{\pi}}H_i(t_i-a_i)]=\frac{1}{\underset{i=1}{\overset{n}{\pi}}u_i}\int_0^\infty\int_0^\infty\cdots\int_0^\infty e^{-\sum_{i=1}^{n}\frac{x_i+a_i}{u_i}}f(\overline{x})\underset{i=1}{\overset{n}{\pi}}dx_i$$
$$=e^{-\sum_{i=1}^{n}\frac{a_i}{u_i}}\frac{1}{\underset{i=1}{\overset{n}{\pi}}u_i}\int_0^\infty\int_0^\infty\cdots\int_0^\infty e^{-\sum_{i=1}^{n}\frac{x_i}{u_i}}f(\overline{x})\underset{i=1}{\overset{n}{\pi}}dx_i \tag{2.15}$$
$$=e^{-\sum_{i=1}^{n}\frac{a_i}{u_i}}F(\overline{u})\,.$$

Corollary 2.6. Suppose that $F_i(u_i)$ be the Sumudu transform of the functions $f_i(t_i)$ for $i=1,2,...,n$. Then

$$S_n[\underset{i=1}{\overset{n}{\pi}}f_i(t_i-a_i)H_i(t_i-a_i)]=e^{-\sum_{i=1}^{n}\frac{a_i}{u_i}}\underset{i=1}{\overset{n}{\pi}}F_i(u_i)\,. \tag{2.16}$$

Proof . If $f(\overline{t-a})=\underset{i=1}{\overset{n}{\pi}}f_i(t_i-a_i)$ then $f(\bar{t})=\underset{i=1}{\overset{n}{\pi}}f_i(t_i)$ and by applying theorem 2.5 and theorem 2.1 we get the desired result .

Theorem *2.7* . For an even number n we have

$$S_n[\underset{j=1}{\overset{n/2}{\pi}}(t_{i_{2j-1}}-c_jt_{i_{2j}})^{m_j}H(t_{i_{2j-1}}-c_jt_{i_{2j}})]=\underset{j=1}{\overset{n/2}{\pi}}\frac{(m_j)!(u_{i_{2j-1}})^{m_j+1}}{u_{i_{2j-1}}+c_ju_{i_{2j}}}, \tag{2.17}$$

where $i_k\neq l_l$ for $k\neq l$, $k,l,i_k,i_l=1,2,...,n$ and $m_j=0,1,2,...$.

Proof . From remark (2.2) we have

$$S_n[\underset{j=1}{\overset{n/2}{\pi}}(t_{i_{2j-1}}-c_jt_{i_{2j}})^{m_j}H(t_{i_{2j-1}}-c_jt_{i_{2j}})]=\underset{j=1}{\overset{n/2}{\pi}}S_2[(t_{i_{2j-1}}-c_jt_{i_{2j}})^{m_j}H(t_{i_{2j-1}}-c_jt_{i_{2j}})]. \tag{2.18}$$

For $k\neq l$ then using definition (1.5) when $n=2$ and definition of the Heaviside function in (2.10) , then integrating $m+1$ times and once with respect to dt_k and dt_l respectively give

$$S_2[(t_k-ct_l)^mH(t_k-ct_l)]=\frac{1}{u_ku_l}\int_0^\infty\int_0^\infty e^{-(\frac{t_k}{u_k}+\frac{t_l}{u_l})}(t_k-ct_l)^mH(t_k-ct_l)dt_k\,dt_l$$
$$=\frac{1}{u_ku_l}\int_0^\infty\int_{ct_l}^\infty e^{-(\frac{t_k}{u_k}+\frac{t_l}{u_l})}(t_k-ct_l)^m dt_k\,dt_l=\frac{m!(u_k)^{m+1}}{u_k+cu_l}\,. \tag{2.19}$$

If we put $k=i_{2j-1}$, $c=c_j$, $l=i_{2j}$ and $m=m_j$ in equation (2.19) and substitution the result in equation (2.18) the desired result is obtained .

Theorem 2.8 . Let $F(\overline{u})$ denote the $n-$dimensional Sumudu transform of the function $f(\bar{t})$. Then

$$S_n[\int_0^{t_n}\int_0^{t_{n-1}}...\int_0^{t_1} f(\bar{x})\, \underset{i=1}{\overset{n}{\pi}} dx_i = \underset{i=1}{\overset{n}{\pi}} u_i F(\bar{u}) . \tag{2.20}$$

Proof . To prove this theorem we shall define the functions $g_i(\bar{t})$ for $i = 0,1,...,n$ as follows

$$g_i(\bar{t}) = \int_0^{t_n}\int_0^{t_{n-1}}...\int_0^{t_{i+1}} f(t_1,...,t_i, x_{i+1},..., x_{n-1}, x_n)\, dx_{i+1}...dx_{n-1}dx_n . \tag{2.21}$$

Therefore ,

$$\int_0^{t_n}\int_0^{t_{n-1}}...\int_0^{t_1} f(\bar{x})\, \underset{i=1}{\overset{n}{\pi}} dx_i = g_0(\bar{t}).$$
$$f(\bar{t}) = g_n(\bar{t}). \tag{2.22}$$

Also

$$g_i(t_1, t_2,..., t_i, 0, t_{i+2},..., t_n) = 0 \text{ for } i = 0,1,..., n-1 . \tag{2.23}$$

Let for $i = 0,1,..., n-1$ that

$$\frac{\partial g_i(\bar{t})}{\partial t_{i+1}} = g_{i+1}(\bar{t}) . \tag{2.24}$$

By definition (1.5) and integrating by parts with respect to dt_1 and dt_2 respectively with using relations (2.23) and (2.24) and change the order of integration after each integration we get

$$S_n[\int_0^{t_n}\int_0^{t_{n-1}}...\int_0^{t_1} f(\bar{x})\, \underset{i=1}{\overset{n}{\pi}} dx_i] = S_n[g_0(\bar{t})]$$
$$= \frac{1}{\underset{i=1}{\overset{n}{\pi}} u_i}\int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} e^{-\sum_{i=1}^{n}\frac{t_i}{u_i}} g_0(\bar{t})\, \underset{i=1}{\overset{n}{\pi}} dt_i$$
$$= \frac{u_1}{\underset{i=1}{\overset{n}{\pi}} u_i}\int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} e^{-\sum_{i=1}^{n}\frac{t_i}{u_i}} g_1(\bar{t})\, dt_2 \underset{i=3}{\overset{n}{\pi}} dt_i\, dt_1 \tag{2.25}$$
$$= \frac{u_1 u_2}{\underset{i=1}{\overset{n}{\pi}} u_i}\int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} e^{-\sum_{i=1}^{n}\frac{t_i}{u_i}} g_2(\bar{t})\ dt_3 \underset{i=4}{\overset{n}{\pi}} dt_i\, dt_1\, dt_2 .$$

After performing $n-1$ integrations yields

$$S_n[g_0(\bar{t})] = \frac{\underset{i=1}{\overset{n-1}{\pi}} u_i}{\underset{i=1}{\overset{n}{\pi}} u_i}\int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} e^{-\sum_{i=1}^{n}\frac{t_i}{u_i}} g_{n-1}(\bar{t})\ dt_n dt_1 dt_2 ...dt_{n-1}$$
$$= \underset{i=1}{\overset{n}{\pi}} u_i \frac{1}{\underset{i=1}{\overset{n}{\pi}} u_i}\int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} e^{-\sum_{i=1}^{n}\frac{t_i}{u_i}} f(\bar{t})\ \underset{i=1}{\overset{n}{\pi}} dt_i , g_n(\bar{t}) = f(\bar{t}) \tag{2.26}$$
$$= \underset{i=1}{\overset{n}{\pi}} u_i F(\bar{u})$$

Theorem 2.9. Let $f(\bar{t})$ is a function with $n-$dimensional Sumudu transform $F(\bar{u})$. Then

$$S_n[\frac{1}{\underset{i=1}{\overset{n}{\pi}} t_i}\int_0^{t_n}\int_0^{t_{n-1}}...\int_0^{t_1} f(\bar{x})\, \underset{i=1}{\overset{n}{\pi}} dx_i] = \frac{1}{\underset{i=1}{\overset{n}{\pi}} u_i}\int_0^{u_n}\int_0^{u_{n-1}}...\int_0^{u_1} F(\bar{v})\, \underset{i=1}{\overset{n}{\pi}} dv_i . \tag{2.27}$$

Proof . From definition (1.4) and setting $(\bar{x}) = (\overline{vt})$ we have

$$S_n[\frac{1}{\prod_{i=1}^{n} t_i}\int_0^{t_n}..\int_0^{t_1} f(\bar{x})\prod_{i=1}^{n} dx_i = \int_0^{\infty}..\int_0^{\infty} e^{-\sum_{i=1}^{n} t_i}[\frac{1}{\prod_{i=1}^{n} u_i t_i}\int_0^{u_n t_n}..\int_0^{u_1 t_1} f(\bar{x})\prod_{i=1}^{n} dx_i]\prod_{i=1}^{n} dt_i$$

$$= \frac{1}{\prod_{i=1}^{n} u_i}\int_0^{\infty}..\int_0^{\infty}\int_0^{u_n}..\int_0^{u_1} e^{-\sum_{i=1}^{n} t_i}\frac{1}{\prod_{i=1}^{n} t_i} f(\overline{vt})\prod_{i=1}^{n} t_i \prod_{i=1}^{n} dv_i \prod_{i=1}^{n} dt_i \quad (2.28)$$

$$= \frac{1}{\prod_{i=1}^{n} u_i}\int_0^{u_n}..\int_0^{u_1}[\int_0^{\infty}..\int_0^{\infty} e^{-\sum_{i=1}^{n} t_i} f(\overline{vt})\prod_{i=1}^{n} dt_i]\prod_{i=1}^{n} dv_i$$

$$= \frac{1}{\prod_{i=1}^{n} u_i}\int_0^{u_n}..\int_0^{u_1} F(\bar{v})\prod_{i=1}^{n} dv_i \,.$$

Theorem 2.10. Let $f(\bar{t})$ is a function with $n-$dimensional Sumudu transform $F(\bar{u})$. Then

$$S_n[\prod_{j=1}^{m} t_{i_j}\frac{\partial^m f(\bar{t})}{\prod_{j=1}^{m}\partial t_{i_j}}] = \prod_{j=1}^{m} u_{i_j}\frac{\partial^m F(\bar{u})}{\prod_{j=1}^{m}\partial u_{i_j}} \,, \quad (2.29)$$

for $1 \le m \le n$, $i_k \ne i_l$ for $k \ne l$ and $k, l, i_k, i_l = 1,2,...,n$

Proof . From definition (1.4) and the relation

$$\prod_{j=1}^{m} t_{i_j} = \prod_{j=1}^{m}\frac{d(t_{i_j}u_{i_j})}{du_{i_j}} = \frac{\prod_{j=1}^{m} d(t_{i_j}u_{i_j})}{\prod_{j=1}^{m} du_{i_j}} \,, \quad (2.30)$$

we have

$$S_n[\prod_{j=1}^{m} t_{i_j}\frac{\partial^m f(\bar{t})}{\prod_{j=1}^{m}\partial t_{i_j}}] = \int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} e^{-\sum_{i=1}^{n} t_i}\prod_{j=1}^{m} u_{i_j} t_{i_j}\frac{\partial^m f(\overline{ut})}{\prod_{j=1}^{m}\partial(u_{i_j}t_{i_j})}\prod_{i=1}^{n} dt_i$$

$$= \prod_{j=1}^{m} u_{i_j}\int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} e^{-\sum_{i=1}^{n} t_i}\frac{\prod_{j=1}^{m} d(t_{i_j}u_{i_j})}{\prod_{i=1}^{m} du_{ij}}\frac{\partial^m f(\overline{ut})}{\prod_{j=1}^{m}\partial(t_{i_j}u_{i_j})}\prod_{i=1}^{n} dt_i \quad (2.31)$$

$$= \prod_{j=1}^{m} u_{i_j}\frac{\partial^m}{\prod_{j=1}^{m}\partial u_{i_j}}\int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} e^{-\sum_{i=1}^{n} t_i} f(\overline{ut})\prod_{i=1}^{n} dt_i$$

$$= \prod_{j=1}^{m} u_{i_j}\frac{\partial^m F(\bar{u})}{\prod_{j=1}^{m}\partial u_{i_j}} .$$

Note that if $m = n$ in theorem 2.10 then

$$S_n[\prod_{i=1}^{n} t_i\frac{\partial^n f(\bar{t})}{\prod_{i=1}^{n}\partial t_i}] = \prod_{i=1}^{n} u_i\frac{\partial^n F(\bar{u})}{\prod_{i=1}^{n}\partial t_i} \quad (2.32)$$

3 . The $n-$dimensional Sumudu transform of some functions

In this section we shall give a table of $n-$dimensional Sumudu transforms of some of the familiar functions which we find them using definition (1.4) or definition (1.5) or the theorems given in section 2 .

a table of $n-$dimensional Sumudu transforms of some functions.

No.	$f(\bar{t})$	$S_n[f(\bar{t})]$
1	1	1
2	$\pi_{j=1}^{n}(t_j)^{i_j}$, $i_1,i_2,...i_n=1,2,...$	$\pi_{j=1}^{n}(i_j)!(u_j)^{i_j}$
3	$e^{\sum_{i=1}^{n}a_it_i}$	$\dfrac{1}{\pi_{i=1}^{n}(1-a_iu_i)}$
4	$\pi_{i=1}^{n}H_i(t_i-a_i)$	$e^{-\sum_{i=1}^{n}\frac{a_i}{u_i}}$
5	$\sinh(\sum_{i=1}^{n}a_it_i)$	$\dfrac{\sum_{m=1}^{N/2}\sum_{i_1,i_2...i_{2m-1}=1}^{n}\frac{1}{(2m-1)!}\pi_{j=1}^{2m-1}a_{i_j}u_{i_j}}{\pi_{i=1}^{n}(1-a_i^2u_i^2)}$, $N=n$ if n is an even and $N=n+1$ if n is an odd number , $i_k\neq i_l$ for $k\neq l$ and $k,l,i_k,i_l=1,2,...,n$.
6	$\cosh(\sum_{i=1}^{n}a_it_i)$	$\dfrac{1+\sum_{m=1}^{N/2}\sum_{i_1,i_2...i_{2m}=1}^{n}\frac{1}{(2m)!}\pi_{j=1}^{2m}a_{i_j}u_{i_j}}{\pi_{i=1}^{n}(1-a_i^2u_i^2)}$, $N=n$ if n is an even and $N=n-1$ if n is an odd number , $i_k\neq i_l$ for $k\neq l$ and $k,l,i_k,i_l=1,2,...,n$.
7	$(\sum_{i=1}^{n}a_it_i)^m$, $m=1,2,.$	$\sum_{m_1+m_2+...+m_n=m}\binom{m}{m_1,m_2,...,m_n}\pi_{i=1}^{n}(m_i)!(a_iu_i)^{m_i}$
8	$\sin(\sum_{i=1}^{n}a_it_i)$	$\dfrac{\sum_{m=1}^{N/2}\sum_{i_1,i_2,...,i_{2m-1}=1}^{n}\frac{(-1)^{m-1}}{(2m-1)!}\pi_{j=1}^{2m-1}a_{i_j}u_{i_j}}{\pi_{i=1}^{n}(1+a_i^2u_i^2)}$, $N=n$ if n is an even and $N=n+1$ if n is an odd number , $i_k\neq i_l$ for $k\neq l$ and $k,l,i_k,i_l=1,2,...,n$.
9	$\cos(\sum_{i=1}^{n}a_it_i)$	$\dfrac{1+\sum_{m=1}^{N/2}\sum_{i_1,i_2...i_{2m}=1}^{n}\frac{(-1)^{m}}{(2m)!}\pi_{j=1}^{2m}a_{i_j}u_{i_j}}{\pi_{i=1}^{n}(1+a_i^2u_i^2)}$, $N=n$ if n is an even and $N=n-1$ if n is an odd number , $i_k\neq i_l$ for $k\neq l$ and $k,l,i_k,i_l=1,2,...,n$.

No.	$f(\bar{t})$	$S_n[f(\bar{t})]$
10	$e^{\sum_{i=1}^{n} a_i t_i} \sinh(\sum_{i=1}^{n} b_i t_i)$	$\dfrac{\pi_{i=1}^{n}(1-a_i u_i)\sum_{m=1}^{N/2}\sum_{i_1,i_2,\dots,i_{2m-1}=1}^{n}\dfrac{1}{(2m-1)!}\pi_{j=1}^{2m-1}\dfrac{b_{i_j}u_{i_j}}{1-a_{i_j}u_{i_j}}}{\pi_{i=1}^{n}[1-2a_iu_i+(a_i^2-b_i^2)u_i^2]}$, $N=n$ if n is an even and $N=n+1$ if n is an odd number , $i_k \neq i_l$ for $k \neq l$ and $k,l,i_k,i_l=1,2,\dots,n$.
11	$e^{\sum_{i=1}^{n} a_i t_i} \cosh(\sum_{i=1}^{n} b_i t_i)$	$\dfrac{\pi_{i=1}^{n}(1-a_i u_i)[1+\sum_{m=1}^{N/2}\sum_{i_1,i_2,\dots,i_{2m}=1}^{n}\dfrac{1}{(2m)!}\pi_{j=1}^{2m}\dfrac{b_{i_j}u_{i_j}}{1-a_{i_j}u_{i_j}}]}{\pi_{i=1}^{n}[1-2a_iu_i+(a_i^2-b_i^2)u_i^2]}$, $N=n$ if n is an even and $N=n-1$ if n is an odd number , $i_k \neq i_l$ for $k \neq l$ and $k,l,i_k,i_l=1,2,\dots,n$
12	$e^{\sum_{i=1}^{n} a_i t_i} (\sum_{i=1}^{n} b_i t_i)^m$ $m=1,2,\dots$	$\dfrac{\sum_{m_1+m_2+\dots+m_n=m}\binom{m}{m_1,m_2,\dots,m_n}\pi_{i=1}^{n}(m_i)!(\dfrac{b_iu_i}{1-a_iu_i})^{m_i}}{\pi_{i=1}^{n}(1-a_iu_i)}$
13	$e^{\sum_{i=1}^{n} a_i t_i} \sin(\sum_{i=1}^{n} b_i t_i)$	$\dfrac{\pi_{i=1}^{n}(1-a_i u_i)\sum_{m=1}^{N/2}\sum_{i_1,i_2,\dots,i_{2m-1}=1}^{n}\dfrac{(-1)^{m-1}}{(2m-1)!}\pi_{j=1}^{2m-1}\dfrac{b_{i_j}u_{i_j}}{1-a_{i_j}u_{i_j}}}{\pi_{i=1}^{n}[1-2a_iu_i+(a_i^2+b_i^2)u_i^2]}$, $N=n$ if n is an even and $N=n+1$ if n is an odd number , $i_k \neq i_l$ for $k \neq l$ and $k,l,i_k,i_l=1,2,\dots,n$.
14	$e^{\sum_{i=1}^{n} a_i t_i} \cos(\sum_{i=1}^{n} b_i t_i)$	$\dfrac{\pi_{i=1}^{n}(1-a_i u_i)[1+\sum_{m=1}^{N/2}\sum_{i_1,i_2,\dots,i_{2m}=1}^{n}\dfrac{(-1)^m}{(2m)!}\pi_{j=1}^{2m}\dfrac{b_{i_j}u_{i_j}}{1-a_{i_j}u_{i_j}}]}{\pi_{i=1}^{n}[1-2a_iu_i+(a_i^2+b_i^2)u_i^2]}$, $N=n$ if n is an even and $N=n-1$ if n is an odd number , $i_k \neq i_l$ for $k \neq l$ and $k,l,i_k,i_l=1,2,\dots,n$.

Now , we shall introduce some explanations about the table . To prove No.5 and No.6 of the table , first we shall use the mathematical induction to prove that

$$\pi_{i=1}^{n}(1+a_iu_i)=1+\sum_{m=1}^{n}\sum_{i_1,i_2,\dots,i_m=1}^{n}\frac{1}{m!}\pi_{j=1}^{m}a_{i_j}u_{i_j}\ , \tag{3.1}$$

where $i_k \neq i_l$ for $k \neq l$ and $k,l,i_k,i_l=1,2,\dots,n$. Note that for each number $1 \le m \le n$ the summation $\sum_{i_1,i_2,\dots,i_m=1}^{n}\frac{1}{m!}\pi_{j=1}^{m}a_{i_j}u_{i_j}$ with the condition $i_k \neq i_l$ for $k \neq l$ and $k,l,i_k,i_l=1,2,\dots,n$ is summation of all permutations of the n objects $a_1u_1,\dots,a_nu_n$ taken m at a time such that each term in that summation occurs $m!$ times .

It is clear that relation (3.1) is satisfied when $n=1$ since

$$1+a_1u_1=1+\sum_{i_1=1}^{1}\frac{1}{1!}\pi_{j=1}^{1}a_{i_j}u_{i_j}\ . \tag{3.2}$$

Suppose that relation (3.1) is true when $n = k$, i.e.

$$\pi_{i=1}^{k}(1+a_i u_i) = 1 + \sum_{m=1}^{k} \sum_{i_1,i_2,\ldots,i_m=1}^{k} \frac{1}{m!} \pi_{j=1}^{m} a_{i_j} u_{i_j} . \tag{3.3}$$

For $n = k+1$ we have

$$\pi_{i=1}^{k+1}(1+a_i u_i) = (1+a_{k+1}u_{k+1}) \pi_{i=1}^{k}(1+a_i u_i) = (1+a_{k+1}u_{k+1})[1+\sum_{m=1}^{k} \sum_{i_1,i_2,\ldots,i_m=1}^{k} \frac{1}{m!} \pi_{j=1}^{m} a_{i_j} u_{i_j}]$$

$$= 1 + a_{k+1}u_{k+1} + \sum_{m=1}^{k} \sum_{i_1,i_2,\ldots,i_m=1}^{k} \frac{1}{m!} \pi_{j=1}^{m} a_{i_j} u_{i_j} + a_{k+1}u_{k+1} \sum_{m=1}^{k} \sum_{i_1,i_2,\ldots,i_m=1}^{k} \frac{1}{m!} \pi_{j=1}^{m} a_{i_j} u_{i_j}$$

$$= 1 + [a_{k+1}u_{k+1} + \sum_{i_1=1}^{k} \frac{1}{1!} \pi_{j=1}^{1} a_{i_j} u_{i_j}] + \sum_{m=2}^{k}[\sum_{i_1,i_2,\ldots,i_m=1}^{k} \frac{1}{m!} \pi_{j=1}^{m} a_{i_j} u_{i_j} + a_{k+1}u_{k+1} \tag{3.4}$$

$$\sum_{i_1,i_2,\ldots,i_{m-1}=1}^{k} \frac{m}{m(m-1)!} \pi_{j=1}^{m-1} a_{i_j} u_{i_j}] + a_{k+1}u_{k+1} \sum_{i_1,i_2,\ldots,i_k=1}^{k} \frac{k+1}{(k+1)k!} \pi_{j=1}^{k} a_{i_j} u_{i_j}$$

$$= 1 + \sum_{i_1=1}^{k+1} \frac{1}{1!} \pi_{j=1}^{1} a_{i_j} u_{i_j} + \sum_{m=2}^{k} \sum_{i_1,i_2,\ldots,i_m=1}^{k+1} \frac{1}{m!} \pi_{j=1}^{m} a_{i_j} u_{i_j} + \sum_{i_1,i_2,\ldots,i_{k+1}=1}^{k+1} \frac{1}{(k+1)!} \pi_{j=1}^{k+1} a_{i_j} u_{i_j}$$

$$= 1 + \sum_{m=1}^{k+1} \sum_{i_1,i_2,\ldots,i_m=1}^{k+1} \frac{1}{m!} \pi_{j=1}^{m} a_{i_j} u_{i_j} ,$$

since , permutations of the objects $a_{i_1}u_{i_1},\ldots,a_{i_m}u_{i_m}$ are $m! = m(m-1)!$, $2 \le m \le k+1$.
Therefore relation (3.1) is satisfied for $n = k+1$. Now , using No.3 of the table we have

$$S_n[\sinh(\sum_{i=1}^{n} a_i t_i)] = \frac{1}{2} S_n[e^{\sum_{i=1}^{n} a_i t_i}] - \frac{1}{2} S_n[e^{-\sum_{i=1}^{n} a_i t_i}]$$

$$= \frac{1}{2\pi_{i=1}^{n}(1-a_i u_i)} - \frac{1}{2\pi_{i=1}^{n}(1+a_i u_i)} \tag{3.5}$$

$$= \frac{\pi_{i=1}^{n}(1+a_i u_i) - \pi_{i=1}^{n}(1-a_i u_i)}{2\pi_{i=1}^{n}(1-a_i^2 u_i^2)} .$$

By replacing $a_i u_i$, in relation (3.1) , by $-a_i u_i$ for $\pi_{i=1}^{n}(1-a_i u_i)$ and substitution in the last equation of equations (3.5) give

$$S_n[\sinh(\sum_{i=1}^{n} a_i t_i)] = \frac{1}{2\pi_{i=1}^{n}(1-a_i^2 u_i^2)} \sum_{m=1}^{n} \sum_{i_1,i_2,\ldots,i_m=1}^{n} \frac{1-(-1)^m}{m!} \pi_{j=1}^{m} a_{i_j} u_{i_j}$$

$$= \frac{1}{\pi_{i=1}^{n}(1-a_i^2 u_i^2)} \sum_{m=1}^{N/2} \sum_{i_1,\ldots,i_{2m-1}=1}^{n} \frac{1}{(2m-1)!} \pi_{j=1}^{2m-1} a_{i_j} u_{i_j} , \tag{3.6}$$

where $N = n$ if n is an even number and $N = n+1$ if n is an odd number , $i_k \ne i_l$ for $k \ne l$ and $k,l,i_k,i_l = 1,2,\ldots,n$. In a similar manner we have

$$S_n[\cosh(\sum_{i=1}^{n} a_i t_i)] = \frac{1}{2\underset{i=1}{\overset{n}{\pi}}(1-a_i^2 u_i^2)}[2+\sum_{m=1}^{n}\sum_{i_1,i_2,\ldots,i_m=1}^{n}\frac{1+(-1)^m}{m!}\underset{j=1}{\overset{m}{\pi}} a_{i_j} u_{i_j}]$$
$$= \frac{1}{\underset{i=1}{\overset{n}{\pi}}(1-a_i^2 u_i^2)}[1+\sum_{m=1}^{N/2}\sum_{i_1,\ldots,i_{2m}=1}^{n}\frac{1}{(2m)!}\underset{j=1}{\overset{2m}{\pi}} a_{i_j} u_{i_j}], \quad (3.7)$$

where $N = n$ if n is an even number and $N = n-1$ if n is an odd number , $i_k \neq i_l$ for $k \neq l$ and $k, l, i_k, i_l = 1,2,\ldots,n$.

For No. 7 of the table then from the multinomial theorem [8] we have

$$S_n[(\sum_{i=1}^{n} a_i t_i)^m] = S_n[\sum_{m_1+m_2+\ldots+m_n=m}\binom{m}{m_1,m_2,\ldots,m_n}\underset{i=1}{\overset{n}{\pi}}(a_i t_i)^{m_i}], \quad (3.8)$$

where the numbers $\binom{m}{m_1,m_2,\ldots,m_n} = \frac{m!}{m_1!m_2!\ldots m_n!}$ are called multinomial coefficients . From No. 2 of the table yields

$$S_n[(\sum_{i=1}^{n} a_i t_i)^m] = \sum_{m_1+m_2+\ldots+m_n=m}\binom{m}{m_1,m_2,\ldots,m_n}\underset{i=1}{\overset{n}{\pi}}(a_i)^{m_i} S_n[\underset{i=1}{\overset{n}{\pi}}(t_i)^{m_i}]$$
$$= \sum_{m_1+m_2+\ldots+m_n=m}\binom{m}{m_1,m_2,\ldots,m_n}\underset{i=1}{\overset{n}{\pi}}(m_i)!(a_i u_i)^{m_i} \quad (3.9)$$

No.8 and No.9 can be concluded by using the relations

$$S_n[\sin(\sum_{i=1}^{n} a_i t_i)] = \frac{1}{2k}S_n[e^{k\sum_{i=1}^{n} a_i t_i}] - \frac{1}{2k}S_n[e^{-k\sum_{i=1}^{n} a_i t_i}], \; k=\sqrt{-1}, \quad (3.10)$$

$$S_n[\cos(\sum_{i=1}^{n} a_i t_i)] = \frac{1}{2}S_n[e^{k\sum_{i=1}^{n} a_i t_i}] + \frac{1}{2}S_n[e^{-k\sum_{i=1}^{n} a_i t_i}], \; k=\sqrt{-1}, \quad (3.11)$$

and No.3 of the table then replacing $a_i u_i$, in relation (3.1) , by $k a_i u_i$ and $-k a_i u_i$ for $\underset{i=1}{\overset{n}{\pi}}(1+k a_i u_i)$ and $\underset{i=1}{\overset{n}{\pi}}(1-k a_i u_i)$ respectively .

It is clear that No. 10 , No.11 ,.. and No.14 can be obtained using theorem 2.4 in addition , No.5 , No.6 , …and No.9 of the table respectively .

Example 1. Here we shall give an example for finding the inverse of the triple Sumudu transform (3- dimension)

$$S_3^{-1}[\frac{1}{16u_1^4-16u_2^2u_1^4+8u_2^2u_1^2-8u_1^2-u_2^2+1}(32u_2u_3^2u_1^5-32u_2^3u_3^2u_1^5+4u_2u_1^3+16u_2^3u_3^2u_1^3$$
$$-16u_2u_3^2u_1^3+4u_1^2+2u_2u_3^2u_1-2u_2^3u_3^2u_1+u_2u_1)]$$
$$=S_3^{-1}[\frac{1}{(16-16u_2^2)u_1^4+(8u_2^2-8)u_1^2-u_2^2+1}\{(32u_2u_3^2-32u_2^3u_3^2)u_1^5+(4u_2+16u_2^3u_3^2$$
$$-16u_2u_3^2)u_1^3+4u_1^2+(2u_2u_3^2-2u_2^3u_3^2+u_2)u_1\}] \qquad (3.12)$$
$$=S_3^{-1}[2u_1u_2u_3^2+\frac{4u_2u_1^3+4u_1^2+u_1u_2}{(16-16u_2^2)u_1^4+(8u_2^2-8)u_1^2-u_2^2+1}]$$
$$=S_3^{-1}[2u_1u_2u_3^2]+S_2^{-1}[\frac{4u_2u_1^3+4u_1^2+u_1u_2}{(1-4u_1^2)^2(1-u_2^2)}]$$
$$=xyz^2+\frac{1}{2}S_2^{-1}[u_1\frac{\partial}{\partial u_1}(\frac{1+2u_1u_2}{(1-4u_1^2)(1-u_2^2)})]$$
$$=xyz^2+\frac{x}{2}\frac{\partial}{\partial x}(\cosh(2x+y))$$
$$=xyz^2+x\sinh(2x+y).$$

Note that for the first term of the last equation of equations (3.12) we used No.2 of the table when $n=3$, $i_1=i_2=1$ and $i_3=2$. For the second term we used theorem 2.10 when $n=2$, $m=1$ and $i_1=1$, in addition we used No.6 of the table when $n=2$, $a_1=2$ and $a_2=1$ to obtain

$$S_2^{-1}[\frac{1+2u_1u_2}{(1-4u_1^2)(1-u_2^2)}]=\cosh(2x+y) \;. \qquad (3.13)$$

Note that $t_1=x$, $t_2=y$ and $t_3=z$.

4 . Applications to PDEs in the 2- dimension

In this section we shall find the double Sumudu transform of some of the partial derivatives of the function $u(x,y)$ and then use them to solve two non – homogenous linear partial differential equations .

To obtain the double Sumudu transform of partial derivatives we use integration by parts . Using definition (1.5) when $n=2$, $t_1=x$, $t_2=y$, $S_2[u(x,y)]=U(u_1,u_2)$ then

$$S_2[u_x(x,y)]=\frac{1}{u_1u_2}\int_0^\infty\int_0^\infty e^{-(\frac{x}{u_1}+\frac{y}{u_2})}u_x(x,y)\,dx\,dy$$
$$=\frac{1}{u_1u_2}\int_0^\infty e^{-\frac{y}{u_2}}[\int_0^\infty e^{-\frac{x}{u_1}}u_x(x,y)\,dx\,]\,dy \qquad (4.1)$$
$$=\frac{1}{u_1}[\frac{-1}{u_2}\int_0^\infty e^{-\frac{y}{u_2}}g(y)\,dy+\frac{1}{u_1u_2}\int_0^\infty\int_0^\infty e^{-(\frac{x}{u_1}+\frac{y}{u_2})}u(x,y)dx\,dy\,]$$
$$=\frac{1}{u_1}\left(-S(g(y))+S_2(u(x,y)\right)=\frac{1}{u_1}U(u_1,u_2)-\frac{1}{u_1}G(u_2),$$

where

$$G(u_2)=S[g(y)],\quad g(y)=u(0,y). \qquad (4.2)$$

Similarly

$$S_2[u_y(x,y)] = \frac{1}{u_2}U(u_1,u_2) - \frac{1}{u_2}F(u_1)\,, \tag{4.3}$$

where

$$F(u_1) = S\,[\,f(x)],\quad f(x) = u(x,0) \tag{4.4}$$

Similarly , using the last equation of equations (4.1) we have

$$\begin{aligned} S_2[u_{xx}(x,y)] &= \frac{1}{u_1u_2}\int_0^\infty\int_0^\infty e^{-(\frac{x}{u_1}+\frac{y}{u_2})}u_{xx}(x,y)\,dx\,dy \\ &= \frac{1}{u_1u_2}\int_0^\infty e^{-\frac{y}{u_2}}[\int_0^\infty e^{-\frac{x}{u_1}}u_{xx}(x,y)\,dx\,]\,dy \\ &= \frac{1}{u_1}[\frac{-1}{u_2}\int_0^\infty e^{-\frac{y}{u_2}}g_1(y)\,dy + \frac{1}{u_1u_2}\int_0^\infty\int_0^\infty e^{-(\frac{x}{u_1}+\frac{y}{u_2})}u_x(x,y)\,dx\,dy] \\ &= \frac{1}{u_1}[-S(g_1(y)) + S_2(u_x(x,y)] \\ &= \frac{1}{u_1^2}U(u_1,u_2) - \frac{1}{u_1^2}G(u_2) - \frac{1}{u_1}G_1(u_2)\,, \end{aligned} \tag{4.5}$$

where

$$G_1(u_2) = S\,[g_1(y)]\ ,\quad g_1(y) = u_x(0,y)\ , \tag{4.6}$$

and $G(u_2)$ is defined in (4.2) .

Example 2. Determination of a solution $u(x,y)$ of the PDE

$$u_x + au_y = 2ay^n\,,\ x>0\ ,\ y>0\ ,\ a>0\ ,\ n=1,2,.. \tag{4.7}$$

under the following initial and boundary conditions

i. $u(x,0) = 0$,

ii. $u(0,y) = y^m$.

From relations (4.4) and (4.2) we get

$$F(u_1) = S\,[0] = 0\ , \tag{4.8}$$

and

$$G(u_2) = S\,[y^m] = m!u_2^m\,, \tag{4.9}$$

respectively . By taking the double Sumudu transform to the PDE (4.7) using the last equation of equations (4.1) and relations (4.3) , (4.8) and (4.9) we get

$$\begin{aligned} U(u_1,u_2) &= \frac{2an!u_1u_2^{n+1}}{u_2+au_1} + \frac{m!u_2^{m+1}}{u_2+au_1} \\ &= 2n!u_2^{n+1} - \frac{2n!u_2^{n+2}}{u_2+au_1} + \frac{m!u_2^{m+1}}{u_2+au_1}\,. \end{aligned} \tag{4.10}$$

By using theorem 2.7 when $n=2$, $c_1 = a$, $i_1 = 2$ and $i_2 = 1$ then the inverse transform S_2^{-1} of the second equation of (4.10) gives the following solution of the PDE (4.7)

$$\begin{aligned} u(x,y) &= \frac{2}{n+1}S^{-1}[(n+1)!u_2^{n+1}] - \frac{2}{n+1}S_2^{-1}[\frac{(n+1)!u_2^{(n+1)+1}}{u_2+a_1u_1}] + S_2^{-1}[\frac{m!u_2^{m+1}}{u_2+au_1}] \\ &= \begin{cases} \dfrac{2}{n+1}y^{n+1}\,, & if\ 0 \le y \le ax\,, \\ \dfrac{2}{n+1}y^{n+1} - \dfrac{2}{n+1}(y-ax)^{n+1} + (y-ax)^m\,, & if\ y > ax\,. \end{cases} \end{aligned} \tag{4.11}$$

Example 3. Consider the PDE

$$u_y = \alpha^2 u_{xx} + \sin 3\pi\, x \ , \ 0 < x < 1 \ , \quad (4.12)$$

with the initial and boundary conditions

i- $u(x,0) = \sin \pi\, x$,

ii- $u(0,y) = 0$,

iii- $u_x(0,y) = \pi e^{-\alpha^2\pi^2 y} + \dfrac{1}{3\alpha^2\pi}(1 - e^{-9\alpha^2\pi^2 y})$.

From relations (4.4) , (4.2) and (4.6) we get

$$F(u_1) = S[\sin \pi\, x] = \frac{\pi\, u_1}{1+\pi^2 u_1^2} \quad (4.13)$$

$$G(u_2) = S\,[0] = 0 \ , \quad (4.14)$$

and

$$\begin{aligned} G_1(u_2) &= S\,[\pi\, e^{-\alpha^2\pi^2 y} + \frac{1}{3\alpha^2\pi}(1 - e^{-9\alpha^2\pi^2 y})] \\ &= \frac{\pi}{1+\alpha^2\pi^2 u_2} + \frac{1}{3\alpha^2\pi} - \frac{1}{3\alpha^2\pi(1+9\alpha^2\pi^2 u_2)} \end{aligned} \quad (4.15)$$

respectively . Applying the double Sumudu transform of the PDE (4.12) by using relations (4.3) , (4.5) , (4.13) , (4.14) and (4.15), simplifications and adding the terms $\mp 9\pi^2\alpha^2 u_2 u_1^3$ and $\mp 3\pi^2\alpha^2 u_2{}^2 u_1^3$ to the denominator of the right hand side give the transformed problem

$$\begin{aligned} U(u_1,u_2) &= \frac{\pi}{(1+\alpha^2\pi^2 u_2)(1+9\alpha^2\pi^2 u_2)(1+\pi^2 u_1^2)(1+9\pi^2 u_1^2)}[(9\pi^2 + 81\alpha^2\pi^4 u_2 \\ &\quad + 3\pi^2 u_2 + 3\alpha^2\pi^4 u_2^2)u_1^3 + (1+9\alpha^2\pi^2 u_2 + 3u_2 + 3\alpha^2\pi^2 u_2^2)u_1] \\ &= \frac{\pi}{(1+\alpha^2\pi^2 u_2)(1+9\alpha^2\pi^2 u_2)}[\frac{Au_1 + B}{1+\pi^2 u_1^2} + \frac{Cu_1 + D}{1+9\pi^2 u_1^2}] , \end{aligned} \quad (4.16)$$

by applying partial fractions with respect to the variable u_1 . Solving the partial fractions gives

$$\begin{aligned} U(u_1,u_2) &= \frac{\pi}{(1+\alpha^2\pi^2 u_2)(1+9\alpha^2\pi^2 u_2)}[\frac{(1+9\alpha^2\pi^2 u_2)u_1}{1+\pi^2 u_1^2} + \frac{(3u_2 + 3\pi^2\alpha^2 u_2^2)u_1}{1+9\pi^2 u_1^2}] \\ &= \frac{\pi\, u_1}{(1+\alpha^2\pi^2 u_2)(1+\pi^2 u_1^2)} + \frac{1}{9\alpha^2\pi^2}(1 - \frac{1}{1+9\alpha^2\pi^2 u_2})\frac{3\pi\, u_1}{1+9\pi^2 u_1^2} \end{aligned} \quad (4.17)$$

Since , from theorem 2.1 we have $S_2^{-1} = S^{-1}.S^{-1}$ then taking S_2^{-1} of the last equation of (4.17) gives

$$\begin{aligned} u(x,y) &= S^{-1}[\frac{1}{1+\alpha^2\pi^2 u_2}]S^{-1}[\frac{\pi\, u_1}{1+\pi^2 u_1^2}] + \frac{1}{9\alpha^2\pi^2}[S^{-1}(1) - S^{-1}(\frac{1}{1+9\alpha^2\pi^2 u_2})]S^{-1}[\frac{3\pi\, u_1}{1+9\pi^2 u_1^2}] \\ &= e^{-\alpha^2\pi^2 y}\sin \pi\, x + \frac{1}{9\alpha^2\pi^2}(1 - e^{-9\alpha^2\pi^2 y})\sin 3\pi x . \end{aligned} \quad (4.18)$$

5. Conclusion

Throught our work in this paper , we note that there is a little work has been done on the single Sumudu transform and a very little work on the double Sumudu transform . In the field of the generalized Sumudu transform we don't find any relating paper or reference . Hence , for advanced reseach , there is many works such as introducing other interesting properties or in applied mathematics via control problems in partial differential equations .

References

1. Watugala , G. K., Sumudu Transform : a New Integral Transform to Solve Differential Equations and Control Engineering Problems , Int. J. Math. Educ. Sci. Technol.,Vol.24 , No.1, (1993)35-43 .
2. Weerakoon , S . , Application of Sumudu Transform to Partial Differential Equations , Int. J. Math. Educ. Sci. Technol., Vol.25, No.2 ,(1994)277-283.
3. Weerakoon , S. , Complex Inversion Formula for Sumudu Transform , Int. J. Math. Educ. Sci. Technol., Vol.29, No.4, (1998) 618-621 .
4. Belgacem , F.B. M., Karaballi , A.A., Kalla , S. L., Analytical Investigations of the Sumudu Transorm and Applications to Integral Production Equations , Math. Probl. Engr., Vol.3 , (2003)103-118 .
5. Watugala , G. K., The Sumudu Transform for Functions of Two Variables , Math. Engr. Indust., Vol.8 , No.4, (2002)293-302 .
6. Tchuenche , J. M., Mbare, N. S., An Application of the Double Sumudu Transform , Appl. Math. Sci. , Vol. 1 , No.1 , (2007)31-39 .
7. Dahiya , R. S., Saberi-Nadjafi, J. , Theorems on $n-$ Dimensional Laplace Transforms and their Applications , Electronic Journal of Differential Equations , Conference 02, (1999)61-74 .
8. Lipschutz , S., Schaum's Outline of Theory and Problems of Probability , (McGraw-Hill) , (1974) .