A Rational Triangle Function as a Model for a Conjugate Gradient Optimization Method

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الملخص

في هذا البحث تم تطوير واستعمال خوارزمية جديدة في مجال ألا مثلية غير المقيدة تعتمد على أحد نماذج المثلثية النسبية غير التربيعية. تم استخدام هذه الخوارزمية بطريقة:ا باستخدام الاتجاهات الخطية الدقيقة. تمت مقارنة هذه الاستخدامات مع طريقة المتجهات المترافقة عدديا. وان النتائج التي تم التوصل أليها أثبتت أن الخوارزمية الجديدة هي اكثر كفاءة من الخوارزمية المعرفة في هذا المجال.

ABSTRACT

This paper presents the development and implementation of a new numberical based on a non-quadratic Triangular rational function model. For solving non-linear optimization problem .The algorithm is implemented in one version, employing exact line search. This version is compared numberically against versions of the CG-method. The results indicate that in general the new algorithm is superior to the previon algorithm.

1. Introduction

A more general model than the quadratic one is proposed in this paper as a basis for a CG algorithm. If q(x) is a quadratic function, then a function f is defined as a non-linear scaling of q(x) if the following condition holds :

$$\mathbf{f = F(q(x)),\ dF/dq = F' > o \text{ and } q(x) > 0} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where x* is the minimizer of q(x) with respect to x [13] .

The following properties are immediately derived from the above condition:

i) Every contour line to q(x) is a contour line of f .
ii) If x* is a minimzer of q(x), then it is a minimizer of f.
iii) That x* is a global minimum of q(x) does not necessarily mean that it is a global minimum of f [5].

Various authors have puplished-related work in the area:

A conjugate method which minimizers the function $f(x) = (q(x))^{\rho}$, and $x \in R^n$ in at most step has been described by Fried[9]. Another special case, namely $F(q(x)) = e_1 q(x) + \frac{1}{2} e_2 \; q^2(x)$ Whereε_1 and ε_2 are scalars, has been investigated by Boland et al, [5].

Another model has been developed by Tassopoulos and Storey, [14] as follows: $F(q(x) = \varepsilon_1 q(x) + 1/\varepsilon_2 q(x)$: $\varepsilon_2 > 0$

AL-Assady in [3] developed a model as follows :(F(q(x)) = In (q(x))

Al-Bayat, [1] has developed a new rational model which is defined as follows: $F(q(x)) = \varepsilon_1\; q(x)/1-\varepsilon_2\, q(x)$.

Also Al-Bayati [4] developed an extended CG algorithm which is based on a general logarithmic model

F(q(x) = log(eq(x) – 1) , e> **0**

And Al-Assady, [2] described there ECG algorithm which is based on the natural log function for the rational q(x) function

$$F(q) = \log \left[e_1 q(x) \big/ e_2 q(x) + 1 \right] , \; \varepsilon_2 < 0$$

In this paper, a new sine model is investigated and tested on a set of standard test function, on the assumed that condition (1) holds. An extended conjugate gradient algorithm is developed which is based on this new model which scales q(x) by the natural sinh function for the rational q(x) functions.

$$F(q(x) = \sin (\varepsilon_1 q(x) / \varepsilon_2 q(x)+1) \qquad \ldots\ldots\ldots\ldots\ldots.(2)$$

We first observe that q(x) and F(q(x)) given by (2) have identical contours, though with different function values, and they have the same unique minimum point denoted by x^*.

2.Theorem

Given an identical starting point $x_{1,}$the method of Fletcher and Reeves [8]defined by

$$\left.\begin{aligned} & d_1 = -g_1 \\ & d_{i+1} = -g_{i+1} + b_i d_i \quad , i \geq 1 \\ & b_i = \frac{\|g_{i+1}\|^2}{\|g_i\|^2} \end{aligned}\right\} \ldots\ldots\ldots\; \ldots\ldots\ldots\; \ldots\ldots\ldots\; \ldots\ldots(3)$$

and $\| \ \|$ is the Euclidean norm applied to f(x)=q(x) and the ECG-method using the following search directions:

$$\left.\begin{aligned} &\bar{d}_1 = -\bar{g}_1 \\ &\bar{d}_{i+1} = -\bar{g}_{i+1} + r_i b_i \bar{d}_i \quad , i \ge 1 \\ &r_i = \frac{f_i'}{f_{i+1}'} \\ &b_i = \frac{\|g_{i+1}\|^2}{\|g_i\|^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

and applied to f(q(x)) generate identical conjugate directions (within a positive multiple f_i') and the identical sequence of approximations x_i to the solution x* for any function satisfying (1).

It is assumed that the one-dimensional searches are exact. The vectors n $\bar{g}_1, \bar{g}_i$ are gradients of f(q(x)) at x_1 and x_i ,respectively.

Proof:

The theorem is true For i=1, because

$$\bar{d}_1 = -\bar{g}_1 = -f_1' g_1 = \bar{f}_1 d_1$$

Now for i=2 , we have

$$\bar{d}_2 = -\bar{g}_2 + r_1 b_1 \bar{d}_2$$

$$= -f_2' g_2 + \left(\frac{f_1'}{f_2'}\right)\left(\frac{\|\bar{g}_2\|^2}{\|\bar{g}_1\|^2}\right) f_1' d_1$$

$$= -f_2' g_2 + \left(\frac{f_1'}{f_2'}\right)\left(\frac{f_{21}'}{f_1'}\right)^2 \left(\frac{\|g_2\|^2}{\|g_1\|^2}\right) f_1' d_1$$

$$= f_2' d_2 .$$

Assume that , for i≥ 2 ,

$$\bar{d}_i = f_i' \left[-g_{i+1} + \left(\|g_{i+1}\|^2 / \|g_i\|^2 \right) d_i \right]$$

$$= f_i' d_i$$

It follows from (4) that

$$\overline{d_{i+1}} = -\overline{g_{i+1}} + r_i b_i \bar{d}_i$$

$$= -f'_{i+1} g_{i+1} + \left(\frac{f'_i}{f'_{i+1}}\right)\left(\frac{f'_{i+1}}{f'_i}\right)^2 \left(\frac{\|g_{i+1}\|}{\|g_i\|^2}^{2}\right) f'_i d_i$$

$$= -f'_{i+1} d_{i+1}$$

Both methods generate the same sequence of approximations x_I , since isocontour curve of q(x) and f(q(x)) are identical. These isontours differ only by the function values on the corresponding curves, and hence the theorem is proved

3. The Derivation of r_i for the New Model:

The implementation of the extended CG method has been performed for general function F(q(x) of the form of equations(2).

The unknown quantities r_i were expressed in terms of available quantities of the algorithm.

The new $\sin\left(\frac{e_1 q(x)+1}{e_2 q(x)}\right)$ model can now be written as

f(x) = F(q(x) **=** $\sin\left(\frac{e_1 q(x)+1}{e_2 q(x)}\right)$

Solving equation (2) for q

Sin^{-1} f(x) $\left(\frac{e_1 q(x)+1}{e_2 q(x)}\right)$

In $\left[if(x)+\sqrt{1-f(x)^2}\right]$ = $\frac{e_1 q(x)+1}{e_2 q(x)}$ Þ **q =** $1/e_2 \ln\left[if(x)+\sqrt{1-f(x)^2}\right] - e_1$

And using the expression for $\mathbf{p_i} = \mathbf{f¢_{i-1}/f¢_i}$

$$r_i = -\frac{\cos(e_1 q_{i-1} + 1/e_2 q_{i-1})\left(-1/e_2 q^2{}_{i-1}\right)}{\cos(e_1 q_i + 1/e_2 q_i)\left(-1/e_2 q^2{}_i\right)}.$$

from the above equation we have

$$r_i = \left[\frac{\dfrac{\left[\left[if_{i-1}+\sqrt{1-f_{i-1}^2}\right]^2+1\right]\left[\ln\left(if_{i-1}+\sqrt{1-f_{i-1}^2}\right)-\dfrac{e_1}{e_2}\right]^2}{if_{i-1}+\sqrt{1-f_{i-1}^2}}}{\dfrac{\left[\left[if_i+\sqrt{1-f_i^2}\right]^2+1\right]\left[\ln\left(if_i+\sqrt{1-f_i^2}\right)-\dfrac{e_1}{e_2}\right]^2}{if_i+\sqrt{1-f_i^2}}} \right] \qquad \text{......... ....(5)}$$

In terms of the known quantities such a function and gradient values, from

$$g_i = F_i' Q(x_i - x^*)$$
$$g_{i-1} = F_{i-1}' Q(x_{i-1} - x^*)$$

Where Q is the Hessian Matrix and x^* is the minimum point, we have:

$$r_i = \left[\frac{\dfrac{\left[\left[if_{i-1}+\sqrt{1-f_{i-1}^2}\right]^2+1\right]\left[\ln\left(if_{i-1}+\sqrt{1-f_{i-1}^2}\right)-\dfrac{e_1}{e_2}\right]^2}{if_{i-1}+\sqrt{1-f_{i-1}^2}}}{\dfrac{\left[\left[if_i+\sqrt{1-f_i^2}\right]^2+1\right]\left[\ln\left(if_i+\sqrt{1-f_i^2}\right)-\dfrac{e_1}{e_2}\right]^2}{if_i+\sqrt{1-f_i^2}}} \right] \qquad \text{......... ....(6)}$$

Furthermore

$$g_{i-1}^T(x_i - x^*) = g_{i-1}^T(x_{i-1} + l_{i-1}d_{i-1} - x^*)$$
$$= g_{i-1}^T(x_{i-1} - x^*) + l_{i-1}g_{i-1}^T d_{i-1} \text{.........................(7)}$$

$$g_i^T(x_i - x^*) = g_i^T(x_i + l_i d_i - x^*)$$
$$= g_i^T(x_i - x^*)$$

Since $g_i^T d_{i-1} = 0$ therefore, we can express r_i as follows:

$$r_i = \frac{g_{i-1}^T(x_{i-1} + l_{i-1}d_{i-1} - x^*)}{g_i^T(x - x^*)} \qquad \text{..(8)}$$

From (7) and (8) , it follows that :

$$r_i = r_i \left[\frac{q_{i-1}}{q_i} \right] + l_{i-1} g_{i-1}^T d_{i-1} / 2 F_i' q_i$$

Where $q = 1 \Big/ e_2 \left[\ln\left(if + \sqrt{1 - f^2} \right) - \frac{e_1}{e_2} \right]$

$$and \;\; f' = \frac{\left[\left[if + \sqrt{1 - f^2} \right]^2 + 1 \right] - e_2 \left[\ln\left(if + \sqrt{1 - f^2} \right) - \frac{e_1}{e_2} \right]^2}{2\left[if + \sqrt{1 - f^2} \right]}$$

The quantities $q_{i-1/}q_i$ and $f_i' q_i$ can be rewritten as: $\dfrac{q_i}{q_i}$

$$\frac{q_{i-1}}{q_i} = \frac{\ln\left[if_i + \sqrt{1 - f_i^2} \right] - \frac{e_1}{e_2}}{\ln\left[if_{i-1} + \sqrt{1 - f_{i-1}^2} \right] - \frac{e_1}{e_2}}$$

$$f_i' q_i = \frac{\left[\left[if_i + \sqrt{1 - f_i^2} \right]^2 + 1 \right] \left[\ln\left(if_i + \sqrt{1 - f_i^2} \right) - \frac{e_1}{e_2} \right]}{2\left[if_i + \sqrt{1 - f_i^2} \right]}$$

From the definition of ρ_i we have:

$$\left[\frac{ \dfrac{ \left[\left[if_{i-1} + \sqrt{1 - f_{i-1}^2} \right]^2 + 1 \right] \left[\ln\left(if_{i-1} + \sqrt{1 - f_{i-1}^2} \right) - \frac{e_1}{e_2} \right]^2 }{ if_{i-1} + \sqrt{1 - f_{i-1}^2} } }{ \dfrac{ \left[\left[if_i + \sqrt{1 - f_i^2} \right]^2 + 1 \right] \left[\ln\left(if_i + \sqrt{1 - f_i^2} \right) - \frac{e_1}{e_2} \right]^2 }{ if_i + \sqrt{1 - f_i^2} } } \right] =$$

$$\left[\frac{\dfrac{\left[\left[if_{i-1}+\sqrt{1-f_{i-1}^{2}}\right]^{2}+1\right]\left[\ln\left(if_{i-1}+\sqrt{f_{i-1}^{2}}\right)-\dfrac{e_1}{e_2}\right]}{if_{i-1}+\sqrt{1-f_{i-1}^{2}}}}{\dfrac{\left[\left[if_{i}+\sqrt{1-f_{i}^{2}}\right]^{2}+1\right]\left[\ln\left(if_{i}+\sqrt{1-f_{i}^{2}}\right)-\dfrac{e_1}{e_2}\right]}{if_{i}+\sqrt{1-f_{i}^{2}}}}\right]-\frac{\left(l_{i-1}g_{i-1}^{T}d_{i-1}\right)}{\dfrac{\left[\left[if_{i}+\sqrt{1-f_{i}^{2}}\right]^{2}+1\right]\left[\ln\left(if_{i}+\sqrt{1-f_{i}^{2}}\right)-\dfrac{e_1}{e_2}\right]}{if_{i}+\sqrt{1-f_{i}^{2}}}}$$

Using the following transformation:

$$\frac{\left[if_i+\sqrt{1-f_i^{2}}\right]^{2}+1}{if_i+\sqrt{1-f_i^{2}}}=x, \qquad \ln\left[\mathrm{if}_{\mathrm{i-1}}+\sqrt{1-f_{i-1}^{2}}\right]-\frac{e_1}{e_2}=y$$

$$\ln\left[\mathrm{if}_{\mathrm{i}}+\sqrt{1-f^{2}}\right]-\frac{e_1}{e_2}=y+w \quad \text{and} \quad \ln\left[\mathrm{if}_{\mathrm{i}}+\sqrt{1-f_i^{2}}\right]-\ln\left[if_{i-1}+\sqrt{1-f_{i-1}^{2}}\right]=w$$

$$c=l_{i-1}g_{i-1}^{T}d_{i-1}$$

then y=cw/xw+c

Therefore

$$\frac{e_1}{e_2}=\ln\left[if_{i-1}+\sqrt{1-f_{i-1}^{2}}\right]-\frac{\left[\ln\left(if_i+\sqrt{1-f_i^{2}}\right)\right]-\ln\left[if_i+\sqrt{1-f_i^{2}}\right].\left[-l_{i-1}g^{T}{}_{i-1}d_{i-1}\right]}{\dfrac{\left[if_{i-1}+\sqrt{1-f_{i-1}^{2}}\right]^{2}+1}{\left[if_{i-1}+\sqrt{1-f_{i-1}^{2}}\right]}.\left[\ln\left(if_{i-1}+\sqrt{1-f_{i-1}^{2}}\right)-\ln\left(if_{i-1}+\sqrt{1-f_{i-1}^{2}}\right)\right]+l_{i-1}g^{T}{}_{i-1}d_{i-1}}$$

4.The Outlines of our New Algorithm Area:

Given $x_0 \in R^n$ an initial estimate of the minimizer x^*.

Step (1): set $d_0 = - g_0$.

Step (2) : For i = 1, 2, ….

Compute $x_i = x_{i-1} + \lambda_{i-1} d_{i-1}$

Where λ_{i-1} is the optimal step size obtained by the line search procedure.

Step (3) : compute

$$r_i = \left[\frac{\dfrac{\left[\left[if_{i-1} + \sqrt{1 - f_{i-1}^2}\right]^2 + 1\right]\left[\ln\left(i\,f_{i-1} + \sqrt{1 - f_{i-1}^2}\right) - \frac{e_1}{e_2}\right]^2}{if_{i-1} + \sqrt{1 - f_{i-1}^2}}}{\dfrac{\left[\left[if_i + \sqrt{1 - f_i^2}\right]^2 + 1\right]\left[\ln\left(if_i + \sqrt{f_i^2}\right) - \frac{e_1}{e_2}\right]^2}{if_i + \sqrt{1 - f_i^2}}} \right]$$

Where the derivation of scaling r_i will be presented below.

Step (4) : calculate the new direction

$\mathbf{d_i = - g_i +}\ b_i\ \mathbf{d_i}$.

where b_i is defined by different formulae according to variation and it is expressed as follows:

$b_i = r_i\, (\|\mathbf{g_i}\|^2 / \|\mathbf{g_{i-1}}\|^2)$[modified Fletcher and Reeves, 1964 F/R,[8]]

$\left| b_i = g_i^T (r_i\, g_i - g_{i-1}) / d_{i-1}^T (r_i\ \ g_i - g_{i-1}) \right|$[modified Hestenes an stiefle 1952, H/s[10]]

$b_i = g_i^T (r_i\, g_i - g_{i-1}) / d_{i-1}^T g_{i-1}$ [modified Polak and Ribiera 1969,[11]]

$b_i = r_i \|g_{i+1}\|^2 / d_i^T g_i$ [modified Dixon 1972,[7]]

Conjugate gradient methods are usually implemented by restarts in order to avoid an accumulation of errors affecting the search directions.

It is therefore generally agreed that restarting is very helpful in practices, so we have used the following restarting criterion in our practical investigations. If the new direction satisfies:

$$\mathbf{d_i^T g_i} \geq -0.8\|\mathbf{g_i}\|^2$$

Then a restart is also initiated. This new direction is sufficiently downhill in Powell [12].

5. The Numerical Experiments:

In order to test the effectiveness of the new algorithm that have used to extend the CG method, a number of functions have been chosen and solved numerically by utilizing the new and established method.

The same line search was employed for all the methods. This was the cubic interpolation procedure described in Bunday [6].

It is found that the NEW method which modifies CG-algorithm is better than the previous algorithm shown in Tables (1) and (2).

Table (1) which uses the H/S formula, presents a comparison between the results of the NEW methods and the classical CG-method. So we can show that the NEW method has less (NOI) and (NOF) than the classical CG. Method and NEW method improve the two measures of performances, vis (NOI) and (NOF) (56.60)% and the (60.16) % for the H/S formula.

Table (1): Comparison between the different ECG – methods by using H/S formula .

Test Function	N	New NOI (NOF)	Classical CG NOI (NOF)
CUBIC	2	18 (51)	19 (53)
	200	12 (35)	14 (40)
	400	13 (32)	14 (40)
ROSEN	2	31 (82)	34 (87)
	10	21 (63)	26 (71)
	100	19 (56)	17 (52)
POWELL	60	48 (102)	125(303)
	80	91 (203)	112 (303)
	400	221 (537)	401 (860)
Non Diagonal	40	16 (44)	22 (73)
	60	17 (47)	22 (61)
	100	16 (46)	22 (60)
MIELE	40	50 (124)	82 (197)
	200	147 (338)	211 (491)
	400	142 (324)	402 (910)
CANTRAL	4	18 (113)	25 (148)
	40	19 (129)	20 (132)
	400	14 (71)	20 (132)
SHALLOW	40	9(21)	9(20)
	400	8(21)	9(21)
Total	NOI (NOF)	930 (2439)	1606 (4054)

Table (2) which uses the P/R formula, presents a comparison between the results of the NEW methods and the classical CG-method. So we can show that the NEW method has less (NOI) and (NOF) than the classical CG. Method and NEW method improve the two measures of performances, vis (NOI) and (NOF) by (49.22)% and the (53.71) % for the P/R formula.

Table (2): Comparison between the differenct ECG – methods by using P/R formula.

Test Function	N	New NOI (NOF)	Classical CG NOI (NOF)
CUBIC	2	18 (51)	19 (53)
	200	12 (33)	15 (40)
	400	11 (32)	15 (40)
ROSEN	2	31 (82)	33 (53)
	200	18 (53)	22 (61)
	400	18 (54)	22 (61)
POWELL	80	52 (117)	118(255)
	200	117 (240)	205 (427)
	400	52 (112)	405 (826)
Non Diagonal	60	17 (49)	18 (53)
	80	15 (43)	25 (70)
	100	17 (47)	22 (62)
MIELE	40	56 (155)	85 (238)
	60	56 (133)	65 (189)
	100	39 (101)	71 (199)
CANTRAL	4	23 (162)	25 (163)
	10	19 (92)	22 (135)
	400	14 (72)	22 (157)
SHALLOW	10	8(21)	8(19)
	400	10(27)	8(19)
Total	**NOI (NOF)**	**603 (1676)**	**1225 (3120)**

APPENDIX

1.Cubic Function :

$$F(x) = 100\left(x_2 - x_1^3\right)^2 + \left(1 - x_1\right)^2, \qquad x_0 = (-1.2, -1.)^T$$

2.Non – Diagonal Variant of Rosenbrock Function :

$$F(x) = \sum_{i=2}^{n} \left[100\left(x_i - x_i^2\right)^2 + \left(1 - x_i\right)^2 \right], \qquad n > 1,$$

3.SHALLOW Function

$$F(x) = \sum_{i=1}^{n} \left[\left(x_{2i-1}\right)^2 - \left(x_{2i}\right)^2 + \left(1 - x_{2i-1} - 1\right)^2 \right]$$

$$x_0 = (-2.0; -2.0; ....)^T$$

4. Generalized Powell Quartics Functions :

$$F(x) = \sum_{i=1}^{n/4} \left[\left(x_{4i-3} + 10x_{4i-2}\right)^2 + 5\left(x_{4i-1} - x_{4i}\right)^2 + \left(x_{4i-2} - 2x_{4i-1}\right)^4 + 10\left(x_{4i-3} - x_{4i}\right)^4 \right]$$

$$x_0 = (3.0; -1.0; 0.0; 1.0)^T$$

5. Rosenbrock Function :

$$F(x) = \sum_{i=1}^{n/2} \left[100\left(x_{2i} - x_{2i-1}^2\right)^2 + \left(1 - x_{2i-1}\right)^2 \right]$$

$$x_0 = (-1.2; 1.0; ....)^T$$

6. Miele Function :

$$F(x) = \sum_{i=1}^{n/4} \left[\exp\left(x_{4i-3}\right) - x_{4i-2}\right]^2 + 100\left(x_{4i-2} - x_{4i-1}\right)^6 +$$

$$\left[\tan\left(x_{4i-1} - x_{4i}\right)\right]^4 + x_{4i-3}^8 + \left(x_{4i} - 1\right)^2,$$

$$x_0 = (1.0; 2.0; 2..0; 2.0, ...)^T$$

7. Cantral Function :

$$F(x) = \sum_{i=1}^{n/4} \left[\exp\left(x_{4i-3}\right) - x_{4i-2}\right]^4 + 100\left(x_{4i-2} - x_{4i-1}\right)^6 +$$

$$\left[\operatorname{a\,tan}\left(x_{4i-1} - x_{4i}\right)\right]^4 + x_{4i-3}^8.$$

$$x_0 = (1.0; 2.0; 2..0; 2.0, ...)^T$$

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