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RESEARCH PAPER

Extended-Cyclic Operators

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A B S T R A C T:
In this paper, we study new classes of operators on separable Banach spaces which are called extended-cyclic operators and extended-transitive operators. We study some properties of their vectors which are called extended-cyclic vectors. We show that if x is an extended-cyclic vector for T, then $T^n x$ is also an extended-cyclic vector for T for all $n \in \mathbb{N}$. Then, we show the extended-cyclicity is preserved under qsuasi-similarity. Moreover, we prove that an operator is extended-cyclic if and only if it is extended-transitive. As a consequence, the set of all extended-cyclic vectors is a dense and G_δ set. Finally, we find some spectral properties of these operators. Particularly, the point spectrum of the adjoint of an extended-cyclic operator has at most one element of modules greater than one. Moreover, if the spectrum of an operator has a connected component subset of $B_0(1)$, then T is not extended-cyclic.

KEY WORDS: Extended-Cyclic, Banach spaces, G_δ set.
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1. INTRODUCTION:

Nowadays the area of linear dynamics is being interested by many researchers due to its connection with the two of the biggest open problems in mathematics which are invariant subspace problem and invariant subset problem. This connection will be clarified in the sequel.

In this paper, we use X to denote a separable infinite dimensional Banach space over scalar field $\mathbb{C}$ unless otherwise stated. Moreover, we use $\mathcal{B}(X)$ to denote the operator algebra consisting of all continuous linear operators $T: X \to X$.

A vector x is said to be cyclic for an operator $T \in \mathcal{B}(X)$ if the linear span of the orbit $span\text{Orb}(T,x) = \text{span}\{T^n x: n \in \mathbb{N}\}$ is dense in X , and T is called cyclic if it has a cyclic vector.

A vector x is said to be hypercyclic for an operator $T \in \mathcal{B}(X)$ if the orbit $\text{Orb}(T,x) = \{T^n x: n \in \mathbb{N}\}$ is dense in$X$, and T is called hypercyclic it has a hypercyclic vector. The study of hypercyclic operators can be traced back to the work of Birkhoff [1] in 1922 on the Frechet space $H(\mathbb{C})$ of entire functions. However, the first example of hypercyclic operators on Banach spaces was constructed by Rolewicz [2] who showed that if T is the unilateral backward shift on the sequence space $\ell^p(\mathbb{N})$ and if c is a scalar with $|c| \leq 1$, then cT is not hypercyclic.

Note that T has no nontrivial closed invariant subspace (resp. subset) if and only if every nonzero vector is cyclic (resp. hypercyclic) for T. This establishes the connection between these concepts and invariant subspace (resp. subset) problem and invariant subset problem.

The supercyclicity origin goes back to 1972 paper of Hilden and Wallen, [3] and it has widely studied since then. A vector x is said to be supercyclic if the scaled orbit $\mathbb{C}Orb(T,x) = \{\alpha T^n x: \alpha \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in X, and T is called supercyclic if it has a supercyclic vector.

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Based on Rolewicz example the disk cyclicity notion was defined by Zeana in 2002 by the same manner [4]. A vector $x \in X$ is called diskcyclic if the disk orbit $\mathbb{D}Orb(T,x) = \{\alpha T^n x : \alpha \in \mathbb{D}, n \geq \mathbb{N}\}$ is dense in X and in this case T is called diskcyclic operator. Good references to learn about hypercyclicity, supercyclicity and disk cyclicity are [5], [6], [7] and [8].

In this paper, we extend the disk cyclicity concept; that is, instead of the closed unit disk we use the closed disk $D_r(0)$ of radius $r \geq 1$ and centered at zero, and we call it extended-cyclic.

First, we find some basic properties of extended-cyclic operators and their vectors. We show that the positive integer powers of an extended-cyclic operator T are also extended-cyclic. We study the extended-cyclicity of similar and quasi-similar operators.

Second, we define extended-transitivity concept and prove that extended-transitive operators are equivalent to extended-cyclic ones. Moreover, we establish an extended-cyclic criterion for an operator to be extended-transitive. Then, we use this result to find an example of an extended-cyclic operator which is not diskcyclic.

Finally, we study the spectral properties of these operators. We show that if T is extended-cyclic, then T^* has at most one eigenvalue of modules greater than 1. On the other hand, the spectrum of an extended-cyclic operator cannot have a component in the open unit ball.

2.MAIN RESULTS:

First, we define extended-cyclicity notion which is an intermediate concept between diskcyclicity and supercyclicity.

Definition 1. Let $T \in \mathcal{B}(X)$, then T is called extended-cyclic operator if there exist $x \in X$ such that the extended- orbit (E-orbit, for short) of x, $\text{EOrb}(T,x) = \{\alpha T^n(x) : \alpha \in D_r(0), n \in \mathbb{N}\}$ is dense in X. In this case, x is called extended-cyclic vector for T.

In what follows, we denote the set of all extended-cyclic vectors for T by $EC(T)$ and the set of all extended-cyclic operators on $\mathcal{B}(X)$ by $EC(X)$.

Remark:

1. Every diskcyclic operator is extended-cyclic;
2. Every extended-cyclic is supercyclic.

We need the following lemma to prove the next proposition.

Lemma 1. If x is an extended-cyclic vector for T, then

$$inf\{\lambda \|T^n x\| : n \geq 0, \lambda \in D_r(0)\} = 0$$
$$\sup\{\|T^n x\| : n \geq 0\} = \infty$$

Proof: We have $\lambda \in D_r(0)$, then it follows that $\inf\{\lambda\|T^n x\| : n \geq 0\} = 0$. First we will show that $\sup\{\lambda\|T^n x\| : n \geq 0, \lambda \in D_r(0)\} = \infty$. By contradiction, suppose that

$$\sup\{\lambda\|T^n x\| : n \geq 0, \lambda \in D_r(0)\} = k, k \in \mathbb{R}$$

and $z \in X$ such that $\| z \| > k$. Since $T \in EC(X)$, then there exist two sequences $\{n_k\} \in \mathbb{N}$ and $\{\lambda_k\} \in D_r(0)$ such that $\lambda_k T^{n_k} x \to z$. It follows that $\| z \| \leq k$ which is contradiction. Now, we have

$$\begin{aligned} \infty &= \sup\{\lambda\|T^n x\| : n \geq 0, \lambda \in D_r(0)\} \\ &\leq r\sup\{\|T^n x\| : n \geq 0\} \end{aligned}$$

It follows that

$$\sup\{\|T^n x\| : n \geq 0\} = \infty.$$

Proposition 1. Let $T \in \mathcal{B}(X)$ and $\| T \| < 1$, then T is not extended-cyclic.

Proof: Suppose that T is an extended-cyclic operator and $x_0 \in EC(T)$. By Lemma 1, we have $\sup\{\|T^n x_0\| : n \geq 0\} = \infty$ that is $\|T^m x_0\| > k$ for some $m \in \mathbb{N}$ and all $k \in \mathbb{R}^+$. Since $\| T \| < 1$ then,

$$k \leq \|T^m x_0\| \leq \|T^m\|\|x_0\| \leq \| T \|^m \|x_0\| \leq \|x_0\|$$

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for all $k \in \mathbb{R}^+$which means that $\|x_0\| = \infty$ a contradiction to the assumption that $x_0 \in EC(T)$. Therefore, T is not extended-cyclic.

Example 1. If $T \in \ell^p(\mathbb{N})$ is the backward shift operator. Then αT is not extended-cyclic for all $\alpha \in \mathbb{C}; |\alpha| \leq 1$.
Proof: Since $\| T \| \leq 1$ then $\| \alpha T \| \leq |\alpha| < 1$. It follows that αT is not extended-cyclic.

Proposition 2. If $T \in EC(X)$ and $S \in \mathcal{B}(X)$ be an operator with dense range. If T commutes with S and $x \in EC(T)$ then $Sx \in EC(T)$
Proof: Since $x \in EC(T)$, then

$$\begin{aligned}\overline{\text{EOrb}(T, Sx)} &= \overline{\{\alpha T^n Sx : \alpha \in D_r(0), n \geq 0\}}\\ &= \overline{\{\alpha S T^n x : \alpha \in D_r(0), n \geq 0\}}\\ &= \overline{S\{\alpha T^n x : \alpha \in D_r(0), n \geq 0\}}\\ &\supseteq S\left(\overline{\{\alpha T^n x : \alpha \in D_r(0), n \geq 0\}}\right)\\ &= S(X).\end{aligned}$$

Thus $E\text{Orb}(T, Sx)$ is dense in X and hence $Sx \in EC(T)$. From the last proposition one can easily deduce that if an operator has one extended-cyclic vector, then it has many extended-cyclic vectors. In other words, we have the following two corollaries

Corollary 1. If x is an extended-cyclic vector for T, then $T^n x$ is also an extended-cyclic vector for T for all $n \in \mathbb{N}$.

Corollary 2. If T is an extended-cyclic operator, then it has a dense range.

Let $T \in \mathcal{B}(X)$ and $S \in \mathcal{B}(Y)$, then T and S are called quasi-similar if there exists an operator $f: X \to Y$ with dense range such that $Sf = fT$. If f is a homeomorphism, then T and S are called similar. The following proposition shows that the extended-cyclicity is preserved under quasi-similarity.

Proposition 3. Let T and S be quasi-similar, and let $T \in EC(X)$ then $S \in EC(Y)$. Moreover, $f(EC(T)) \subset EC(S)$.
Proof: Since T and S are quasi-similar, then there exists an operator $f: X \to Y$ with dense range such that $Sf = fT$. Let A be a non-empty subset of Y, then $f^{-1}(A)$ is also open and non-empty.

Now, let $x \in EC(T)$, then there exists $\alpha \in D_r(0), n \in \mathbb{N}$ such that $\alpha T^n x \in f^{-1}(A)$ which means that $\alpha f(T^n x) \in A$. It follows that $\alpha S(fx) = \alpha f(Tx) \in A$; that is, EOrb$(S, fx)$ intersects every open sets in Y and so it is dense in Y. It follows that $S \in EC(Y)$ and $fx \in EC(S)$.

Corollary 3. Let T and S be similar, then $T \in EC(X)$ if and only if $S \in EC(Y)$. Moreover, $f(EC(T)) \subset EC(S)$.

Proposition 4. Let $\{X_i\}_{i=1}^n$ be a family of Banach spaces and $T_i \in \mathcal{B}(X_i)$ for all $1 \leq i \leq n$. If $\oplus T_i$ is extended-cyclic in $\oplus X_i$ then T_i is extended-cyclic in X_i for all $1 \leq i \leq n$.
Proof: Since $\oplus T_i$ is quasi-similar to T_k; $1 \leq k \leq n$ then the proof follows by Proposition 3.
The following theorem shows that the set of all extended-cyclic vectors for an operator can be written as a countable intersection of open sets.

Theorem 1. Let T be an extended-cyclic operator, then

$$EC(T) = \bigcap_k \left(\bigcup_{\lambda \in D_r(0)} \bigcup_n T^{-n}\left(\frac{1}{\lambda} B_k\right) \right)$$

where $\{B_k\}$ is a countable open basis for X. Moreover, $EC(T)$ is a G_δ set.

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Proof: First, we will show that $EC(T) \subseteq \bigcap_k \left(\bigcup_{\lambda \in D_r(0)} \bigcup_n T^{-n}\left(\frac{1}{\lambda} B_k\right)\right)$. Let $x \in EC(T)$ then $\{\lambda T^n x : n \geq 0, \lambda \in D_r(0)\}$ is dense in X and so for each $k > 0$, there exist $\lambda \in D_r(0)$, and $n \in \mathbb{N}$ such that $\lambda T^n x \in B_k$. It follows that $x \in \bigcap_k \left(\bigcup_{\lambda \in D_r(0)} \bigcup_n T^{-n}\left(\frac{1}{\lambda} B_k\right)\right)$ and so $EC(T) \subseteq \bigcap_k \left(\bigcup_{\lambda \in D_r(0)} \bigcup_n T^{-n}\left(\frac{1}{\lambda} B_k\right)\right)$. The proof of the second inclusion is just the reverse of the former. Therefore, we get

$$EC(T) = \bigcap_k \left(\bigcup_{\lambda \in D_r(0)} \bigcup_n T^{-n}\left(\frac{1}{\lambda} B_k\right)\right)$$

Since $EC(T)$ can be written as a countable intersection of open sets, then $EC(T)$ is a G_δ set.

Now, we define extended-transitive operators. Then we find its relation with extended-cyclic operators in order to find some more properties of extended-cyclic operators.

Definition 2. A bounded linear operator $T: X \to X$ is called extended-transitive if for any pair U, V of nonempty open subsets of X, there exist $\lambda \in D_r(0)$, and $n \geq 0$ such that $T^n(\lambda U) \cap V \neq \phi$ or equivalently, $T^{-n}(\frac{1}{\lambda} U) \cap V \neq \phi$.

Proposition 5. An operator T is extended-transitive if and only if $EC(T)$ is dense in X.

Proof: By Theorem 1, we have $EC(T) = \bigcap_k \left(\bigcup_{\lambda \in D_r(0)} \bigcup_n T^{-n}(\lambda B_k)\right)$. Suppose that $G_k = \bigcup_{\lambda \in D_r(0)} \bigcup_n T^{-n}(\lambda B_k)$, then by Baire Theorem $EC(T)$ is dense if and only if each open set $G_k = \bigcup_{\lambda \in D_r(0)} \bigcup_n T^{-n}(\lambda B_k)$ is dense; i.e, if and only if for each non-empty open set U and any $k \in \mathbb{N}$ there exist n and $\lambda \in D_r(\mathbf{0})$ such that $U \cap T^{-n}(\lambda B_k) \neq \phi$ if and only if T is extended-transitive.

Theorem 2. An operator $T \in \mathcal{B}(X)$ is extended-cyclic if and only if it is extended-transitive.

Proof: Let T be extended-transitive. By Proposition 5, EC (T) is a dense set and so T is extended-cyclic. Conversely, let T be extended-cyclic, and let U and V be two open sets. By the density of the E-orbit of T there exist an $\lambda \in D_r(0)$ and $p \in \mathbb{N}$ such that $\lambda T^p x \in U$. Also one can find $\beta \in D_r(0), s \in \mathbb{N}$, such that $s \geq p, |\beta| \leq r|\lambda|$ and $\beta T^s x \in V$. Thus, $(\beta/\lambda) T^{s-p} U \cap V \neq \phi$. It follows that T is extended-transitive.

The following two propositions give some necessary and sufficient conditions for an operators to be extended-transitive.

Proposition 6. Let $T \in \mathcal{B}(X)$. The following statements are equivalent.

1. T is extended-transitive.
2. For each $x, y \in X$, there exist sequences $\{x_k\}$ in $X, \{n_k\}$ in $\mathbb{N}$, and $\{\alpha_k\}$ in $D_r(0)$ such that $x_k \to x$ and $T^{n_k} \alpha_k x_k \to y$.
3. For each $x, y \in X$ and each neighborhood W of 0, there exist $z \in X, n \in \mathbb{N}$, and $\alpha \in D_r(0)$ such that $x - z \in W$ and $T^n \alpha z - y \in W$.

Proof: $1 \Rightarrow 2$: Let $x, y \in X$. For all $k \geq 1$, let $U_k = \mathbb{B}(x, 1/k), V_k = \mathbb{B}(y, 1/k)$. Then both U_k and V_k are non-empty open sets in X. Since T is extended-transitive, there exist $n_k \in \mathbb{N}$, $\alpha_k \in D_r(0)$ such that $\alpha_k T^{n_k} U_k \cap V_k \neq \phi$. So, for all $k \geq 1$ there exists $x_k \in U_k$ such that $\alpha_k T^{n_k} x_k \in V_k$ which means that $\|x_k - x\| < 1/k$ and $\|T^{n_k} \alpha_k x_k - y\| < 1/k$ for all $k \geq 1$. It follows that $x_k \to x$ and $T^{n_k} \alpha_k x_k \to y$.

$2 \Rightarrow 3$: Follows directly by taking $z = x_k$ for a large enough $k \in \mathbb{N}$.

$3 \Rightarrow 1$: Let U and V be two non-empty open subsets of X, and let $x \in U$ and $y \in V$. Suppose that $W_k = B\left(0, \frac{1}{k}\right)$ is a neighborhood for 0 for all $k \geq 1$. Then, there exist $z_k \in X, n_k \in \mathbb{N}$, $\alpha_k \in D_r(0)$ such that $x - z_k \in W_k$ and $T^{n_k} \alpha_k z_k - y \in W_k$. This implies that $z_k \in U$ and $T^{n_k} \alpha_k z_k \in V$ which follows $T^{n_j} \alpha_j z_j U \cap V \neq \phi$ for some $j \in \mathbb{N}$ and so T is extended-transitive.

Proposition 7. An operator $T \in \mathcal{B}(X)$ is extended-cyclic if and only if the set $\{(x, \lambda T^n x) : x \in X, n \geq 0, \lambda \in D_r(0)\}$ is dense in $X \oplus X$.

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Proof: Let $(a,b) \in X \oplus X$ and $\epsilon > 0$. Since $T \in EC(X)$, then there exist $c \in X, m \geq 0$ and $\lambda \in D_r(0)$ such that $\| c - a \| \leq \frac{\epsilon}{2}$ and $\|\lambda T^n c - b\| \leq \frac{\epsilon}{2}$. It follows that

$$\|(c, \lambda T^n c) - (a,b)\| = \|(c-a, \lambda T^n c - b)\| = \| (c-a) \| + \|(\lambda T^n c - b)\| \leq \epsilon$$

Conversely: Let $a, b \in X$. By hypothesis, there exist sequences $c_k \in X, \lambda_k \in D_r(0)$ and $n_k \in \mathbb{N}$ such that $(c_k, \lambda_k T^{n_k} c_k) \to (a,b)$ as $k \to \infty$. Therefore, there exist a large number $m \in \mathbb{N}$ and $\epsilon > 0$ such that $\|(c_k, \lambda_k T^{n_k} c_k) - (a,b)\| \leq \epsilon$ for all $k \geq m$ which means $\|c_k - a\| \leq \epsilon$ and $\|\lambda_k T^{n_k} c_k - b\| \leq \epsilon$ that is $c_k \to a$ and $\lambda_k^{n_k} c_k \to b$ as $k \to \infty$, Hence T is extended-cyclic.

Proposition 8. Let $T \in \mathcal{B}(X)$ and W be a 0-neighborhood in X, and let U and V be two nonempty open sets in X. If there exist $n \geq 0, \lambda \in D_r(0)$ such that $\lambda T^n U \cap W \neq \phi$ and $\lambda T^n W \cap V \neq \phi$, then T is extended-cyclic.

Proof: Let $x, y \in X$, and let $A_k = B_{\frac{1}{k}}(x)$ and $B_k = B_{\frac{1}{k}}(y)$ for all $k \geq 1$. By hypothesis, there exist sequences $n_k \in \mathbb{N}, \lambda_k \in D_r(0)$ and $w_k \in W$ such that $z_k \in A_k$ and $\lambda_k T^{n_k} z_k \in W$ and $\lambda_k T^{n_k} w_k \in B_k$ for all $k \geq 1$. Therefore,

$$z_k \to x, \text{ and } \lambda_k T^{n_k} z_k \to 0$$

and

$$w_k \to 0, \text{ and } \lambda_k T^{n_k} w_k \to y$$

By taking $x_k = z_k + w_k$ the proof is completed.

Definition 3. An operator $T \in \mathcal{B}(X)$ is said to be satisfied extended-cyclic criterion if there exist two dense sets A and B in X, a sequence n_k of positive integers, a sequence $\lambda_k \in D_r(0)$ and a map $S: B \to X$ such that

1 $\lambda_k T^{n_k} x \to 0$ for all $a \in A$,

2 $\frac{1}{\lambda_k} S^{n_k} y \to 0$ for all $b \in B$,

3 $T^{n_k} S^{n_k} y \to y$ for all $b \in B$

The following theorem gives a sufficient condition for an operator to be extended-cyclic. Then, we use this theorem to find an example of an extended-cyclic operator.

Theorem 3. If $T \in \mathcal{B}(X)$ satisfies extended-cyclic criterion, then T is extended-cyclic.

Proof: Let U and V be two nonempty open sets. Since A and B are dense sets then there exist a and b in X such that $a \in A \cap U$ and $b \in B \cap V$. Consider the sets

$c_k = a + \frac{1}{\lambda_k} S^{n_k} b \ldots$ (1)

for all $k \geq 1$. By hypothesis, we have $\frac{1}{\lambda_k} S^{n_k} b \to 0$ as $k \to \infty$ and so $c_k \to a$ which follows that $c_k \in G$ for all $k \geq k_1; k_1 \in \mathbb{N}$. Using Equation (1), we get $\lambda_k T^{n_k} c_k = \lambda_k T^{n_k} a + T^{n_k} S^{n_k} b$. Again, by hypothesis, we have $\lambda_k T^{n_k} c_k \to b$ as $k \to \infty$. It follows that $\lambda_k T^{n_k} c_k \in V$ for all $k \geq k_2; k_2 \in \mathbb{N}$; and so, $\lambda_k T^{n_k} U \cap V \neq \phi$ for some $k \geq \max\{k_1, k_2\}$. Therefore, T is extended-transitive and so extended-cyclic.

We have seen in Example 1 if $T \in \ell^p(\mathbb{N})$ is the backward shift operator, then αT is not extended-cyclic whenever $\alpha \in \mathbb{C}; |\alpha| \leq 1$. However, the next example shows different thing when $|\alpha| > 1$.

Example 2. If $T \in \ell^p(\mathbb{N})$ is the backward shift operator. Then αT is extended-cyclic for all $\alpha \in \mathbb{C}; |\alpha| > 1$.

Proof: We will verify extended-cyclic criterion. Let $A = B$ be the dense sets in $\ell^p(\mathbb{N})$ consisting of all points with only finitely many non-zero coordinates. Let $n_k = n$ be the set of all nonnegative integers, $F \in \ell^p(\mathbb{N})$ be the unilateral forward shift operator and $\lambda_k = \frac{r}{\alpha} \in D_r(0)$ for some $r \in \mathbb{C}; |r| > 1$. Consider the map $S = \frac{1}{\alpha} F: B \to X$ and $T_1 = \alpha T$ then

1 $\lambda_k T_1^k x = \frac{r^k}{\alpha^k} \alpha^k T^k x \to 0$ for all $x \in A$,

2 $\frac{1}{\lambda_k} S^k y = \frac{\alpha^k}{r^k} \frac{1}{\alpha^k} F y \to 0$ for all $y \in B$,

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3 $T_1^k S^k y = (\alpha T)^k \frac{1}{\alpha^k} F^k y = y$ for all $y \in B$.

By Theorem 3, αT is extended-cyclic. The following example shows that not every extended-cyclic operator is diskcyclic

Example 3. Let $F: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the bilateral forward weighted shift with the weight sequence

$$w_n = \begin{cases} 2 & \text{if } n \geq 0 \\ 5 & \text{if } n < 0 \end{cases}$$

Let $p \in \mathbb{N}; p > 2$, and let

$$\lambda_n = \begin{cases} 4^n & \text{if } n \leq p \\ \frac{1}{4^n} & \text{if } n > p \end{cases}$$

Then F is extended-cyclic but not diskcyclic with respect to the sequence λ_n.

Proof: We will verify extended-cyclic criterion. It is clear that $\lambda_n \in B_r(0)$ for some $r > 4^p$ and all $n \geq 1$. Let $A = \{x \in \ell^2(\mathbb{Z}): x \text{ has only finitely many non-zero coordinates}\}$ and (n) be the sequence of all non-negative integers. Let $\{e_n\}_{n\in\mathbb{Z}}$ be the canonical basis of $\ell^2(\mathbb{Z})$, and let $x \in A$. Suppose that $S = F^{-1}$ be the bilateral backward weighted shift such that $Se_n = (1/w_{n-1})e_{n-1}$. Then S has the weight sequence

$$\frac{1}{w_{n-1}} = \begin{cases} \frac{1}{2} & \text{if } n > 0 \\ \frac{1}{5} & \text{if } n \leq 0 \end{cases}$$

Without loss of generality, we will suppose that $x = e_0$ then by [b, Lemma 3.1.], if $F^n e_0 \to 0$ as $n \to \infty$ then $F^n e_k \to 0$ for all $k \in \mathbb{Z}$ and so by triangle inequality, $F^n x \to 0$. Since

$$\lim_{n\to\infty} \|\lambda_n F^n e_0\| = \lim_{n\to\infty} \left(\frac{1}{4^n} \prod_{k=1}^{n} 2 \right) = \lim_{n\to\infty} \frac{2^n}{4^n} = 0$$

Then

$$\lambda_n F^n x = 0 \text{ as } n \to \infty \text{ ……………………(1)}$$

Again, by [2, Lemma 3.1.], if $S^n e_0 \to 0$ as $n \to \infty$ then $S^n e_k \to 0$ for all $k \in \mathbb{Z}$ and so by triangle inequality, $S^n x \to 0$.

Now, since $\lim_{n\to\infty} \left\| \frac{1}{\lambda_n} S^n e_0 \right\| = \lim_{n\to\infty} 4^n \prod_{i=1}^{n} (1/5) = \lim_{n\to\infty} (4^n/5^n) = 0$ then

$$\frac{1}{\lambda_n} S^n x \to 0 \text{ as } n \to \infty \text{ ………………………(2)}$$

Then by equations (1) and (2), and the fact that $F^n S^n x = x, F$ satisfies extended-cyclic criterion with respect to the sequence λ_n and so T is extended-cyclic.

On the other hand, since $\lambda_n \notin \mathbb{D}$ then by [5, Theorem 2.6.] T is not diskcyclic with respect to λ_n. In what follows, we will suppose that X is an infinite dimensional separable Hilbert space. First, we need the following lemma.

Lemma 2. Let x be an extended-cyclic vector for T and $z \in X$. Then the set $A = \{\langle \alpha T^n x, z \rangle : n \geq 0, \alpha \in D_0(r)\}$ is dense in $\mathbb{C}$.

Proof: Let $c \in \mathbb{C}$. It is clear that the vector $\frac{cz}{\|z\|^2} \in X$. Since $T \in EC(X)$, then there exists a sequence $\alpha_k T^{n_k} x \in$ E Or $b(T, x)$ such that

$$\alpha_k T^{n_k} x \to \frac{cz}{\| z \|^2}$$

Which follows that

$$\langle \alpha_k T^{n_k} x, z \rangle \to \left\langle \frac{cz}{\| z \|^2}, z \right\rangle = \frac{c}{\| z \|^2} \langle z, z \rangle = c.$$

Therefore, the set A is dense in $\mathbb{C}$.

Theorem 4. Let T is extended-cyclic. Then T^* has at most one eigenvalue of modules greater than 1.
Proof: Since T is extended-cyclic then it is supercyclic and so either $\sigma_p(T^*)$ contains at most one non-zero eigenvalue [9, Proposition 3.1.]. Suppose that $\sigma_p(T^*) = \{\lambda\}$. Hence, there is a unit vector $z \in X$ such that $T^*z = \lambda z$. Let $x \in EC(T)$, then by Lemma 2 it is easy to show that

$$A = \{|\langle \alpha T^n x, z\rangle| : n \geq 0, \alpha \in D_0(r)\}$$

is dense in $\mathbb{R}^+ \cup \{0\}$.

Now, suppose that $|\lambda| < 1$. Then for all $|\langle \alpha T^n x, z\rangle| \in A$, we have

$$\begin{aligned} |\langle \alpha T^n x, z\rangle| &= |\alpha| \langle T^n x, z\rangle \\ &\leq r\langle T^n x, z\rangle = r\langle x, T^{*n} z\rangle \\ &= r\lambda^n \langle x, z\rangle \\ &< r\langle x, z\rangle \end{aligned}$$

which contradicts (2) since $r\langle x, z\rangle$ is a constant. Therefore, $|\lambda| \geq 1$.

Proposition 9. Let T be extended-cyclic, then either the Weyl-spectrum $\sigma_w(T) = \sigma(T)$ or $\sigma_w(T) = \sigma(T) \setminus \{\alpha\}; |\alpha| > 1$
Proof: For any $T \in \mathcal{B}(X)$, we have

$$\sigma(T^*) \setminus \sigma_w(T^*) \subseteq \sigma_p(T^*)$$

By Theorem 4, we get

$$\sigma_w(T^*) = \sigma(T^*) \text{ or } \sigma_w(T^*) = \sigma(T^*) - \{\lambda\}; |\lambda| > 1$$

So, either,

$$\sigma_w(T) = \overline{\sigma_w(T^*)} = \overline{\sigma(T^*)} = \sigma(T)$$
$$\sigma_w(T) = \overline{\sigma_w(T^*)} = \overline{\sigma(T^*) - \{\lambda\}} = \sigma(T) - \{\bar{\lambda}\}; |\bar{\lambda}| > 1$$

From equations (3) and (4), we get the desired result.

Proposition 10. Let $T \in \mathcal{B}(X)$. If $\sigma(T)$ has a connected component σ such that $\sigma \subset B_0(1)$, then T is not extended-cyclic.
Proof: Suppose that σ is a connected component $\sigma(T)$ such that $\sigma \subset B_0(1)$. Then, by Riesz decomposition Theorem, $T = T_1 \oplus T_2$ such that $\sigma(T_1) = \sigma$. It follows that $\lim_{n\to\infty} \|T^n x\| \to 0$ for all $x \in X$ which means that $\sup\{\|T^n x\| : n \geq o\} \neq \infty$ a contradiction to Lemma 1. Therefore T is not extended-cyclic.

Corollary 4. Let T be an extended-cyclic operator, then $\sigma(T) \cap D_p(0)$ is connected for all $p \leq 1$.
Proof: Suppose that $\sigma(T) \cap D_p(0)$ is not connected for some $p \leq 1$. Then there exists a closed and open set $\sigma \subseteq \sigma(T) \cap D_p(0)$. Since $p \leq 1$, then $\sigma \subseteq B_0(1)$. Then by Proposition10, T is not extended-cyclic which contradicts the hypothesis. Therefore $\sigma(T) \cap D_p(0)$ is connected for all $p \leq 1$.

3.CONCLUSIONS

We defined and studied two new types of operators which are called extended-cyclic operators and extended-transitive operators. Also, we studied their corresponding vectors and investigated their properties. Moreover, various results regarding spectral properties were derived.

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