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Statistical and Visual Analysis of Error In Interpolating Sculptured Surfaces

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Abstract

This paper presents a general method for automatic surface interpolation from range data and describes the implementation of high order 2D langrangian methods for interpolating surfaces from 3rd to 8th order. It uses a modified 2D lagrangian method to interpolating, reconstructing and modeling several surfaces, and due to this interpolation there will be an error between interpolated surface and the original one, this paper present an algorithm to detect, represent and analyze these interpolating errors and statistical criteria to compare the error in representing the sculptured surfaces that interpolated with different order basis functions. The comparison is performed using a mathematically defined data as real data obtained from the proposed models.

The system has been implemented for the design of several sculptured surfaces to illustrate the system flexibility. Then the design results have been implemented for manufacturing two of these surfaces using three axis vertical CNC milling machine tool with ball end mill cutter. The method can be used in a variety of CAD/CAM applications and it has proven to be effective as demonstrated by a number of examples using real data from mathematical functions. By applying the proposed surface interpolating models the percent error was found to be ranged between 0.00001% for some model to 3.5% for some other interpolated models with Lagrange method.

Keywords: Sculptured Surfaces, Bivariate Interpolating, Computer Graphics, Error Analysis, Lagrange Interpolating

التحليل الاحصائي والبصري لقيمة الخطأ الناتج من استكمال السطوح المتعرجة

الخلاصة

يقدم هذا البحث طريقة عامة لاستكمال وتمثيل السطوح الهندسية المتعرجة باستخدام طريقة لاكرانج للتحليلات العددية وتطويرها لجعلها ملائمة للتطبيقات الهندسية, وقد تم اعتماد عدة درجات للاستكمال من الدرجة الثالثة لغاية الدرجة الثامنة لتوليد بيانات السطوح وتحليل الخطأ الناتج من جراء الاستكمال ثم مقارنة النتائج مع البيانات الحقيقية للسطوح المقترحة في هذا البحث حيث تم التطبيق لأربعة أسطح ممثلة بدالة الجيب و دالة أسية و دالة تكعيبية , تم استثمار البيانات التصميمية المستنتجة للسطوح المتعرجة من خلال استكمال البيانات بطريقة لاكرانج المقترحة لتصنيع سطحين متعرجين وبدرجات استكمال مختلفة باستخدام ماكنة تفريز عمودية ذات تحكم عددي بثلاث محاور وعدد تفريز ذات نهاية كروية

وتمت المقارنة من خلال التحليل الاحصائي للبيانات و اظهرت النتائج فاعلية الطر يقة المطورة المقترحة لتمثيل السطوح في تطبيقات الهندسية العكسية و التصميم المعان بالحاسوب وغيرها

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من التطبيقات, وعند مقارنة النتائج المختلفة وجد ان نسبة الخطأ تراوحت 0.00001% لاحد النماذج الى 3.5% لنموذج اخر)

1. Introduction

Interpolation, a fundamental topic in numerical analysis, is the problem of constructing a function which goes through a given set of data points. In some applications, these data points are obtained by sampling a function or Process; subsequently, the values of the function can be used to construct an interpolant, which must agree with the interpolated function at the data points.

The simplest kind of interpolation, in which most development has been made, is interpolation by means of univariate polynomials [1].

Multivariate interpolation has applications in computer graphics, numerical quadrature, cubature, and numerical solutions to differential equations [2 and 3]. The sub purpose of this paper is to submit a multivariate Lagrange formula, under conditions which we will specify.

Given a finite collection of points in affine space, we shall investigate 2D lagrangian methods for generating polynomial curves and surfaces that pass through the points. We begin with schemes for curves and later extend these techniques to surfaces.

The Surface Modeling process includes a set of operations that allow transforming spatial data representing a three-dimensional surface from one form to another. The most familiar example of such data is probably the variation in elevation surface [4] and [5]. However, any variable can be visualized and analyzed as a three-dimensional surface as long as it varies relatively smoothly at the chosen map scale and has only a single value at each location.

A three-dimensional surface can be approximated in a number of forms, including irregularly-spaced point observations, a regular grid of values, or contour lines of equal value (isolines). Surface interpolates a regular grid of values from data in the input object and outputs the grid as a meshed object.

The input data can be in the form of points stored in a vector object or in a database that has X and Y coordinate fields for each record. The input object used in this work is a 3D vector object regularly-spaced sample elevation points from a sculptured surface. The elevation is stored as a Z-value for each point.

The proposed interpolation method can be used in a variety of CAD/CAM applications and reverse engineering

Interpolation

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1),......, (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points . Then one can find the value of y at any other value of x. This is called *interpolation*. If x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation* [1] and [3].

A polynomial function $f(x)$ is a common choice for an interpolating function because polynomials are easy to evaluate, differentiate, and integrate,

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ data points. One of the methods used to find this polynomial is called the Lagrangian method of interpolation. Other methods include Newton's divided difference polynomial method and the direct method.

Lagrange Interpolation

The Lagrange interpolating polynomial is given by [6]:

$$f_n(x)=\sum_{i=0}^{n} L_i(x)f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y=f(x)$ given at $n+1$ data points as $(x_0,y_0),(x_1,y_1),\ldots\ldots,(x_{n-1},y_{n-1}),(x_n,y_n)$, and

$$L_i(x)=\prod_{\substack{j=0\\ j\neq i}}^{n}\frac{x-x_j}{x_i-x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j=i$ omitted.

(1- D) Interpolation *(y =f(x)):*

Interpolating polynomial of nth degree of a sequence of planar points defined by $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$ where $x_i<x_j$ for $i<j$ can be calculated as:[6]

$$F_n(x)=\sum_{i=1}^{n} y_1 L_{i,n}(x) \quad \ldots\ldots\ldots(1)$$

Where:

$$L_{i,n}(x)=\frac{(x-x_1)\ldots\ldots x(-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_1)\ldots\ldots x_i(-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)} \quad \ldots\ldots(2)$$

Short notation for this formula is:

$$F_n(x)=\sum_{i=1}^{n} y_i\prod_{\substack{j=0\\ j\neq i}}^{n}\left(\frac{x-x_j}{x_i-x_j}\right) \quad \ldots\ldots(3)$$

where $\prod$ denotes multiplication of the n factors obtained by varying j from (o to n), excluding $i=j$, it can be observed that the function multiplying y_i is equal to unit when $x=x_i$ but becomes zero when (x) equals any of the other coordinates.

Methodology

The methodology of this work to represent and analyze the error caused by using modified 2D Lagrange interpolating of sculptured surfaces with different interpolating order (3rd to 8th), can divide into the following six steps:

1. Definition of the input data point set.
2. Surface representation for a mathematically defined function.
3. Construction of 2D lagrangian models for surface interpolating.
4. Surface interpolating and representing using a modified 2D lagrangian model.
5. Statistical and visual analysis of the error caused by interpolation.
6. Implementation

1. Definition of the input control point Set

The first step for modeling surfaces is the definition of the input data point set that will be used to reconstruct the surfaces. This sample set must be representative of the phenomenon to be modeled [7] and [8]. To compare the high order interpolating approaches for surface interpolating (3^{rd} to 8^{th} order), four different patterns have been chosen.

The first pattern of comparison is the following mathematically defined function that will be called sine function:

$$z = \sin(4*\sqrt{x^2+y^2})/4*\sqrt{x^2+y^2}) \quad .....(4)$$

The second pattern of comparison is the following mathematically defined function that will called polynomial function

$$z = 0.015*((x-2)^4*(x+1)^3*(x-1)+ 0.5*(8*y^4-14*y^3-9*y^2+11*y-1)) \quad .....(5)$$

The third pattern of comparison is the following mathematically defined function that will called exponential function

$$z = x*e^{(-x^2-y^2)} \quad ....(6)$$

Finally, the fourth pattern of comparison is a free form surface.

2. Surface interpolating for a mathematically defined function

Engineering surfaces can be modeled and rendered in several ways [9].Four demonstrative surface construction examples have been provided in this work. Mathematical models were rendered using the sine, polynomial, exponential and free form, to perform this task, rectangular grids 50x50 points were constructed using (z) values calculated from the models. Figures 1, 2, 3 and 4 show the perspective view of the rendered surfaces as real surfaces represent the four different functions.

3. Construction of 2D lagrangian models for surface interpolating

Bilinear interpolation is an extension of linear interpolation for interpolating functions of two variables (x and y) on a regular grid.

The key idea is to perform linear interpolation first in one direction, and then again in the other direction.

In this paper adopted method 2D Lagrange interpolation with different high order basis functions of surface data generation has been implemented, tested and evaluated. The adopted method is an extension to the earlier interpolation method (1D) but by computing both (x) and (y) data to drive the (z) data as in general form of $z = F(x, y)$

The mathematical solution steps of the proposed method can be formulated as follows:

Step-1- Considering a sequence of planar points (input data set points) defined by:-

$$(x_1, y_1, z_1), (x_2, y_2, z_2), ..., (x_n, y_n, z_n)$$

where $(x_i, y_k) < (x_j, y_l)$ for $i < j, k < l$.

Step-2- The interpolating polynomial of n^{th}, m^{th} degree can be formulated as:

$$F(x,y)=\sum_{i=1}^{n}L_i(x)*\sum_{k=1}^{m}L_k(y)*F(x_i,y_k) \quad ......(8)$$

Where:

$$L_i(x)=\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{x-x_j}{x_i-x_j} \quad .... (9)$$

$$L_k(y)=\sum_{\substack{k=1\\ k\neq l}}^{m}\frac{y-y_l}{y_k-y_l} \quad .... (10)$$

Step-3-

The determination of Lagrange coefficients in both direction (x,y) can be invested to determine the interior data of the desired surface depending on a few initial data set points.

In more details the flow chart of the proposed technique illustrated in Figure (5) explain each step of the program that have been built to generate the interior data of the desired 3D sculptured surface while a suitable program was linked with MATLAB(V.7) software to represent the desired surfaces in graphical mode with different orders (3^{rd} to 8^{th}).

4. Surface interpolating and representing using a modified 2D lagrangian model

Based on the data set of initial points derived from the mathematical formulas of the original surfaces, the previous 2D Lagrange formulation have been invested to generate the interior data of the interpolated 3D surfaces and the surface can be rendered and represented, with the aid of (MATLAB V.7) soft-ware. The Figures (6, 7, 8 and 9) illustrate the representation of the selected surfaces by using proposed 2D Lagrange of 3^{rd} to 8^{th} order interpolating models

5.Statistical and visual analysis of the error caused by interpolation

a. Statistical analysis

In order to perform a statistical analysis of the surfaces rendered by the proposed 2D lagrangian interpolating approaches (3^{rd} to 8^{th} order), we compared them with the original surfaces. This was achieved by comparing the regular rectangular grids created by the 2D lagrangian method with the real grids.

The approximation error in some data is the discrepancy between an exact value and some approximation to it. An approximation error can occur because the approximations are used instead of the real data.

One commonly distinguishes between the relative error and the absolute error. The absolute error is the magnitude of the difference between the exact value and the approximation. The relative error is the absolute error divided by the magnitude of the exact value. The percent error is the relative error expressed in terms of per 100.

For each point of a regular rectangular grid, we calculate the absolute error function E_f defined as the difference between the real elevation of the function Z_f and the approximated elevation Z_{approx} at that point. The absolute error function is defined as:

$$E_f=\left|Z_f-Z_{approx}\right| \quad ......(11)$$

Where the vertical bars denote the absolute value.

If $E_f \neq 0$ the relative error is

$$E_r = \frac{|Z_f - Z_{approx}|}{|Z_f|} = \left|\frac{Z_f - Z_{approx}}{Z_f}\right| \quad \ldots\ldots\ldots (12)$$

and the percent error is

$$E_{percent} = \frac{|Z_f - Z_{approx}|}{|Z_f|} \times 100 = \left|\frac{Z_f - Z_{approx}}{Z_f}\right| \times 100 \quad . (13)$$

If there are n points representing the surface, then the average A_v and standard deviation S_d of the error function can be evaluated according to the following equations [2],[3]:

$$A_v = \sum_{i=0}^{n} (E_{f_i} / n) \quad \ldots\ldots. (14)$$

$$STD = \left(\frac{1}{n-1}\sum_{i=1}^{n}(E_{f\,i} - A_V)^2\right)^{\frac{1}{2}} \quad .. (15)$$

In this paper sine, polynomial, exponential and free form surfaces data file have been used as source of real data Z_f on each point of the grid. For several tested surfaces the value of Z_{approx} in each point of the grid, has been approximated using the 3rd to 8th

b. Visual Analysis

In this work the absolute error in each grid of the generated sculptured surfaces have been calculated as difference between the elevation of each grid of the original surface model and those created from the 3rd to 8th proposed 2D Lagrange interpolating model , then this calculated data (elevation difference) have been manipulated , rendered and represented as an error map that give good expressions about the distribution of the error among each interpolated surface. A visual analysis of the elevation error due to interpolating of the models (sine, polynomial , exponential and free form) are illustrated in the Figures 10, 11, 12 and 13, respectively.

Implementation

In addition to generation of the interior data of desired interpolated sculptured surfaces the adopted technique have the ability of:

- Tool path generation
- CNC program generation to machine the desired surfaces (finishing)

The techniques have been implemented for the design of several different sculptured surfaces to illustrates the system flexibility, the design result have been implemented for manufacturing two of these surfaces using CNC vertical machine .

The simplest approach used in finish machining is iso-parametric tool path generation. Surface points are calculated as a function of (***u***, ***w***) parameter space, the tool is then indexed along the surface by incrementing (***u***) and (***w***). Tool path planning is accomplished by holding the ***w*** parameter constant and indexing the ***u*** parameter, hence the term iso-parametric machining [10].

Step-forward increments in ***u*** must be carefully chosen since tool movements are linearly interpolated and the chordal deviation between the straight lines and the actual surface must be less than the desired tolerance. Step-over increments in ***w*** must be small enough to keep the cusps between the spherically shaped cutter paths to less than the desired tolerance. The Figures (14 and 15) illustrates the tool path and the machined model of two selected sculptured surfaces.

Results and Conclusions

After the elevation comparison have been made between the interpolated surfaces 3^{rd} – 8^{th} order with the original models, the elevation error in each grid of the interpolated surfaces are calculated then the result are analyzed according to the equations (11 to 15) as maximum error, absolute error, relative error, percent error average error and standard deviation as illustrated in the tables 1, 2, 3 and 4. Each table lists coefficients of error for 2500 (50 x 50 points) counties of 6 maps (3^{rd} to 8^{th} order) examined for each interpolated sculptured surface in the analysis.

Conclusions

In this work a Lagrange's interpolation polynomial has been derived. Specifically, the proposed algorithm showed how to interpolate multinomial function of degree *n* given distinct points; also the algorithm provided numerical examples to illustrate the use of the derived expression.

1. This work presents a general framework for a higher-order 2D Lagrange method and apply this to smooth surface constructions. Starting from a third order, the general level set formulation has been derived , and provide an efficient solution of a high order (8^{th} order).
2. High order interpolation scheme have been presented in this paper to generate sculptured surfaces characterizing the shape of given data mesh with low error. The presented interpolating scheme give good continuity on the resulting surfaces and it's visible on test models, the overall surface shape is very good. Further, the interpolating scheme is not only cubic and it will offer great advantage for easy implementation and stability
3. A comparison between the calculated data from 2D Lagrange interpolation model and that obtained from original mathematical model, are in good agreement with each other and the coincided between generated sculptured surface and the original one reach's up to 99.99% for some model.
4. By applying the proposed surface interpolating models the percent error was found to be ranged between 0.00001% for some model to 3.5% for some other interpolated models with Lagrange method.
5. The visual representation of the error due to the interpolation scheme give a

good impression and clear idea about the distribution of the error function among the desired surface

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Table1. The error statistical analysis of the 1st interpolated surface

Fitting order	Data set points (rectangular grid)	Maximum deviation	Average A_v	Standard deviation STD	Relative error	Percent error
$z = \sin(4*\sqrt{x^2+y^2})/4*\sqrt{x^2+y^2})$						
3rd	2500	0.49306	-0.07143	0.056628	3.5017	3.50%
4th	2500	0.64062	0.087343	0.081979	2.0148	2.01%
5th	2500	0.29289	0.087343	0.019644	1.7908	1.79%
6th	2500	0.48576	0.036009	0.062961	2.2835	2.28%
7th	2500	0.272	0.013266	0.037113	1.3765	1.37%
8th	2500	0.27038	-0.02523	0.030362	0.93868	0.94%

Table2. The error analysis of the 2nd interpolated surface

Fitting order	Data set points (rectangular grid)	Maximum deviation	Average A_v	Standard deviation STD	Relative error	Percent error
$z = 0.015*((x-2)^4*(x+1)^3*(x-1)+0.5*(8*y^4-14*y^3-9*y^2+11*y-1))$						
3rd	**2500**	**0.30119**	**-0.0442**	**3.8917×10^{-7}**	**20.547**	**20.5%**
4th	**2500**	**0.16749**	**0.01325**	**2.3128×10^{-7}**	**5.1064**	**5.1%**
5th	**2500**	**0.055397**	**0.002218**	**2.563×10^{-7}**	**1.5256**	**1.5%**
6th	**2500**	**0.093046**	**-0.01142**	**8.2049×10^{-7}**	**6.5526**	**6.5%**
7th	**2500**	**0.051172**	**-0.00695**	**6.2738×10^{-7}**	**3.5023**	**3.5%**
8th	**2500**	**4.8094×10^{-5}**	**3.78×10^{-7}**	**3.3207×10^{-6}**	**0.000373**	**0.0004%**

Table3. The error analysis of the 3rd interpolated surface

Fitting order	Data set points (rectangular grid)	Maximum deviation	Average A_v	Standard deviation STD	Relative error	Percent error
$z = x * e^{(-x^2-y^2)}$						
3rd	2500	0.21586	$2.585\text{x}10^{-18}$	0.03548	2.4378	2.44%
4th	2500	0.17568	$-2.83\text{x}10^{-18}$	0.034175	1.3313	1.33%
5th	2500	0.17808	$-1.655\text{x}10^{-18}$	0.023946	0.77937	0.78%
6th	2500	0.10818	$3.727\text{x}10^{-18}$	0.014144	0.54755	0.55%
7th	2500	0.085166	$1.801\text{x}10^{-18}$	0.0094618	0.42252	0.42%
8th	2500	0.053374	$-5.481\text{x}10^{-19}$	0.0057927	0.20329	0.2%

Table4. The error analysis of the 4th interpolated surface

Fitting order	Data set points (rectangular grid)	Maximum deviation	Average A_v	Standard deviation STD	Relative error	Percent error
Free form Surface						
3rd	2500	0.015568	0.029627	1.0813	0.059903	0.06%
4th	2500	3.7612	$2.6\text{x}10^{-5}$	0.34816	0.030681	0.03%
5th	2500	0.00049975	$1.26\text{x}10^{-5}$	0.00045	$1.01\text{x}10^{-5}$	0.00001%
6th	2500	0.0037928	$10\text{x}10^{-5}$	0.0002658	$2.29\text{x}10^{-5}$	0.00002%
7th	2500	0.0034941	$3.15\text{x}10^{-5}$	0.00034859	$2.06\text{x}10^{-5}$	0.00002%
8th	2500	0.015568	$-22\text{x}10^{-5}$	0.0010141	$3.39\text{x}10^{-5}$	0.00003%

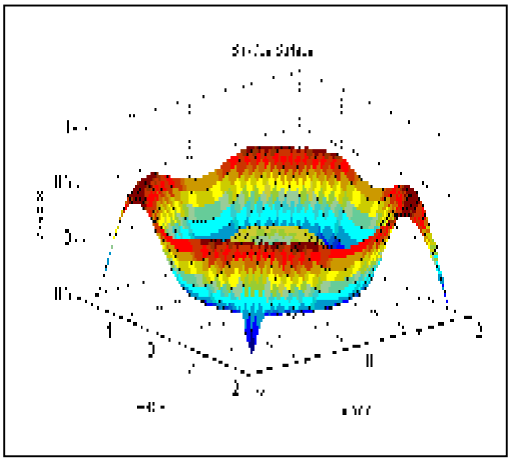

Figure (1) perspective view of the 1st surface.

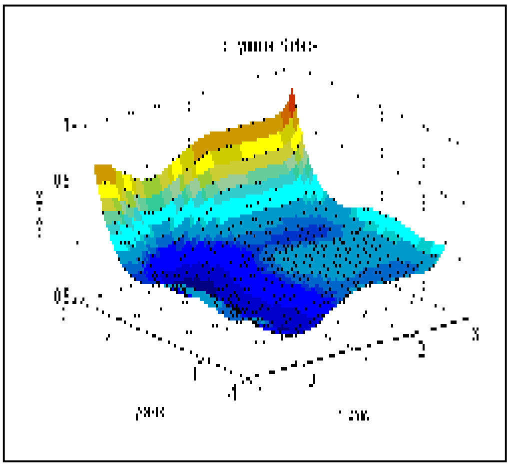

Figure (2) perspective view of the 2nd surface

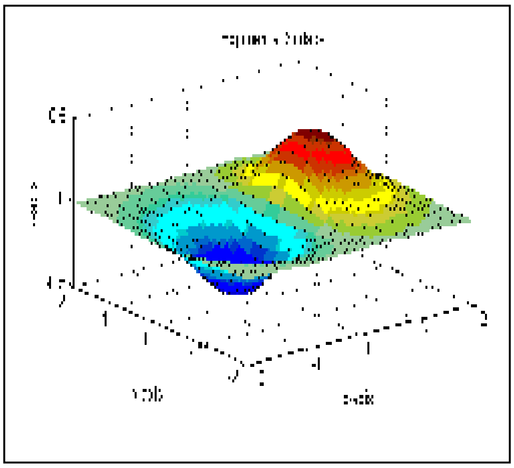

Figure (3) perspective view of the 3rd surface.

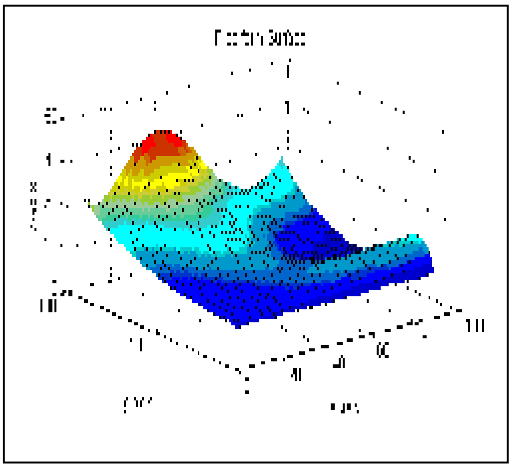

Figure (4) perspective view of the 4th surface

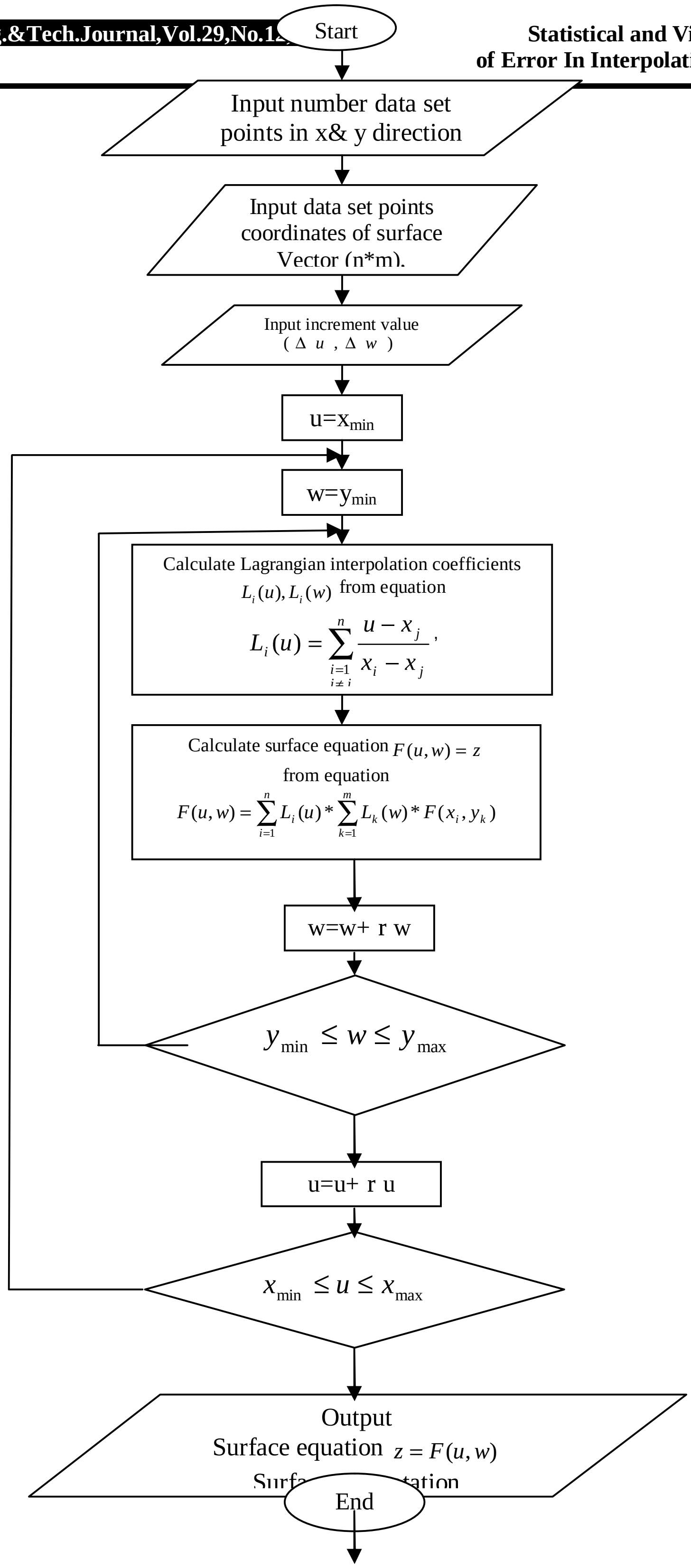

Figure (5) Flowchart of the proposed program depending on Lagrangian technique

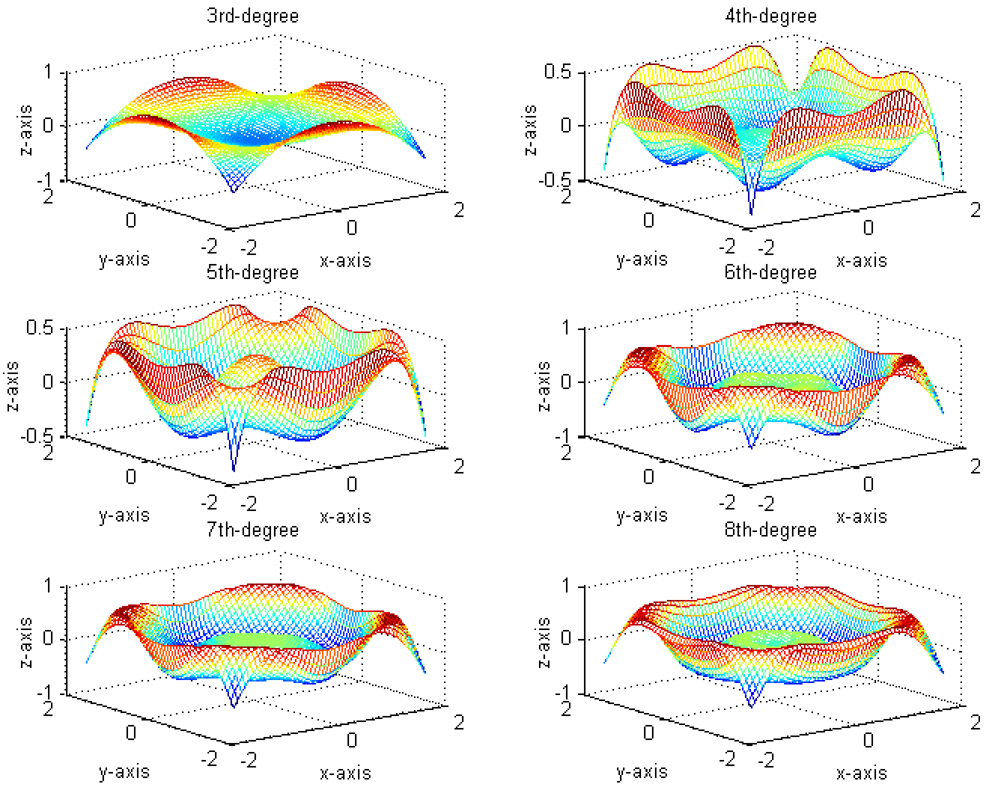

Figure (6) 1st sculptured surface interpolated with different orders (3rd to 8th)

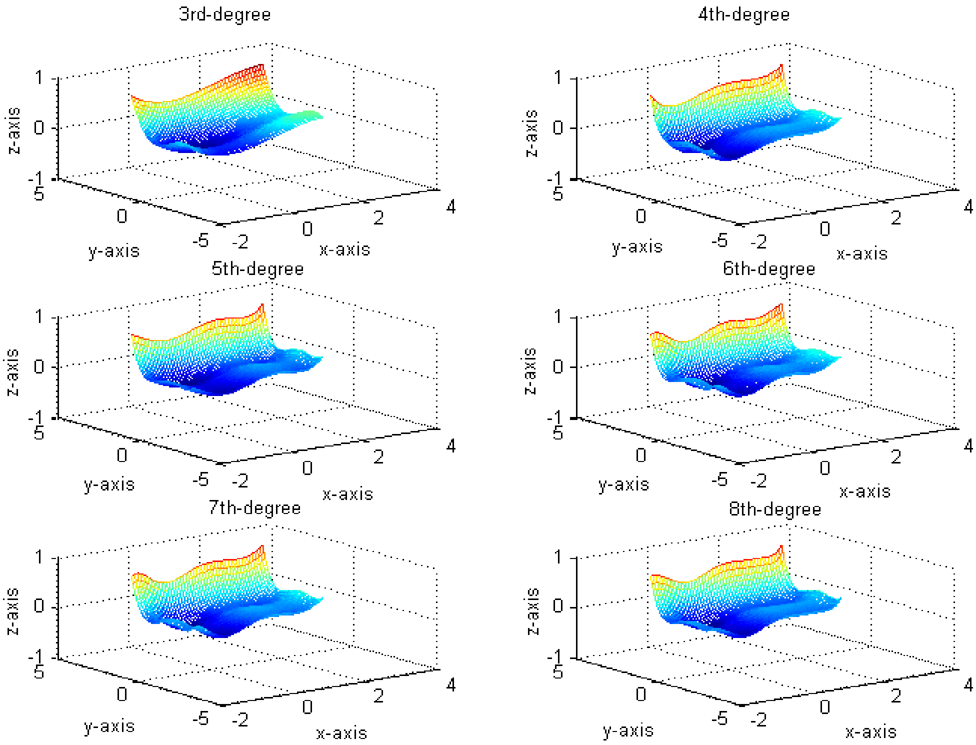

Figure (7) 2^{nd} sculptured surface interpolated with different orders (3^{rd} to 8^{th})

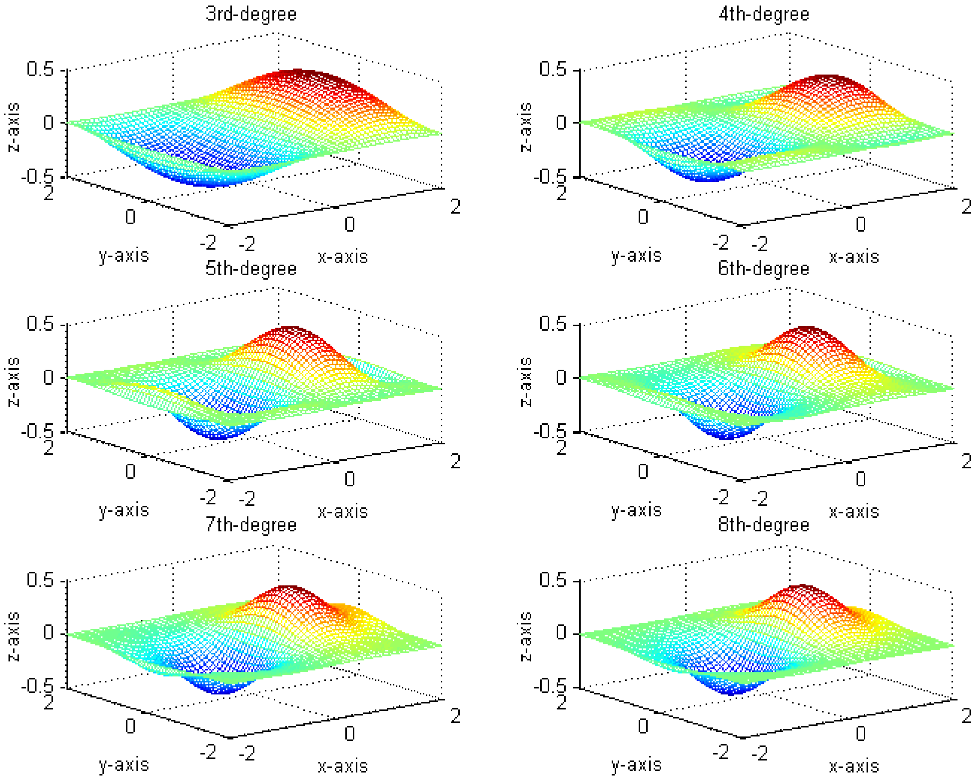

Figure (8) 3rd sculptured surface interpolated with different orders (3rd to 8th)

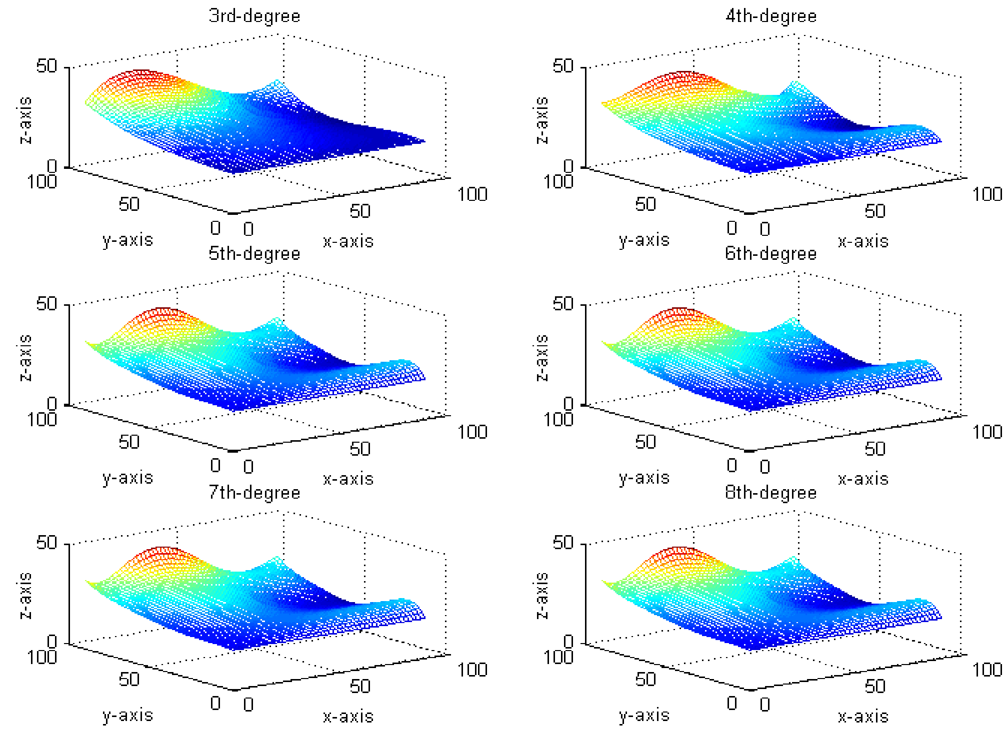

Figure (9) 4th sculptured surface interpolated with different orders (3rd to 8th)

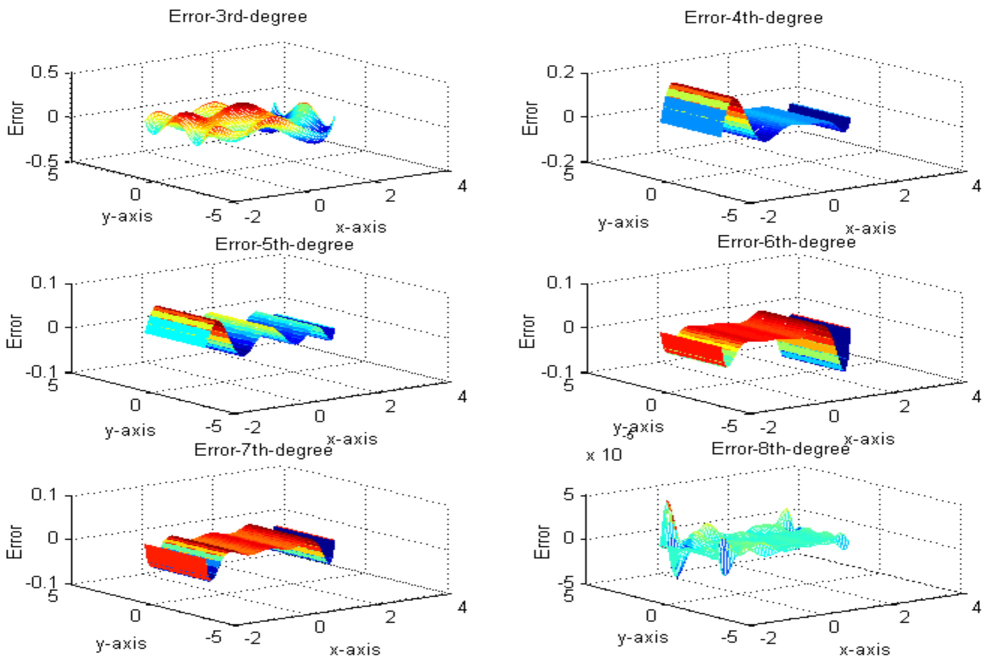

Figure (10) Error map of the 1st sculptured surface interpolated 3rd order to 8th order

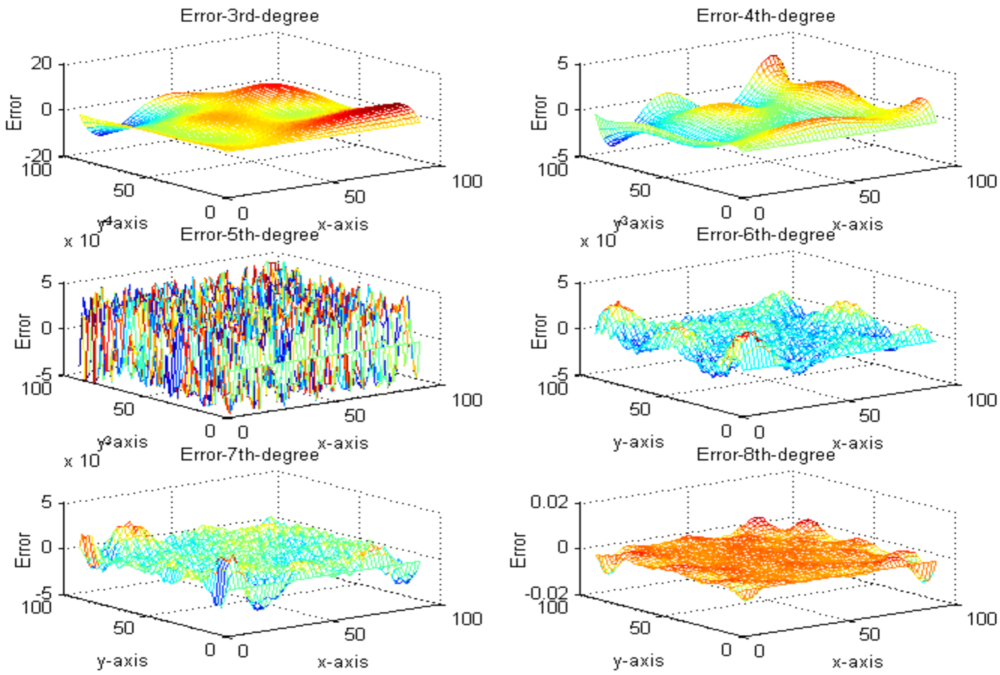

Figure (11) Error map of the 2nd sculptured surface interpolated 3rd order to 8th order

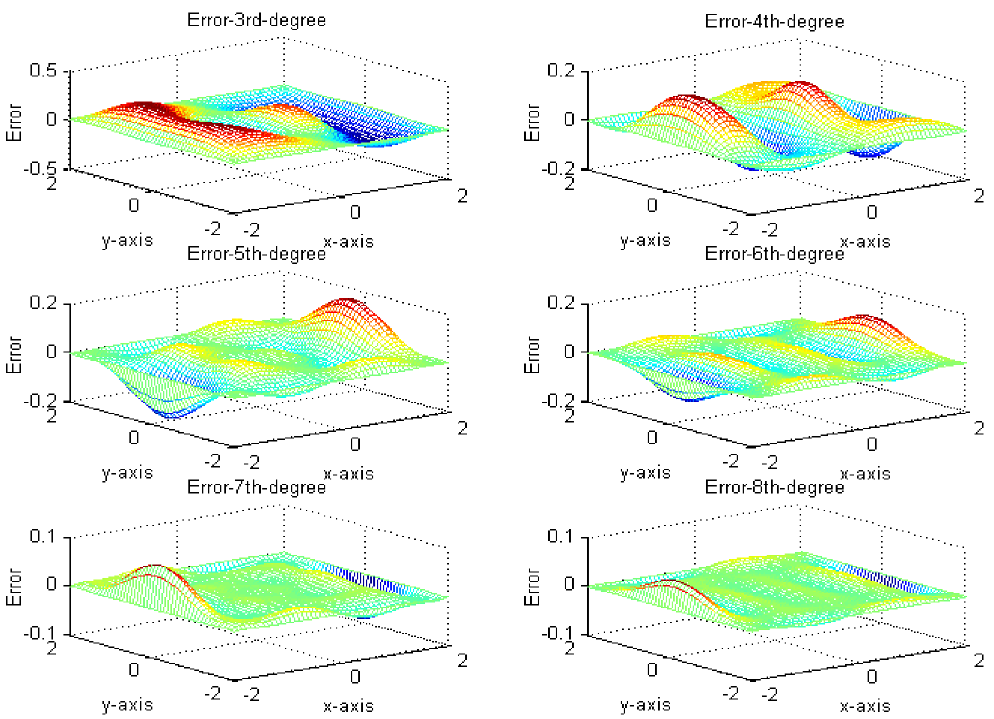

Figure (12) Error map of the 3rd sculptured surface interpolated 3rd order to 8th order

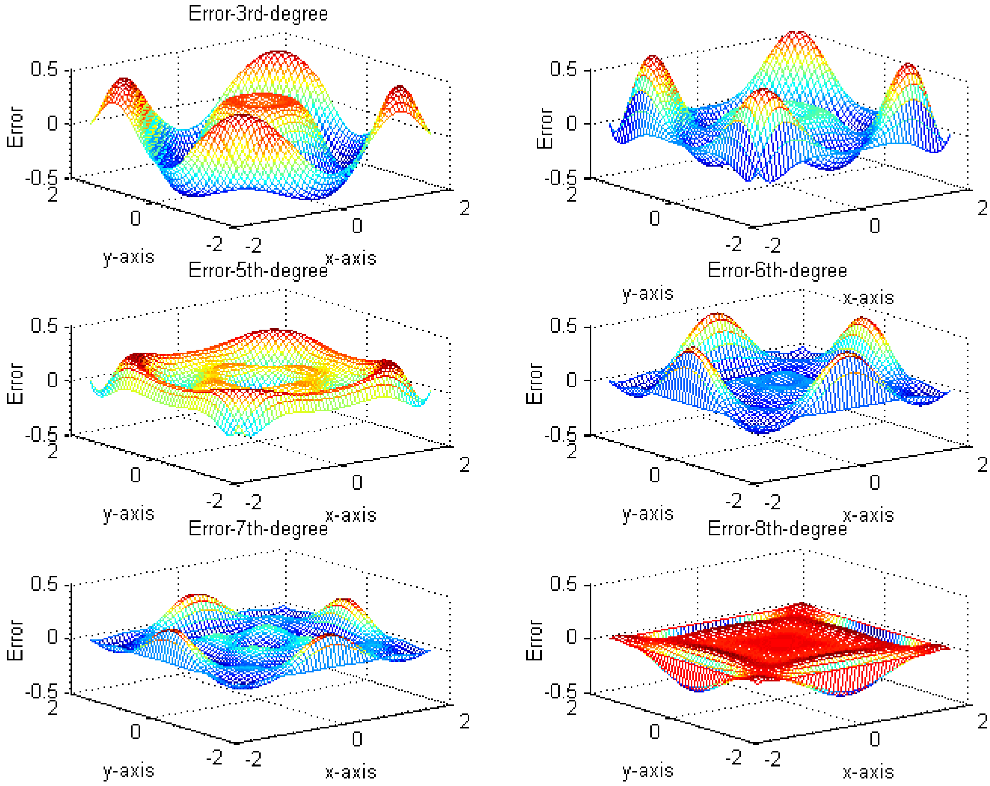

Figure (13) Error map of the 4th sculptured surface interpolated 3rd order to 8th order

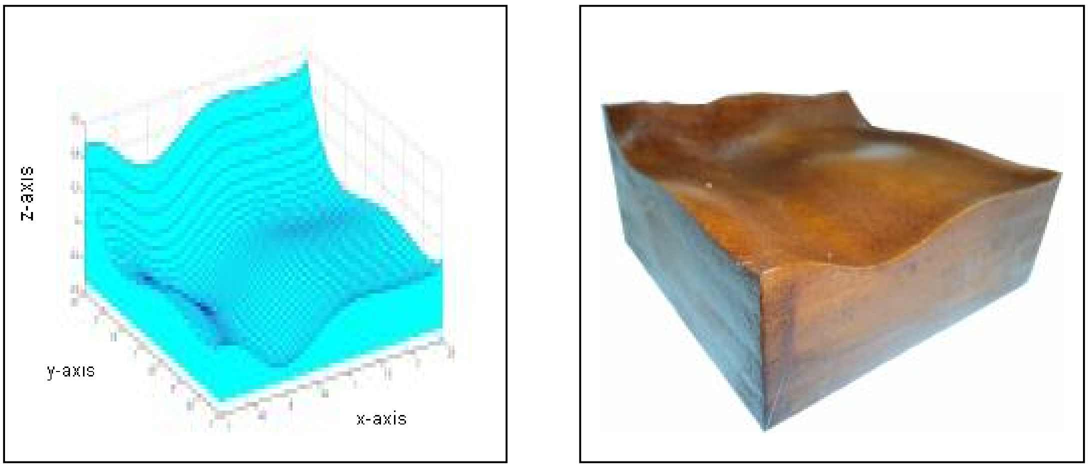

Figure (14) a- Tool path of the 2nd Sculptured Surface. b- Machined Surface

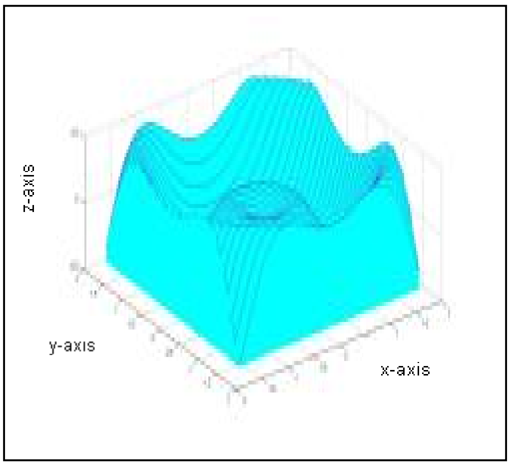

Figure (15) a- Tool path of the 3rd Sculptured Surface. b- Machined Surface