Solution of Seventh Order Boundary Value Problem by Differential Transformation Method

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Abstract:

In this paper, we will study the numerical solution of seventh order fuzzy boundary value problems using Differential transformation method .The approximate solution of the problem is calculated in the form of a rapid convergent series.. Also, comparison between the obtained results is made, as well as with the crisp solution, when the□-level equals one.

Keywords: Differential transformation method , seventh order fuzzy boundary value problems , linear and nonlinear problems , series solution

1.INTRODUCTION

Fuzzy set theory is a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models, which has been applied to a wide variety of real problems, for instance, the golden mean [1], practical systems [2], quantum optics and gravity [3], medicine [4] and engineering problems.

The concept of fuzzy sets which was originally introduced by Zadeh [5] led to the definition of the fuzzy number and its implementation in fuzzy control [6] and approximate reasoning problems [7], [8]. The basic arithmetic structure for fuzzy numbers was later developed by Mizumoto and Tanaka [9], [10], Nahmias [11], Dubois and Prade [12], [13] and Ralescu [14], all of which observed the fuzzy number as a collection of

α-levels, $0 \leq \alpha \leq 1$, [15].

In this paper, the differential transformation method is applied to solve the seventh order fuzzy boundary value problems. The given problem can be transformed into a recurrence relation, using differential transformation operations, which leads to a series solution. Consider the following seventh order fuzzy boundary value problem

$$\left.\begin{array}{l} y^{(7)}(x) = f\big(x, y(x)\big), a \leq x \leq b \\ y^{(i)}(a) = \tilde{A}_i \ , \ i = 0,1,2,3 \\ y^{(j)}(b) = \tilde{B}_j \ , \ j = 0,1,2 \end{array}\right\} \qquad \ldots\ldots\ldots\ldots\ldots (1.1)$$

where $\tilde{A}_i$, i = 0,1,2,3 and $\tilde{B}_j$, j = 0,1,2 are finite fuzzy constants, also $f\big(x, y(x)\big)$ is a continuous function on [a, b].

Zhou [16] introduced the differential transformation method to solve linear and nonlinear initial value problems in electric circuit analysis. Differential transformation method has been used to solve integral and integro-differential systems, differential algebraic equation, ordinary differential equation, system of ordinary differential equations, Lane-Emden type equations arising in astrophysics, Helmholtz equation [17]. Differential transformation method has been also applied to study the free vibration analysis of rotating ronprismatic beams , unsteady rolling motion of spheres equation in inclined tubes [18]. This method constructs approximate solution in the form of a polynomial and finally gives a series solution. It is different from the traditional high order Taylor's series, as, it requires a long time in calculation and needs computation of the necessary derivatives of the data functions.

2. Fuzzy Sets:

In this section, we shall present some basic definitions of fuzzy sets including the definition of fuzzy numbers and fuzzy functions.

Definition (1), [5]:

Let X be any set of elements. A fuzzy set $\tilde{A}$ is characterized by a membership

function $\mu_{\tilde{A}}$: X→ [0, 1], and may be written as the set of points

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1\}$$

Definition (2), [19]:

The crisp set of elements that belong to the set $\tilde{A}$ at least to the degree α is called the weak α -level set (or weak α -cut), and is defined by:

$$A_\alpha = \{ x \in X: \mu_{\tilde{A}}(x) \ge \alpha\}$$

while the strong α -level set (or strong α -cut) is defined by:

$$A'_\alpha = \{ x \in X: \mu_{\tilde{A}}(x) > \alpha\}$$

Definition (3)[19]:

A fuzzy number $\tilde{M}$ Is a convex normalized fuzzy set $\tilde{M}$ Of the real line R, such that:

1. There exists exactly one x_0 R, with $\mu_{\tilde{M}}(x_0) = 1$ (x_0 Is called the mean value of $\tilde{M}$).

2. $\mu_{\tilde{M}}(x)$ Is piecewise continuous

Now, in applications, the representation a fuzzy number in terms of its membership function is so difficult to use, therefore two approaches are given for representing the fuzzy number in terms of its □-level sets, as in the following remark:

Remark (1):

A fuzzy number $\widetilde{M}$may be uniquely represented in terms of its □-level sets, as the following closed intervals of the real line:

$M_\alpha = [m - \sqrt{1-\alpha}\, , m + \sqrt{1-\alpha}]$ or$M_\alpha = [\alpha m, \frac{1}{\alpha} m]$

Where m is the mean value of $\widetilde{M}$and □□[0, 1]. This fuzzy number may be written as

$M_\alpha = [\underline{\widetilde{M}}\, , \overline{\widetilde{M}}\,]$ where $\underline{\widetilde{M}}$ refers to the greatest lower bound of $M_\alpha$and $\overline{\widetilde{M}}$to the least upper bound of M_α.

Remark (2):

Similar to the second approach given in remark (1), one can fuzzyfy any crisp or nonfuzzy function f, by letting:

$$\underline{f}(x) = \alpha f(x), \quad \overline{f}(x) = \frac{1}{\alpha} f(x) \quad , x \in X, \text{□□}(0, 1]$$

and hence the fuzzy function $\tilde{f}$in terms of its□-levels is given by $f_\alpha = [\underline{f}, \overline{f}]$.

3-Differetial Transformation Method [20]

The differential transformation of the kth derivative of a function f(x) is defined by

$$F(K) = \frac{1}{K!}\left[\frac{d^K f(x)}{dx^K}\right]_{x=x_0} \quad \ldots \ldots \ldots \quad (3.1)$$

and the inverse differential transformation of $F(K)$ is defined by

$$f(x) = \sum_{K=0}^{\infty} F(K)\,(x - x_0)^k \quad \ldots \ldots \ldots \quad (3.2)$$

In real applications, the function $f(x)$ can be expressed as a finite series and Eq. (3.2) can be written as:

$$f(x) = \sum_{K=0}^{n} F(K)\,(x - x_0)^k \quad \ldots \ldots \ldots \quad (3.3)$$

Substituting Eq. (3.1) into Eq. (3.2), gives

$$f(x) = \sum_{K=0}^{\infty} (x - x_0)^k \,\frac{1}{K!}\left[\frac{d^K f(x)}{dx^K}\right]_{x=x_0} \quad \ldots \ldots \ldots \quad (3.4)$$

which is the Taylor's series for $f(x)$ at $x = x_0$. From Eq. (3.1) and Eq. (3.2) following theorems can be deduced

Theorem 1:

If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$, where F, G and H are the differential

transforms of f, g *and* h*, respectively.*

Theorem 2:

If $f(x) = cg(x)$ then $F(k) = cG(k)$, where c is aconstant

Theorem 3:

If $f(x) = \frac{d^m g(x)}{dx^m}$, then $F(k) = \frac{(k+m)!}{k!} G(k+m)$

Theorem 4:

If $f(x) = g(x)h(x)$, then $F(k) = \sum_{k_1=0}^{k} G(k_1)H(k-k_1)$, where F, G and H are the differential *transforms of* f, g *and* h, *respectively.*

Theorem 5:

If $f(x) = e^{\lambda x}$, then $F(k) = \frac{\lambda^k}{k!}$

Theorem 6:

if $f(x) = x^n$, then $\delta(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$

Theorem 7:

If $f(x) = g_1(x)\, g_2(x) \ldots . g_n(x)$ then

$$F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \ldots . \sum_{k_1}^{k_2} G_1(k_1) G_2(k_2-k_1) \ldots . G_n(k-k_{n-1})$$

To implement the method, two numerical examples are considered in the following section.

4- Numerical Examples

Example(4.1):

Consider the linear seventh order boundary value problem

$$y^{(7)}(x) = xy(x) + e^x(x^2 - 2x - 6), \ \ 0 \le x \le 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
(4.1)

Subject to the fuzzy boundary conditions

$$\left.\begin{array}{l} y(0) = \tilde{1}, \quad y^{(1)}(0) = \tilde{0}\ , \quad y^{(2)}(0) = -\tilde{1}, \quad y^{(3)}(0) = -\tilde{2} \\ y(1) = \tilde{0}, \quad y^{(1)}(1) = -\tilde{e}\ , \quad y^{(2)}(1) = -2\tilde{e} \end{array}\right\} \ldots\ldots\ldots\ldots (4.2)$$

The exact solution of the problem (4.1) is $y(x) = (1 - x)e^x$.

approximate solution $\tilde{y}$With □-levels:$y_x(x)$ □[$\underline{y}$,$\overline{y}$], □□(0, 1], Hence, to find the solution in the lower case of solution $\underline{y}$, consider the problem:

$$\underline{y}^{(7)}(x) = x\,\underline{y}(x) + e^x(x^2 - 2x - 6), \ \ 0 \le x \le 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
(4.3)

With lower bound of boundary conditions:

$\underline{y}(0)$ □$\tilde{1}\sqrt{1-\alpha}$, $\underline{y}^{(1)}$ (0) □$\tilde{0}\sqrt{1-\alpha}$, $\underline{y}^{(2)}$ (0) □-$\tilde{1}\sqrt{1-\alpha}$

$\underline{y}^{(3)}(0)$ □-$\tilde{2}\sqrt{1-\alpha}$, $\underline{y}(1)$ □$\tilde{0}\sqrt{1-\alpha}$, $\underline{y}^{(1)}$ (1) □-$\tilde{e}\sqrt{1-\alpha}$

$\underline{y}^{(2)}$ (1) □-2e $\sqrt{1-\alpha}$ ……………………………………………………………….(4.4)

Applying the above theorems, the differential transformation of Eq. (4. 3) is obtained as

$$\underline{Y}(k+7) = \frac{k!}{(k+7)!}\left[\sum_{k_1=0}^{k} \delta(k_1-1)\,\underline{Y}(k-k_1) + \sum_{k_1=0}^{k} \frac{\delta(k_1-2)}{(k-k_1)!} - 2\sum_{k_1=0}^{k} \frac{\delta(k_1-1)}{(k-k_1)!} - \frac{6}{k!}\right]$$

(4.5)

Using Eq. (3.1), the boundary conditions (4.4) can be transformed at $x_0 = 0$ as:

$$\left.\begin{array}{c} \underline{Y}(0) = 1-\sqrt{1-\alpha} \quad ,\underline{Y}(1) = 0-\sqrt{1-\alpha}, \quad \underline{Y}(2)\dfrac{-1-\sqrt{1-\alpha}}{2!} \\ \underline{Y}(3) = \dfrac{-2-\sqrt{1-\alpha}}{3!} \;, \sum_{k=0}^{n} \underline{Y}(k) = 0-\sqrt{1-\alpha}, \sum_{k=0}^{n} k\underline{Y}((k) = -e-\sqrt{1-\alpha} \\ \sum_{k=0}^{n} k(k-1)\underline{Y}((k) = -2e-\sqrt{1-\alpha} \end{array}\right\} \quad \ldots..(4.6)$$

where n is a sufficiently large integer. Using the recurrence relation (4.5) and the transformed boundary conditions(4.6), the following series solution up to $O(x^{14})$ can be obtained as:

$$\underline{y}(x) = 1-\sqrt{1-\alpha} + \frac{-1-\sqrt{1-\alpha}}{2}x^2 + \frac{-2-\sqrt{1-\alpha}}{6}x^3 + Ax^4 + Bx^5 + Cx^6 - \frac{x^7}{840} - \frac{x^8}{5760} - \frac{x^9}{45360} - \frac{x^{10}}{403200} - \frac{x^{11}}{3991680} + \frac{(\frac{1}{30}+A)\,x^{12}}{3991680} + \frac{(\frac{1}{60}+B)\,x^{13}}{8648640} + O(x^{14}) \ldots\ldots\ldots$$

(4.7)

where the constants A, B and C can be determined, using Eq. (3.1), as:

$$A = \frac{\underline{y}^{(4)}(0)}{4!} = \underline{Y}(4)\,, B = \frac{\underline{y}^{(5)}(0)}{5!} = \underline{Y}(5)\,, C = \frac{\underline{y}^{(6)}(0)}{6!} = \underline{Y}(6) \ldots\ldots\ldots$$

(4.8)

Taking n = 13, an algebraic system of linear equations in terms of A, B and C is obtained. The solution of the Eq. (4.8) yields

A =−0.1250000058927127, B = −0.033333320250896525,C =−0.006944451794759032.(4.9)

Finally, the series solution can be written as:

$$\underline{y}(x) = 1-\sqrt{1-\alpha}+\frac{-1-\sqrt{1-\alpha}}{2}x^2+\frac{-2-\sqrt{1-\alpha}}{6}x^3-0.125x^4-0.033333x^5-0.00694445x^6-\frac{x^7}{840}-\frac{x^8}{5760}-\frac{x^9}{45360}-\frac{x^{10}}{403200}-\frac{x^{11}}{3991680}+(2.29644\times10^{-8})x^{12}+(1.92708\times10^{-9})x^{13}+O(x^{14})\ldots\ldots\ldots \quad (4.10)$$

Similarly, for the upper solution $\overline{y}$, consider the problem:

$$\overline{y}^{(7)}(x) = x\,\overline{y}(x)+e^x(x^2-2x-6),\ \ 0\le x\le 1 \ \ldots\ldots\ldots \quad (4.11)$$

With upper bound of boundary conditions:

$\overline{y}(0)$ □1$\sqrt{1-\alpha}$, $\overline{y}^{(1)}$ (0) □0$\sqrt{1-\alpha}$, $\overline{y}^{(2)}$ (0) □-1$\sqrt{1-\alpha}$

$\overline{y}^{(3)}$(0) □-2$\sqrt{1-\alpha}$, $\overline{y}$(1) □0$\sqrt{1-\alpha}$, $\overline{y}^{(1)}$ (1) □$-e+\sqrt{1-\alpha}$

$\overline{y}^{(2)}$ (1) □ $-2e+\sqrt{1-\alpha}$ ……………………………………… (4.12)

Applying the above theorems, the differential transformation of Eq. (4. 3) is obtained as

$$\bar{Y}(k+7) = \frac{k!}{(k+7)!}\left[\sum_{k_1=0}^{k} \delta(k_1-1)\bar{Y}(k-k_1) + \sum_{k_1=0}^{k} \frac{\delta(k_1-2)}{(k-k_1)!} - 2\sum_{k_1=0}^{k} \frac{\delta(k_1-1)}{(k-k_1)!} - \frac{6}{k!}\right]$$

(4.13)

Using Eq. (3.1), the boundary conditions (4.12) can be transformed at $x_0 = 0$ as:

$$\left.\begin{array}{l} \bar{Y}(0) = 1\sqrt{1-\alpha}, \bar{Y}(1) = 0\sqrt{1-\alpha}, \quad \bar{Y}(2) = \frac{-1+\sqrt{1-\alpha}}{2!}, \quad \bar{Y}(3) = \frac{-2+\sqrt{1-\alpha}}{3!} \quad \ldots (4.14) \\ \sum_{k=0}^{n} \bar{Y}(k) = 0+\sqrt{1-\alpha}, \quad \sum_{k=0}^{n} k\bar{Y}(k) = -e+\sqrt{1-\alpha}, \\ \sum_{k=0}^{n} k(k-1)\bar{Y}(k) = -2e+\sqrt{1-\alpha} \end{array}\right\}$$

where *n* is a sufficiently large integer. Using the recurrence relation (4.5) and the transformed boundary conditions(4.6), the following series solution up to $O(x^{14})$ can be obtained as:

$$\bar{y}(x) = 1+\sqrt{1-\alpha}+\frac{-1+\sqrt{1-\alpha}}{2}x^2+\frac{-2+\sqrt{1-\alpha}}{6}x^3+Ax^4+Bx^5+Cx^6-\frac{x^7}{840}-\frac{x^8}{5760}-\frac{x^9}{45360}-\frac{x^{10}}{403200}-\frac{x^{11}}{3991680}+\frac{(\frac{1}{30}+A)\,x^{12}}{3991680}+\frac{(\frac{1}{60}+B)\,x^{13}}{8648640}+O(x^{14})\ldots$$

.(4.15)

where the constants A, B and C can be determined, using Eq. (3.1), as:

$$A = \frac{\overline{y}^{(4)}(0)}{4!} = \overline{Y}(4)\,, B = \frac{\overline{y}^{(5)}(0)}{5!} = \overline{Y}(5)\,, C = \frac{\overline{y}^{(6)}(0)}{6!} = \overline{Y}(6) \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.16)$$

Taking n = 13, an algebraic system of linear equations in terms of A, B and C is obtained. The solution of the Eq.(4.16) yields

A = − 0.1250000058927127, B = − 0.033333320250896525,C = − 0.006944451794759032

Finally, the series solution can be written as:

$$\overline{y}(x) = 1 + \sqrt{1-\alpha} + \frac{-1+\sqrt{1-\alpha}}{2}x^2 + \frac{-2+\sqrt{1-\alpha}}{6}x^3 - 0.125x^4 - 0.033333x^5 - 0.00694445x^6 - \frac{x^7}{840} - \frac{x^8}{5760} - \frac{x^9}{45360} - \frac{x^{10}}{403200} - \frac{x^{11}}{3991680} + (2.29644 \times 10^{-8})x^{12} + (1.92708 \times 10^{-9})x^{13} + O(x^{14}) \ldots\ldots\ldots \quad (4.17)$$

Combining $\underline{y}$ and $\overline{y}$ yields the fuzzy solution of the fuzzy boundary value problem (4.1) as $y_\alpha(x) \in [\underline{y}(x), \overline{y}(x)]$, $\alpha \in (0, 1]$, x $\in$ [0, 1]. In addition, it is clear that for $\alpha = 1$, we get $\underline{y}(x) = \overline{y}(x)$, which is the same as the exact solution of the related nonfuzzy boundary value problem .The results of the calculations are given in table(4.1):

Table(4.1): the results of example (4.1)

x	$\underline{y}$	$\overline{y}$	exact solution
0	1	1	1
0.1	0.9875	0.9875	0.9946
0.2	0.9771	0.9771	0.9771
0.3	0.9449	0.9449	0.9449
0.4	0.8950	0.8950	0.8950
0.5	0.8243	0.8243	0.8243
0.6	0.7288	0.7288	0.7288
0.7	0.6041	0.6041	0.6041
0.8	0.4451	0.4451	0.4451
0.9	0.2459	0.2459	0.2459
1	0	0	0

Also, the fuzzy solution $y$in terms of the lower bound of solution $\underline{y}$and upper bound of solution $\overline{y}$and for different□-levels (where α□(0, 1]) are presented in Fig.(4.1):

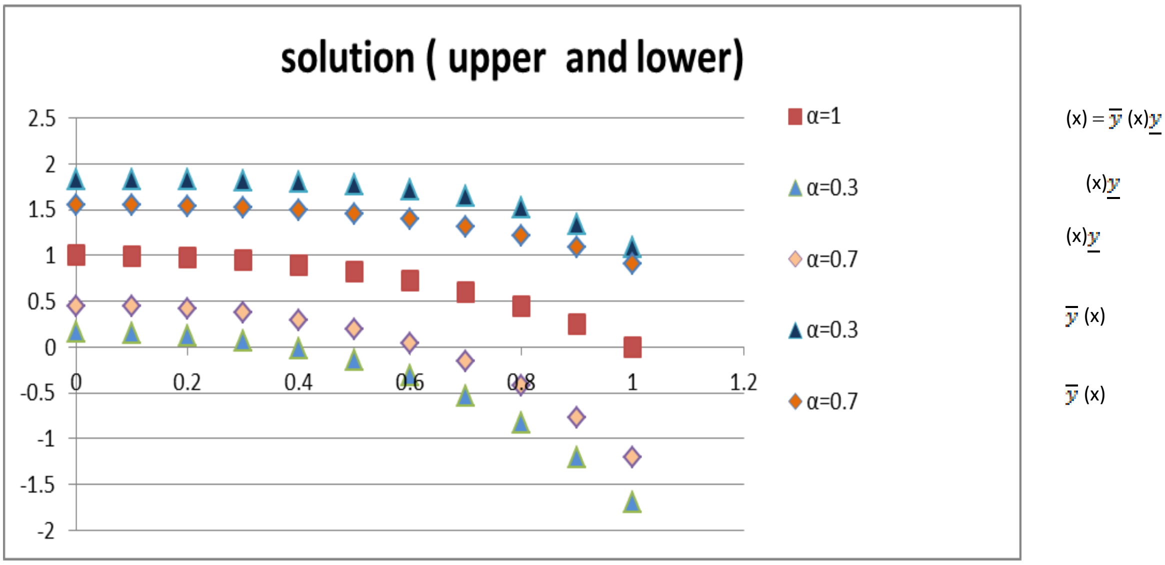


Fig.(4.1) Upper and lower solutions of example (4.1) for different values of α .

Example 4.2

Consider the following seventh order nonlinear boundary value problem

$$y^{(7)}(x) = -e^{x} y^{2}(x), \quad 0 \leq x \leq 1 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (4.18)$$

subject to the fuzzy boundary conditions

$$\left.\begin{array}{l} y(0) = \tilde{1}, \quad y^{(1)}(0) = -\tilde{1}, \quad y^{(2)}(0) = \tilde{1}, \quad y^{(3)}(0) = -\tilde{1} \\ y(1) = \tilde{e}^{-1}, \quad y^{(1)}(1) = -\tilde{e}^{-1}, \quad y^{(2)}(1) = \tilde{e}^{-1} \end{array}\right\} \dots\dots\dots\dots (4.19)$$

The exact solution of the problem (4.18) is $y(x) = e^{-x}$.

approximate solution y With □-levels: $y_x(x)$ □[$\underline{y}$, $\overline{y}$], □□(0, 1], Hence, to find the solution in the lower case of solution $\underline{y}$, consider the problem:

$$\underline{y}^{(7)}(x) = -e^{x}\underline{y}^{2}(x), \qquad 0 \le x \le 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.20)$$

With lower bound of boundary conditions:

$\underline{y}(0)$ □ĩ$\sqrt{1-\alpha}$, $\quad \underline{y}^{(1)}(0)$ □-ĩ$\sqrt{1-\alpha}$, $\quad \underline{y}^{(2)}(0)$ □ĩ$\sqrt{1-\alpha}$, $\quad \underline{y}^{(3)}(0)$ □-ĩ$\sqrt{1-\alpha}$, $\underline{y}(1)$ □$e^{-1}\sqrt{1-\alpha}$, $\underline{y}^{(1)}(1)$ □-$e^{-1}\sqrt{1-\alpha}$

$\underline{y}^{(2)}(1)$ □e^{-1} $\sqrt{1-\alpha}$ ……………………………………………………………(4.21)

Applying the above theorems, the differential transformation of Eq. (4. 20) is obtained as

$$\underline{Y}(k+7) = \frac{k!}{(k+7)!}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{\underline{Y}(k-k_2)\underline{Y}(k_2-k_1)}{k!} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.22)$$

Using Eq. (3.1) the boundary conditions (4.21) can be transformed at $x_0 = 0$ as:

$$\left.\begin{array}{l} \underline{Y}(0) = 1-\sqrt{1-\alpha}, \quad \underline{Y}(1) = -1-\sqrt{1-\alpha}\,, \quad \underline{Y}\,(2) = \dfrac{1-\sqrt{1-\alpha}}{2!}, \\ \underline{Y}(3) = \dfrac{-1-\sqrt{1-\alpha}}{3!} \quad , \displaystyle\sum_{k=0}^{n}\underline{Y}(k) = e^{-1}-\sqrt{1-\alpha}, \\ \displaystyle\sum_{k=0}^{n} k\underline{Y}((k) = -e^{-1}-\sqrt{1-\alpha}\,, \sum_{k=0}^{n} k(k-1)\underline{Y}((k) = e^{-1}-\sqrt{1-\alpha} \end{array}\right\} \ldots (4.23)$$

where n is a sufficiently large integer. Using the recurrence relation (4.22) and the transformed boundary conditions(4.23), the following series solution up to $O(x^{14})$ can be obtained as:

$$\underline{y}(x) = 1-\sqrt{1-\alpha}+\left(-1-\sqrt{1-\alpha}\right)x+\frac{1-\sqrt{1-\alpha}}{2}x^2+\frac{-1-\sqrt{1-\alpha}}{6}x^3+Ax^4+Bx^5+Cx^6-\frac{x^7}{5040}-\frac{x^8}{40320}-\frac{x^9}{362880}-\frac{x^{10}}{3628800}-\frac{(-1+48A)x^{11}}{3991680}+\frac{(1+240B)\,x^{12}}{479001600}+\frac{(\frac{1}{120}+2C)\,x^{13}}{8648640}+O(x^{14})\ldots$$
(4.24)

where the constants A, B and C can be determined, using Eq. (3.1), as:

$$A=\frac{\underline{y}^{(4)}(0)}{4!}=\underline{Y}(4)\,, B=\frac{\underline{y}^{(5)}(0)}{5!}=\underline{Y}(5)\,, C=\frac{\underline{y}^{(6)}(0)}{6!}=\underline{Y}(6)$$ ....
…..........…(4.25)

Taking n = 13, an algebraic system of linear equations in terms of A, B and C is obtained. The solution of the

system yields

A = 0.04166666704765717 , B= −0.008333334180647071 , C = 0.001388889365963449

Finally, the series solution can be written as:

$$\underline{y}(x) = 1-\sqrt{1-\alpha}+\left(-1-\sqrt{1-\alpha}\right)x+\frac{1-\sqrt{1-\alpha}}{2}x^2+\frac{-1-\sqrt{1-\alpha}}{6}x^3+0.0416667x^4+0.0083333x^5+0.00138889x^6-\frac{x^7}{5040}-\frac{x^8}{40320}-\frac{x^9}{362880}-\frac{x^{10}}{3628800}-(2.50521\times10^{-8})x^{11}+(2.08768\times10^{-9})x^{12}-(1.60591\times10^{-10})x^{13}+O(x^{14})$$
… ………(4.26)

Similarly, for the upper solution $\overline{y}$, consider the problem:

$$\overline{y}^{(7)}(x) = -e^{x}\overline{y}^{2}(x), \quad 0 \le x \le 1$$
…………………………………………….....(4.27)

With upper bound of boundary conditions:

$\overline{y}(0)$ □1$\sqrt{1-\alpha}$, $\overline{y}^{(1)}(0)$ □-1$\sqrt{1-\alpha}$, $\overline{y}^{(2)}(0)$ □1$\sqrt{1-\alpha}$ $\overline{y}^{(3)}(0)$ □-1$\sqrt{1-\alpha}$

, $\overline{y}(1)$ □$\tilde{e}^{-1}\sqrt{1-\alpha}$, $\overline{y}^{(1)}(1)$ □-$\tilde{e}^{-1}\sqrt{1-\alpha}$, $\overline{y}^{(2)}(1)$ □$\tilde{e}^{-1}$ $\sqrt{1-\alpha}$……(4.28)

Applying the above theorems, the differential transformation of Eq. (4. 27) is obtained as

$$\overline{Y}(k+7) = \frac{k!}{(k+7)!}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{\overline{Y}(k-k_2)\overline{Y}(k_2-k_1)}{k!} \quad \ldots\ldots(4.29)$$

Using Eq. (3.1) the boundary conditions (4.28) can be transformed at $x_0 = 0$ as:

$\overline{Y}(0)$ □1$\sqrt{1-\alpha}$, $\overline{Y}(1)$ □-1$\sqrt{1-\alpha}$, $\overline{Y}(2)$ □$\frac{1+\sqrt{1-\alpha}}{2!}$, $\overline{Y}(3) = \frac{-1+\sqrt{1-\alpha}}{3!}$, $\sum_{k=0}^{n}\overline{Y}(k) = \tilde{e}^{-1} + \sqrt{1-\alpha}$, $\sum_{k=0}^{n}k\overline{Y}((k) = -\tilde{e}^{-1} + \sqrt{1-\alpha}$, $\sum_{k=0}^{n}k(k-1)\overline{Y}((k) = \tilde{e}^{-1} + \sqrt{1-\alpha}$
………………………………………………………………………
..(4.30).

where n is a sufficiently large integer. Using the recurrence relation (4.29) and the transformed boundary conditions(4.30), the following series solution up to $O(x^{14})$ can be obtained as:

$$\overline{y}(\mathrm{x}) = 1 + \sqrt{1-\alpha} + \left(-1 + \sqrt{1-\alpha}\right)\mathrm{x} + \frac{1+\sqrt{1-\alpha}}{2}\mathrm{x}^2 + \frac{-1+\sqrt{1-\alpha}}{6}\mathrm{x}^3 + \mathrm{A}\mathrm{x}^4 + Bx^5 + Cx^6 - \frac{x^7}{5040} - \frac{x^8}{40320} - \frac{x^9}{362880} - \frac{x^{10}}{3628800} - \frac{(-1+48A)x^{11}}{3991680} + \frac{(1+240B)\,x^{12}}{479001600} + \frac{(\frac{1}{120}+2C)\,x^{13}}{8648640} + O(x^{14}) \quad \ldots(4.31)$$

where the constants A, B and C can be determined, using Eq. (3.1), as:

$$A = \frac{\overline{y}^{(4)}(0)}{4!} = \overline{Y}(4)\,, B = \frac{\overline{y}^{(5)}(0)}{5!} = \overline{Y}(5)\,, C = \frac{\overline{y}^{(6)}(0)}{6!} = \overline{Y}(6) \quad \ldots(4.32)$$

Taking n = 13, an algebraic system of linear equations in terms of A, B and C is obtained. The solution of the system yields

A = 0.041666667047657l7 , B= −0.008333334180647071 , C = 0.001388889365963449

Finally, the series solution can be written as:

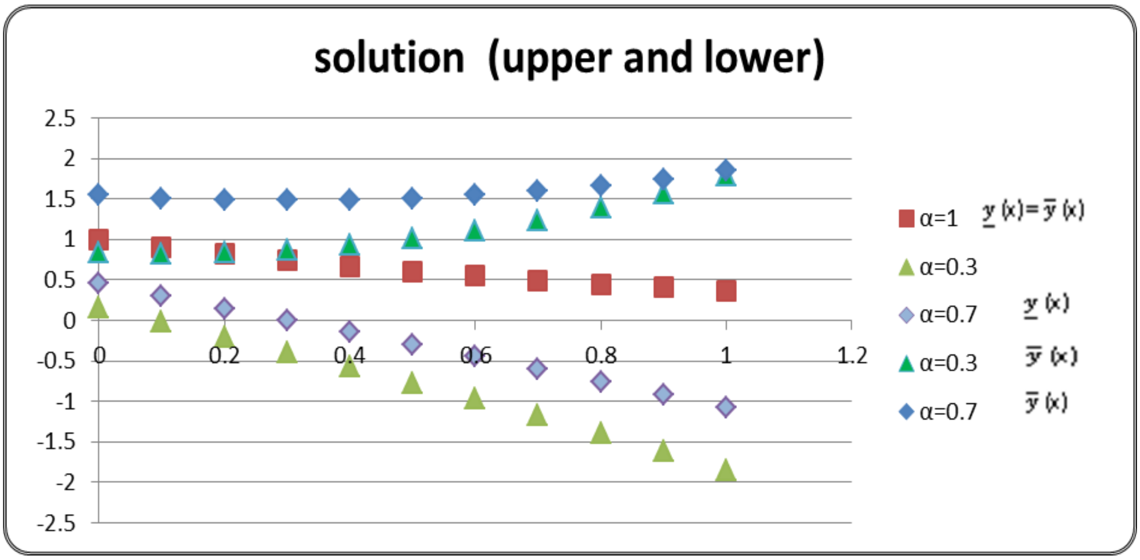
$$\overline{y}(\mathrm{x}) = 1 + \sqrt{1-\alpha} + \left(-1 + \sqrt{1-\alpha}\right)\mathrm{x} + \frac{1+\sqrt{1-\alpha}}{2}\mathrm{x}^2 + \frac{-1+\sqrt{1-\alpha}}{6}\mathrm{x}^3 + 0.0416667\mathrm{x}^4 + 0.0083333x^5 + 0.00138889x^6 - \frac{x^7}{5040} - \frac{x^8}{40320} - \frac{x^9}{362880} - \frac{x^{10}}{3628800} - (2.50521 \times 10^{-8})x^{11} + (2.08768 \times 10^{-9})x^{12} - (1.60591 \times 10^{-10})x^{13} + O(x^{14}) \quad \ldots(4.33)$$

Combining $\underline{y}$ and $\overline{y}$ yields the fuzzy solution of the fuzzy boundary value problem (4.1) as $y_\alpha(\mathrm{x}) = [\underline{y}(\mathrm{x}), \overline{y}(\mathrm{x})]$, $\alpha \in (0, 1]$, x $\in [0, 1]$. In addition, it is clear that for $\alpha = 1$, we get $\underline{y}(\mathrm{x}) = \overline{y}(\mathrm{x})$, which is the same as the exact solution of the related nonfuzzy boundary value problem. The results of the calculations are given in table(4.2):

Table(4.2): the results of example(4.2)

x	$\underline{y}$	$\overline{y}$	exact solution
0	1	1	1
0.1	0.8765	0.8765	0.9048
0.2	0.7346	0.7346	0.8187
0.3	0.6789	0.6789	0.7408
0.4	0.5768	0.5768	0.6703
0.5	0.5987	0.5987	0.6065
0.6	0.4964	0.4964	0.5488
0.7	0.4987	0.4987	0.4965
0.8	0.4321	0.4321	0.4493
0.9	0.3987	0.3987	0.4065
1	0	0	0

Also, the fuzzy solution $y$in terms of the lower bound of solution $\underline{y}$and upper bound of solution $\overline{y}$and for different□-levels (where α□(0, 1]) are presented in Fig.(4.2):



(x)$\underline{y}$

Fig.(4.2) Upper and lower solutions of example (4.2) for different values of α.

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الخلاصة:

في هذا البحث, تم دراسة الحل العددي للمعادلات التفاضلية الحدودية الضبابية من الدرجة السابعة باستخدام طريقة التحول التفاضلية(Differential transformation method). الحل التقريبي للمسالة يتم حسابه في شكل سلسلة متقاربة..

كذلك يتم إجراء المقارنة بين النتائج التي تم الحصول عليها مع الحل الواضح (exact solution) عندما يبلغ مستوى القطع(□-level) واحد .