(δ,p)-Continuous Multifunctions and (δ,p)-o-Closed Multifunctions

Amer Kh. Abed Al-shypany, samer Th. Abaas and Bassim Ka. Mihsin

Department of mathematics and computer science
Kufa university

Abstract

In this paper, the concept of upper and lower (δ,p)-continuous multifunctions and (δ,p)-o-closed multifunctions are introduced and studied, and obtain some characterizations and several properties concerning upper and lower (δ,p)-continuous multifunctions. The relationship between these multifunctions and (δ,p)-o-closed multifunctions.

1. Introduction


The aim of this paper a new form of continuous multifunction called (δ,p)-continuous multifunctions, and a new type of multifunction called (δ,p)-o-closed multifunction are introduced and studied.

2. Preliminaries and Definitions

Throughout the present paper, X and Y are always topological spaces. Let A be a subset of X. We denote the interior and the closure of a set A by Int(A) and CL(A), respectively. A subset A of a space X is said to be regular open [5] (respectively regular closed) if A = Int(CL(A)) (respectively A = CL(Int(A))). A subset A in a space X is called pre-open [6] if A ⊆ Int(CL(A)). The complement of a pre-open set is called pre-closed. The intersection of all pre-closed sets containing a subset A is called the pre-closure of A and is denoted by pCL(A). The pre-interior of A is the union of all pre-open sets of X contained in A and denoted by pInt(A).

Let X, Y be a topological spaces, the corresponding F : X → Y is called a multifunction if given any x ∈ X, then F(x) is a non-empty subset of Y [3]. Let F : X → Y be a multifunction from a space X to a space Y and A ⊆ X, B ⊆ Y then F(A) = {x ∈ X : F(x) ⊆ B} is called image of the set A. F−1(B) = {x ∈ X : F(x) ⊆ B} is called the upper inverse of the set B. F′(B) = {x ∈ X : F(x) ∩ B ≠ ∅} is called the lower inverse of the set B [7]. We will use the symbol ◦ to indicate end of the proof.

Definition 2.1 [5]

A subset A of a space X is said to be pre-
regular p-open (resp. pre-regular p-closed) if \( A = p\text{Int}(p\text{CL}(A)) \) (resp. \( A = p\text{CL}(p\text{Int}(A)) \)). It is clear that a pre-regular p-open set is pre-open.

**Definition 2.2 [2]**

- A point \( x \in X \) is called the \((\delta,p)\)-cluster point of \( A \) if \( A \cap U = \emptyset \) for every pre-regular p-open set \( U \) of \( X \) containing \( x \).
- The set of all \((\delta,p)\)-cluster points of \( A \) is called the \((\delta,p)\)-closure of \( A \), denoted by \( \delta\text{CL}_p(A) \).
- If \( \delta\text{CL}_p(A) = A \), then \( A \) is called \((\delta,p)\)-closed.
- The complement of a \((\delta,p)\)-closed set is called \((\delta,p)\)-open.
- A point \( x \in X \) is called the \((\delta,p)\)-Interior point of \( A \) if there exists pre-regular p-open set \( U \) of \( X \) containing \( x \) and contained in \( A \).
- The set of all \((\delta,p)\)-Interior points of \( A \) is called the \((\delta,p)\)-Interior of \( A \), denoted by \( \delta\text{Int}_p(A) \).
- If \( \delta\text{Int}_p(A) = A \), then \( A \) is called \((\delta,p)\)-open.

We say that a set \( U \) in a space \( X \) is a \((\delta,p)\)-neighborhood of a point \( x \) if \( U \) contains a \((\delta,p)\)-open set to which \( x \) belongs.

**Lemma 2.3 [4]**

If \( A \) and \( B \) are pre-regular p-open sets of the spaces \( X \) and \( Y \), respectively, then \( A \times B \) is a pre-regular p-open set of \( X \times Y \).

- If a space \( X \) is submaximal, then any finite intersection of pre-regular p-open sets is pre-regular p-open.

**Definition 2.4**

- The net \( \chi_\alpha \) is eventually in \( W \) if there exists \( \alpha_0 \in I \) such that \( \chi_\alpha \in W \), for each \( \alpha \geq \alpha_0 \) [7].
- The net \( \chi_\alpha \) in a space \( X \) is called converges to a point \( x \), if \( \chi_\alpha \) is eventually in every neighborhood of \( x \) [7].
- The net \( \chi_\alpha \) in a space \( X \) is called \((\delta,p)\)-converges to a point \( x \), if \( \chi_\alpha \) is eventually in every \((\delta,p)\)-neighborhood of \( x \).

**Definition 2.5 [2]**

A space \( X \) is called \((\delta,p)\)-Hausdorff space if for each two points \( x_1, x_2 \) in \( X \), there exist two \((\delta,p)\)-open sets \( V_1, V_2 \) such that \( x_1 \in V_1 \), \( x_2 \in V_2 \) and \( V_1 \cap V_2 = \emptyset \).

**Definition 2.6 [3]**

Let \((D,\geq)\) be a directed set, \( \{F_\alpha : \alpha \in D\} \) be a net of multifunction \( F_\alpha : X \to Y \) and \( F \) a multifunction on \( X \) into \( Y \), \( \{F_\alpha : \alpha \in D\} \) is said to be

Upper pointwise convergent to \( F \), if for each \( x \in X \) and each open set \( U \subseteq Y \) containing \( F(x) \), there exists \( \beta \in D \) such that \( x \in F_\alpha^*(U) \), for each \( \alpha \geq \beta \).
• Lower pointwise convergent to \( F \), if for each \( x \in X \) and each open set \( U \subseteq Y \) meeting \( F(x) \), there exists \( \beta \in D \) such that \( x \in F_{\alpha \gamma}(U) \), for each \( \alpha \geq \beta \).

• Pointwise convergent if it is both upper pointwise convergent and lower pointwise convergent.

**Definition 2.7 [2]**

A subset \( A \) of a space \( X \) is said to be \((\delta,p)\)-compact relative to \( X \) if every cover of \( A \) by \((\delta,p)\)-open sets of \( X \) has a finite sub cover.

**Definition 2.8**

Let \( X \) be a space, we say that \( X \) is \((\delta,p)\)-disconnected if it is the union of two non-empty \((\delta,p)\)-open sub sets, otherwise is \((\delta,p)\)-connected.

**Definition 2.9 [7]**

Let \( F : X \to Y \) be a multifunction from a space \( X \) to a space \( Y \); then

- \( F \) is called upper \((\delta,p)\)-continuous \((u.\,(\delta,p).c)\) at a point \( x \in X \), if for each open set \( V \) in \( Y \) with \( F(x) \subseteq V \) there exists a pre-regular \( p \)-open set \( U \) in \( X \) containing \( x \) such that \( F(U) \subseteq V \).

- \( F \) is called lower \((\delta,p)\)-continuous \((l.\,(\delta,p).c)\) at a point \( x \in X \), if for each open set \( V \) in \( Y \) with \( F(x) \subseteq V \) there exists a pre-regular \( p \)-open set \( U \) in \( X \) containing \( x \) such that \( F(U) \subseteq V \), for each \( u \in U \).

- \((\delta,p)\)-continuous \((\,(\delta,p).C)\) at a point \( x \in X \), if it is both \((u.\,(\delta,p).c)\) and \((l.\,(\delta,p).c)\) at \( x \).

- \((\delta,p)\)-continuous if it is \((\delta,p)\)-continuous at each point \( x \) in \( X \).

The following four theorems gives some characterizations of upper and lower \((\delta,p)\)-continuous multifunction:

**Theorem 3.2**

Let \( F : X \to Y \) be a multifunction from \( X \) to \( Y \); then the following statements are equivalent:

3. **\((\delta,p)\)-continuous multifunctions**

In this section, the new concept of \((\delta,p)\)-continuous multifunctions, introduced and studied and several characterization and properties of these forms are proved.

**Definition 3.1**

Let \( F : X \to Y \) be a multifunction from \( X \) to \( Y \), \( F \) is said to be

1) Upper \((\delta,p)\)-continuous \((u.\,(\delta,p).c)\) at a point \( x \in X \), if for each open set \( V \) in \( Y \) with \( F(x) \subseteq V \) there exists a pre-regular \( p \)-open set \( U \) in \( X \) containing \( x \) such that \( F(U) \subseteq V \).

2) Lower \((\delta,p)\)-continuous \((l.\,(\delta,p).c)\) at a point \( x \in X \), if for each open set \( V \) in \( Y \) with \( F(x) \cap V \neq \emptyset \) there exists a pre-regular \( p \)-open set \( U \) in \( X \) containing \( x \) such that \( F(U) \cap V \neq \emptyset \), for each \( u \in U \).

3) \((\delta,p)\)-continuous \((\,(\delta,p).C)\) at a point \( x \in X \), if it is both \( u.\,(\delta,p).c \) and \( l.\,(\delta,p).c \) at \( x \).

4) \((\delta,p)\)-continuous if it is \((\delta,p)\)-continuous at each point \( x \) in \( X \).

The following four theorems gives some characterizations of upper and lower \((\delta,p)\)-continuous multifunction:
1) \( F \) is \( l.(\delta,p).c \).

2) \( F^{-}(V) \) is \( (\delta,p) \)-open, for each open set \( V \) in \( Y \).

3) \( F^{+}(K) \) is \( (\delta,p) \)-closed set in \( X \), for each closed set \( K \) in \( Y \).

4) \( F(\delta\text{CL}_{p}(A)) \subseteq \text{CL}(F(A)) \).

5) \( \delta\text{CL}_{p}(F^{-}(B)) \subseteq F^{-}(\text{CL}(B)) \), for each subset \( B \) of a space \( Y \).

6) \( F^{-}(\text{Int}(B)) \subseteq \delta\text{Int}_{p}(F^{-}(B)) \), for each subset \( B \) of a space \( Y \).

7) For each \( x \in X \) and for each open set \( V \) in \( Y \) such that \( F(x) \cap V \neq \emptyset \), there exists a \( (\delta,p) \)-open set \( U \) in \( X \) containing \( x \) such that if \( u \in U \); then \( F(u) \cap V \neq \emptyset \).

8) For each \( x \in X \) and for each net \( \chi_{\alpha} \) which \( (\delta,p) \)-converges to \( x \) in \( X \) and for each open set \( V \) in \( Y \), such that \( x \in F^{-}(V) \); then the net \( \chi_{\alpha} \) is eventually in \( F^{-}(V) \).

**Proof:** (1) \( \Rightarrow \) (2) Let \( x \in F^{-}(V) \) in a space \( X \), \( V \) any open subset of \( Y \), from (1) we get there exists a pre-regular \( p \)-open set \( U \) in a space \( X \) such that \( x \in F^{-}(V) \); therefore, \( F^{-}(V) \) is \( (\delta,p) \)-open.

(2) \( \Rightarrow \) (3) Let \( K \) be a closed set in a space \( Y \); then \( K^{\complement} \) is open set in \( Y \) from (2) we obtain \( F^{-}(K^{\complement}) \) is \( (\delta,p) \)-open set in a space \( X \), since \( F^{-}(K^{\complement}) \) is \( (\delta,p) \)-open; then \( F^{-}(K) \) is \( (\delta,p) \)-closed set in \( X \).

(3) \( \Rightarrow \) (4) Let \( A \subseteq X \); then \( F(A) \) is subset of \( Y \) and \( \text{CL}(F(A)) \) is closed set in \( Y \). since \( F \) is \( l.(\delta,p).c \); then \( F^{+}(\text{CL}(F(A))) \) is \( (\delta,p) \)-closed subset of \( X \), in view fact that \( F(A) \subseteq \text{CL}(F(A)) \) implies that \( A \subseteq F^{+}(\text{CL}(F(A))) \). It follows that \( \delta\text{CL}_{p}(A) \subseteq \delta\text{CL}_{p}(F^{+}(\text{CL}(F(A)))) \); therefore, \( F(\delta\text{CL}_{p}(A)) \subseteq F(F^{+}(\delta\text{CL}_{p}(F(A))) \subseteq \text{CL}(F(A)) \).

(4) \( \Rightarrow \) (5) Let \( B \subseteq Y \), from (4) we get \( F(\delta\text{CL}_{p}(F^{-}(B))) \subseteq \text{CL}(F(F^{-}(B))) \subseteq \text{CL}(B) \), this implies \( \delta\text{CL}_{p}(F^{-}(B)) \subseteq F^{-}(\text{CL}(F(F^{-}(B)))) \subseteq F^{-}(\text{CL}(B)) \).

(5) \( \Rightarrow \) (1) Let \( K \) be any closed subset of \( Y \); then \( \delta\text{CL}_{p}(F^{+}(K)) \subseteq F^{+}(\text{CL}(K)) \), since \( K \) is closed set; then \( \delta\text{CL}_{p}(F^{+}(K)) \subseteq F^{+}(K) \) and \( F^{+}(K) \subseteq \delta\text{CL}_{p}(F^{+}(K)) \); then \( \delta\text{CL}_{p}(F^{+}(K)) = F^{+}(K) \), this means that \( F^{+}(K) \) is \( (\delta,p) \)-closed subset of \( X \); then \( F \) is \( l.(\delta,p).c \).

(6) \( \Rightarrow \) (1) Let \( W \) be any open subset of \( Y \); then \( \text{Int}(W) = W \) from (6) we get \( F^{-}(\text{Int}(W)) \) is \( (\delta,p) \)-open set in \( X \), thus , \( F^{-}(\text{Int}(W)) = \delta\text{Int}_{p}(F^{-}(\text{Int}(W))) \). It follows that \( F^{-}(\text{Int}(W)) \subseteq \delta\text{Int}_{p}(F^{-}(\text{Int}(W))) \).

(7) \( \Rightarrow \) (3) Let \( x \in X \) and \( V \) be any open set in a space \( Y \) such that \( F(x) \cap V \neq \emptyset \); then \( V^{\complement} \) is closed set in \( Y \) from (3) we get \( F^{+}(V^{\complement}) = (F^{-}(V))^{\complement} \) is \( (\delta,p) \)-closed set in a space \( X \). It follows that \( F^{-}(V) \) is \( (\delta,p) \)-open set in \( X \),
we take \( U= F^{-}(V) \) ; then for each \( u \in U \), \( F(u) \cap V \neq \emptyset \).

(7) \( \Rightarrow \) (8) Let \( x \in X \) and let \( \chi_{a} \) be a net which \((\delta,p)\)-converges to \( x \), \( V \) any open set in \( Y \) such that \( x \in F^{-}(V) \) from (7) there exists \((\delta,p)\)-open set \( U \) containing \( x \) and \( F(u) \cap V \neq \emptyset \) for each \( u \in U \); then \( U= F^{-}(V) \), since \( \chi_{a} \) is \((\delta,p)\)-converges to \( x \) in \( X \); then \( \chi_{a} \) is eventually in every \((\delta,p)\)-neighborhood of a point \( x \); therefore, the net \( \chi_{a} \) is eventually in \( F^{-}(V) \).

(8) \( \Rightarrow \) (1) Suppose that (1) is not true , there exists a point \( x \) in \( X \) and \( V \) an open set in \( Y \) with \( x \in F^{-}(V) \) such that for each \((\delta,p)\)-open set \( U \) in \( X \) containing \( x \), \( U \subseteq F^{-}(V) \). Let \( \chi_{U} \in U \) and \( \chi_{U} \notin F^{-}(V) \) for each \((\delta,p)\)-open set \( U \) containing \( x \) in \( X \); therefore, the net \( \chi_{U} \) \((\delta,p)\)-converges to \( x \), but \( \chi_{U} \) is not eventually in \( F^{-}(V) \). This is contradiction , hence \( F \) is \( l.(\delta,p).c. \)

**Theorem 3.3**

Let \( F : X \rightarrow Y \) be a multifunction from \( X \) to \( Y \); then the following statements are equivalent :
1) \( F \) is \( u.(\delta,p).c. \).
2) \( F^{-}(V) \) is \((\delta,p)\)-open , for each open set \( V \) in \( Y \).
3) \( F^{-}(K) \) is \((\delta,p)\)-closed set in \( X \), for each closed set \( K \) in \( Y \).
4) \( \delta CL_{p}(F^{-}(B)) \subseteq F^{-}(CL(B)) \), for each subset \( B \) of a space \( Y \).
5) \( F^{-}(Int(B)) \subseteq \delta Int_{p}(F^{-}(B)) \), for each subset \( B \) of a space \( Y \).

For each \( x \in X \) and for each open set \( V \) in \( Y \) such that \( F(x) \subseteq V \), there exists a \((\delta,p)\)-

6) open set \( U \) in \( X \) containing \( x \) such that if \( y \in U \); then \( F(y) \subseteq V \).
7) For each \( x \in X \) and for each net \( \chi_{a} \) which \((\delta,p)\)-converges to \( x \) in \( X \) and for each open set \( V \) in \( Y \), such that \( x \in F^{-}(V) \); then the net \( \chi_{a} \) is eventually in \( F^{-}(V) \).

**Proof :** The proof is similar to the proof of theorem (3.2) .

The following theorem shows that the restriction of lower (upper) \((\delta,p)\)-continuous multifunction on pre-regular \( p \) open set is also lower (upper) \((\delta,p)\)-continuous multifunction :

**Theorem 3.4**

Let \( F : X \rightarrow Y \) be a multifunction from submaximal space \( X \) into a space \( Y \) and let \( A \) be a pre-regular \( p \)-open subset of \( X \), if
1) \( F \) is \( u.(\delta,p).c. \) ; then \( F\mid_{A} : A \rightarrow Y \) is also \( u.(\delta,p).c. \).
2) \( F \) is \( l.(\delta,p).c. \) ; then \( F\mid_{A} : A \rightarrow Y \) is also \( l.(\delta,p).c. \).
3) \( F \) is \((\delta,p).C \) ; then \( F\mid_{A} : A \rightarrow Y \) is also \((\delta,p).C \).

**Proof :** 1) Let \( x \in A \) and \( V \) be any open set of \( Y \) such that \( x \in (F\mid_{A})^{+}(V) \). Since \( F \) is \( u.(\delta,p).c. \) ; then there exists a pre-regular \( p \)-open set \( U \) in \( X \) such that \( x \in U \subseteq F^{-}(V) \), from here we obtain that \( x \in A \cap U \) and \( A \cap U \subseteq (F\mid_{A})^{+}(V) \), since \( A \) is pre-regular \( p \)-open set and \( X \) is submaximal space ;
then \( A \cap U \) is pre-regular p-open set; therefore, \( F|_A \) is u.(\( \delta \),p).c.

The proof of number (2) is similar, from (1) and (2) we obtain the proof of number (3). \( \blacksquare \)

**Theorem 3.5**

Let \( F : X \rightarrow Y \) be a multifunction, if \( F \) is \( \text{u.(\( \delta \),p).c} \); then \( G : X \rightarrow F(X) \) is also \( \text{u.(\( \delta \),p).c} \) where \( G(x) = F(x) \).

**Proof:** Let \( V \) be any open subset of a space \( Y \); then \( F(X) \cap V \) is open set in a space \( F(X) \), \( G^{-1}(V) = X \cap G^{-1}(V) = F^{-1}(V) \), since \( F \) is \( \text{u.(\( \delta \),p).c} \); then \( F^{-1}(V) \) is \( (\delta,p) \)-open set in a space \( X \); therefore, \( G \) is \( \text{u.(\( \delta \),p).c} \). \( \blacksquare \)

**Theorem 3.6**

If \( X \) is a submaximal space and \( F_{\alpha} : X \rightarrow Y \) is \( \text{u.(\( \delta \),p).c} \) multifunction, for each \( \alpha = 1, 2, \ldots, n \); then \( F = \bigcup_{\alpha=1}^{n} F_{\alpha} \) is also \( \text{u.(\( \delta \),p).c} \) multifunction.

**Proof:** Let \( V \) be any open subset of \( Y \), we have \( F^{-1}(V) = \{ x : x \in X, \bigcup_{\alpha=1}^{n} F_{\alpha}(x) \subseteq V \} = \bigcap_{\alpha=1}^{n} (F_{\alpha})^{-1}(V) \); therefore, \( F^{-1}(V) \) is a \( (\delta,p) \)-open set in \( X \); then \( F \) is \( \text{u.(\( \delta \),p).c} \) on \( X \). \( \blacksquare \)

**Theorem 3.7**

Let \( F : X \rightarrow Y \) be an \( \text{u.(\( \delta \),p).c} \) multifunction and a point compact, if \( Y \) is a Hausdorff space and \( F(x_1) \cap F(x_2) = \emptyset \), for all \( x_1 \neq x_2 \) in \( X \); then \( X \) is \( (\delta,p) \)-Hausdorff space.

**Proof:** Let \( x_1, x_2 \) be any points of \( X \) such that \( x_1 \neq x_2 \), since \( F(x_1) \cap F(x_2) = \emptyset \) in \( Y \) and \( Y \) is a Hausdorff space, there exist two open set \( W_1, W_2 \) such that \( W_1 \) containing \( F(x_1) \) and \( W_2 \) containing \( F(x_2) \) and \( W_1 \cap W_2 = \emptyset \). Since \( F \) is \( \text{u.(\( \delta \),p).c} \); then \( F^*(W_1) \) and \( F^*(W_2) \) are \( (\delta,p) \)-open subsets of \( X \) such that \( x_1 \in F^*(W_1) \) and \( x_2 \in F^*(W_2) \) such that \( F^*(W_1) \cap F^*(W_2) = F^*(W_1 \cap W_2) = F^*(\emptyset) = \emptyset \). It follows that \( X \) is \( (\delta,p) \)-Hausdorff space. \( \blacksquare \)

**Theorem 3.8**

Let \( F \) and \( G : X \rightarrow Y \) be two \( \text{u.(\( \delta \),p).c} \) multifunctions and point closed, if \( X \) is a submaximal space and \( Y \) is a normal space; then the set \( E = \{ x \in X : F(x) \cap G(x) = \emptyset \} \) is \( (\delta,p) \)-closed set in \( X \).

**Proof:** Let \( x \in E^C = \{ x \in X : F(x) \cap G(x) = \emptyset \} \), since \( F \) and \( G \) are point closed and \( Y \) is a normal space; then there exist disjoint two open set \( U \) and \( V \) containing \( F(x) \) and \( G(x) \) respectively. Since \( F \) and \( G \) are \( \text{u.(\( \delta \),p).c} \); then \( F^+(U) \) and \( G^+(V) \) are \( (\delta,p) \)-open subset of \( X \) such that \( x \in F^+(U) \) and \( x \in G^+(V) \). It follows that there exist pre-regular p-open sets \( W_1 \) and \( W_2 \) such that \( x \in W_i \subseteq F^+(U) \) and \( x \in W_2 \subseteq G^+(V) \). Let \( D = W_1 \cap W_2 \); then \( D \) is pre-regular p-open set containing \( x \) and \( D \cap E = \emptyset \), which means \( D \subseteq E^C \), hence \( E \) is a \( (\delta,p) \)-closed subset of \( X \). \( \blacksquare \)

**Theorem 3.9**

Let \( X_i, i=1,2 \) be submaximal space and \( Y \) be a normal space and \( F_i : X_i \rightarrow Y \) are
u.\((\delta,p)\).c multfunctions and point closed , the set \(D = \{(x_1,x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) = \emptyset\}\) is \((\delta,p)\)-closed set in \(X_1 \times X_2\).

**Proof**: Let \((x_1,x_2) \in D^C = \{(x_1,x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) = \emptyset\}\), since \(Y\) is a normal space and \(F_i\) are point closed , for each \((i=1,2)\), there exist disjoint open sets \(W_1\) and \(W_2\) such that \(W_1\) containing \(F(x_1)\) and \(W_2\) containing \(F(x_2)\). Since \(F_i\) are u.\((\delta,p)\).c , then \(F'(W_1)\) and \(F'(W_2)\) are \((\delta,p)\)-open subsets of \(X_1\) and \(X_2\) respectively. It follows that there exists pre-regular p-open sets \(H_1\) and \(H_2\) such that \(x_1 \in H_1 \subseteq F_1^{-1}(U)\) and \(x_2 \in H_2 \subseteq F_2^{-1}(V)\). Let \(S = H_1 \times H_2\); then \(S\) is pre-regular p-open set in \(X_1 \times X_2\) and \((x_1,x_2) \in S \subseteq D^C\), from this we obtain \(D^C\) is \((\delta,p)\)-open set in \(X_1 \times X_2\), implies that to \(D\) is \((\delta,p)\)-closed set in \(X_1 \times X_2\).

**Theorem 3.10**

Let \(F : X \to Y\) be a multifunction from a \((\delta,p)\)-connected topological space \(X\) onto a topological space \(Y\), if \(F\) is u.\((\delta,p)\).c and point connected; then \(Y\) is connected.

**Proof**: Suppose \(Y\) is disconnected , then there exists nonempty open sets \(U_1\) and \(U_2\) in \(Y\) such that \(Y = U_1 \cup U_2\) and \(U_1 \cap U_2 = \emptyset\), since \(F\) is u.\((\delta,p)\).c ; then \(F'(U_1)\) and \(F'(U_2)\) are \((\delta,p)\)-open subset of \(X\), such that \(F'(U_1) \cap F'(U_2) = F'(U_1 \cap U_2) = F'(\emptyset) = \emptyset\) and \(F'(U_1) \cup F'(U_2) = F'(U_1 \cup U_2) = F'(Y) = X\), so, by definition (2.8) \(X\) is \((\delta,p)\)-disconnected, this is contraction, therefore \(Y\) is connected.

The following result studied cartesian product of finite upper (lower) \((\delta,p)\)-continuous multfunctions.

**Theorem 3.11**

Suppose that for each \((i = 1,2,\ldots,n)\), \(X_i\) and \(Y_i\) are topological spaces, let \(F_i : X_i \to Y_i\) be a multifunction, for each \((i = 1,2,\ldots,n)\), consider the multifunction \(F : \prod_{i=1}^{n} X_i \to \prod_{i=1}^{n} Y_i\):

1. \(F_i\) is u.\((\delta,p)\).c ; then \(F\) is also u.\((\delta,p)\).c.
2. \(F_i\) is l.\((\delta,p)\).c ; then \(F\) is also l.\((\delta,p)\).c.
3. \(F_i\) is \((\delta,p)\).c ; then \(F\) is also \((\delta,p)\).c.

**Proof**: Let \(W\) be any open set in \(\prod_{i=1}^{n} Y_i\), then there exist open sets \(U_i\), for each \((i = 1,2,\ldots,n)\), such that \(W = \prod_{i=1}^{n} U_i\). It follows that \(F(W) = F^{-1}(\prod_{i=1}^{n} U_i) = \prod_{i=1}^{n} F_i^{-1}(U_i)\). Since \(F_i\) is u.\((\delta,p)\).c, for each \((i = 1,2,\ldots,n)\), then \(F_i^{-1}(U_i)\) is \((\delta,p)\)-open, for each \((i = 1,2,\ldots,n)\), then \(\prod_{i=1}^{n} F_i^{-1}(U_i)\) is \((\delta,p)\)-open set; therefore, \(F\) is u.\((\delta,p)\).c.

The proof of number (1) is similar, from (1) and (2) we obtain the proof of number (3).

87
**Theorem 2.12**

Let \{F_\alpha : X \to Y \mid \alpha \in D\} be a net of u.((\delta,p),c) multifunctions, \(F_\alpha\) be an upper pointwise convergent to \(F : X \to Y\) and \(F\) is point closed, if \(Y\) is normal and for each open set \(W\) of \(Y\) with \(F^{-1}(W)\neq\emptyset\) and \(\beta \in D\), there exists \(\gamma \geq \beta\) such that \(F_\gamma(x) \subseteq W\), for all \(x \in F^{-1}(W)\); then \(F\) is also u.((\delta,p),c).

**Proof**: Suppose \(F\) is not u.((\delta,p),c) at a point \(x^*\) in \(X\); then there exists an open set \(V\) in \(Y\) containing \(F(x^*)\), such that for each pre-regular \(p\)-open set \(U\) in \(X\) containing \(x^*\) and there exists \(x_U \in U\) and \(F(x_U) \subseteq V\). Let \(y^* \in F(x^*) \cap V\), since \(Y\) is normal space, there exists an open set \(V_1\) such that \(y^* \in V_1 \subseteq CL(V_1) \subseteq V\), let \(W = Y - CL(V_1)\), as \(y^* \in F(x^*) \cap V_1\), which means \(y^* \in F(V_1)\). Since \(\{F_\alpha : X \to Y \mid \alpha \in D\}\) is upper pointwise convergent to \(F\) at \(x^*\). It follows that there exists \(\beta \in D\) such that \(F_\beta(x^*) \subseteq V_1\), for each \(\alpha \preceq \beta\), since \(F(x_U) \cap V \neq \emptyset\); then \(F(x_U) \cap W \neq \emptyset\), this implies \(x_U \in F^{-1}(W)\neq\emptyset\). Hence there exists \(\gamma \geq \beta\) such that \(F_\gamma(x) \cap W \neq \emptyset\), for all \(x \in F^{-1}(W)\), this implies \(F_\gamma(x_U) \cap W \neq \emptyset\), therefore \(F_\gamma(x_U) \subseteq V_1\), hence \(F_\gamma\) is not u.((\delta,p),c) at \(x^*\), this contradiction; therefore, \(F\) is u.((\delta,p),c).

**Theorem 3.1**

Let \(\{F_\alpha : X \to Y \mid \alpha \in D\}\) be a net of l.((\delta,p),c) multifunctions, \(F_\alpha\) be a lower pointwise convergent to \(F : X \to Y\), if \(Y\) is regular and for each open set \(W\) of \(Y\) with \(F^{-1}(W)\neq\emptyset\) and \(\beta \in D\), there exists \(\gamma \preceq \beta\) such that \(F_\gamma(x) \subseteq W\), for all \(x \in F^{-1}(W)\); then \(F\) is also l.((\delta,p),c).

**Proof**: Assume \(F\) is not l.((\delta,p),c) at a point \(x^*\) in \(X\), then there exists an open set \(V\) in \(Y\), \(F(x^*) \cap V \neq \emptyset\) such that, for each pre-regular \(p\)-open set \(U\) in \(X\) containing \(x^*\) and there exists \(x_U \in U\) and \(F(x_U) \cap V = \emptyset\). Let \(y^* \in F(x^*) \cap V\), since \(Y\) is regular space, there exists an open set \(V_1\) such that \(y^* \in V_1 \subseteq CL(V_1) \subseteq V\), as \(y^* \in F(x^*) \cap V_1\), which means \(y^* \in F(V_1)\). Since \(\{F_\alpha : X \to Y \mid \alpha \in D\}\) is lower pointwise convergent to \(F\) at \(x^*\). It follows that there exists \(\beta \in D\) such that \(x^* \in F_\beta(V_1)\), for each \(\alpha \preceq \beta\), since \(F(x_U) \cap V = \emptyset\); then \(F(x_U) \subseteq V \subseteq W\), this implies \(x_U \in F^{-1}(W)\neq\emptyset\). Hence there exists \(\gamma \preceq \beta\) such that \(F_\gamma(x) \subseteq W\), for all \(x \in F^{-1}(W)\), this implies \(F_\gamma(x_U) \subseteq W\); therefore, \(F_\gamma(x_U) = \emptyset\); then \(F_\gamma\) is not l.((\delta,p),c) at \(x^*\), this contradiction; therefore, \(F\) is l.((\delta,p),c).

4. ((\delta,p))-o-closed multifunctions

In this section, the new concept of ((\delta,p))-o-closed multifunctions, introduced and studied, and several properties of these new concept are proved.

**Definition 4.1**

A multifunction \(F : X \to Y\) is said to be ((\delta,p))-o-closed if for each \(x \in X\) and \(y \in Y\), for which \(y \in F(x)\), there exists ((\delta,p))-open set \(U\) in \(X\) containing \(x\) and open set \(V\) in \(Y\) containing \(y\), such that \(F(x) \cap V = \emptyset\), for each \(x_0 \in U\).
The following theorem gives the relationships between the concept \((\delta,p)\)-continuous multifunctions and \((\delta,p)\)-o-closed graph.

**Theorem 4.2**

Let \( F : X \rightarrow Y \) be a multifunction from a space \( X \) into a Hausdorff topological space \( Y \), if \( F \) is \( u.(\delta,p)c \) and point compact; then \( F \) is \((\delta,p)\)-o-closed.

**Proof:** Let \( F : X \rightarrow Y \) be \( u.(\delta,p)c \) on a space \( X \) and \( y \in F(x) \), since \( F(x) \) is compact and \( Y \) is Hausdorff space; then there exist disjoint two open set \( V_1 \) and \( V_2 \) in \( Y \) such that \( y \in V_1 \) and \( F(x) \subseteq V_2 \). Since \( F \) is \( u.(\delta,p)c \), there exists \( U \) \((\delta,p)\)-open set such that \( x \in U \) implies \( F(U) \subseteq V_2 \); then \( F(U) \cap V_1 = \emptyset \). It follows that \( F \) is \((\delta,p)\)-o-closed.

**Theorem 4.3**

If \( F : X \rightarrow Y \) be a \((\delta,p)\)-o-closed multifunction, then \( F(B) \) is closed subset of a space \( Y \), for each \( B \) \((\delta,p)\)-compact relative to a space \( X \).

**Proof:** Let \( y \notin F(B) \); then for each \( x \in B \), \( y \notin F(x) \). By definition (3.1) there exist \((\delta,p)\)-open set \( U_x \) containing \( x \) and an open set \( V_y \) containing \( y \) such that \( f(U_x) \cap V_y = \emptyset \). The family \( \{U_x \mid x \in B\} \) is a cover of a set \( B \) by \((\delta,p)\)-open sets of \( X \), since \( B \) is \((\delta,p)\)-compact there exists a finite subset \( B_0 \) of \( B \) such that \( B \subseteq \bigcup \{U_x \mid x \in B_0\} \). Take \( V = \bigcap \{V_x \mid x \in B_0\} \); then \( V \) is an open set containing \( y \) and \( F(B) \cap V = \emptyset \), this means that \( V \subseteq (F(B))^c \); therefore, \( F(B) \) is closed in \( Y \).

**Definition 4.4**

A multifunction \( F : X \rightarrow Y \) is called contra \((\delta,p)\)-open if the image of every \((\delta,p)\)-open set in a space \( X \) is closed set in a space \( Y \).

**Theorem 4.5**

If \( F : X \rightarrow Y \) is contra \((\delta,p)\)-open multifunction such that the inverse image of each point of a space \( Y \) is a \((\delta,p)\)-closed set in a space \( X \); then \( F \) is \((\delta,p)\)-o-closed.

**Proof:** Let \( y \notin F(x) \); then \( x \notin F^*(y) \), since \( F^*(y) \) is \((\delta,p)\)-closed; then there exists \( U \) \((\delta,p)\)-open set containing \( x \) such that \( U \cap F^*(y) = \emptyset \). Since \( F \) is contra \((\delta,p)\)-open; then \( F(U) \) is closed set in a space \( Y \), this implies that there exists an open set \( S \) in a space \( Y \) such that \( y \in S \) and \( F(U) \cap S = \emptyset \). Hence \( F \) is \((\delta,p)\)-o-closed.

**Theorem 4.6**

If \( \{F_\alpha : \alpha \in \Gamma\} \) is a family of \((\delta,p)\)-o-closed multifunction from a space \( X \) into a space \( Y \); then \( F = \bigcap_{\alpha \in \Gamma} F_\alpha \) (defined by \( F(x) = \bigcap_{\alpha \in \Gamma} F_\alpha(x) \)) is \((\delta,p)\)-o-closed.

**Proof:** Let \( y \notin F(x) \); then there exists \( \alpha \in \Gamma \) such that \( y \notin F_\alpha(x) \), since \( F_\alpha \) is \((\delta,p)\)-o-closed; then there exists a \((\delta,p)\)-open set \( U \) of \( X \) containing \( x \) and open set \( V \) in \( Y \) containing \( y \) such that \( F_\alpha(U) \cap V = \emptyset \), from this we obtain \( F(U) \cap V = \emptyset \); therefore, \( F \) is
Theorem 4.7
If $F_1 : X \to Y$ is u.($\delta, p$) and point compact multifunction from a submaximal space $X$ into a Hausdorff space $Y$ and $F_2 : X \to Y$ be ($\delta, p$)-o-closed multifunction; then $F = F_1 \cap F_2$ is u.($\delta, p$) .

Proof: By theorem (4.2) $F_1$ is ($\delta, p$)-o-closed, since $F_2$ is ($\delta, p$)-o-closed; then by theorem (4.6) $F = F_1 \cap F_2$ is ($\delta, p$)-o-closed. Let $x_0 \in X$ and $V$ open subset in $Y$ containing $F(x_0) = F_1(x_0) \cap F_2(x_0)$. If $F_1(x_0) \subseteq V$ implies there exists $U_x$ ($\delta, p$)-open set such that $V \subseteq U_x \subseteq F_m(V)$, when $F(x_0) \subseteq V$.

Now, let $F_1(x_0) \not\subseteq V$; then $A = F_1(x_0) \cap V = \phi$, let $y \not\in F_2(x_0)$, since $F_2$ is ($\delta, p$)-o-closed, there exists an ($\delta, p$)-open set $U_y$ in $X$ containing $y$ and an open set $W_y$ in $Y$ containing $y$ such that $F_2(W_y) \cap W_y = \phi$. Since $F_1(x)$ is compact subset of $Y$ and $V^C$ is closed, then $A$ is closed subset of $Y$. It follows that $A$ is compact subset of $Y$; therefore, there exists points $y_1, y_2, ..., y_n$ in $A$ such that $A \subseteq U_{i=1}^{n} W_i = W'$, since $F_1$ is u.($\delta, p$) and $V \cup W'$ is open set and $F_1(x) \subseteq V \cup W'$; then there exists an open set $U_x$ containing a point $x$ such that $F_1(U_x) \subseteq V \cup W'$, let $U' = U_{y_1} \cap U_{y_2} \cap \ldots \cap U_{y_n} \cap U_x$, we have $F_2(U') \cap W = \phi$ and $F_1(U') \subseteq V \cup W'$; therefore, $(F_1 \cap F_2)(U') \subseteq V$. Hence $F$ is u.($\delta, p$) at a point $x$.

Theorem 4.8
If $F : X \to Y$ is ($\delta, p$)-o-closed multifunction from a submaximal space $X$ to a space $Y$; then $F^-(B)$ is ($\delta, p$)-closed, for each $B$ compact set in a space $Y$.

Proof: Let $B$ be arbitrary compact set in a space $Y$ and $x \not\in F^{-1}(B) = \{x : F(x) \cap B \neq \phi\}$; then $x \not\in F^{-1}(B)$, for each $b \in B$. Since $F$ is ($\delta, p$)-o-closed, then there exist ($\delta, p$)-open sets $U_b(b)$, and open set $U_b$ such that $x \in U_b(b)$ and $b \in U_b$ implies $F(U_b(b)) \cap U_b = \phi$. The family $\{U_b : b \in B\}$ is open cover of $B$, since $B$ is compact; then there exists $U_{b_1}, U_{b_2}, ..., U_{b_n}$ is finite sub cover such that $B \subseteq \bigcup_{i=1}^{n} U_{b_i}$, let $U_x = \bigcap_{i=1}^{n} U_{x(b_i)}$ such that $x \in U_x$. If $U_x \cap F^{-1}(B) = \phi$; then $F^{-1}(B)$ is ($\delta, p$)-closed.

Now, for showing this, suppose $U_x \cap F^{-1}(B) \neq \phi$, there exists $x_0 \in U_x$ and $x_0 \in F^{-1}(B)$; then $x_0 \in U_x(b_i)$, for each $i=1,2,...,n$ and $F(x_0) \cap B \neq \phi$, this implies there exists $z \in F(x_0)$ and $z \in B$, since $B \subseteq \bigcup_{i=1}^{n} U_{b_i}$; then $z \in U_{b_i}$ when $i=j$, there exists $U_z(b_i)$ such that $F((U_z(b_i)) \cap U_{bj} \neq \phi$, this is contradiction; therefore, $U_x \cap F^{-1}(B) = \phi$ in $X$ and $F^{-1}(B)$ is ($\delta, p$)-closed in $X$.

Now, the following theorem is study the converse of theorem 4.2.

Theorem 4.9
If $F : X \to Y$ is a ($\delta, p$)-o-closed multifunction and $Y$ is a compact space; then $F$ is u.($\delta, p$) .

Proof: Let $H$ be any closed subset of $Y$, since $Y$ is a compact space; then $H$ is compact set in $Y$ from theorem (4.8) we get
$F^-(H)$ is $(\delta,p)$-closed, this implies $F$ is $u.(\delta,p).c$.

References


