Using variational iterate method for solving 1D-2D integral equations.

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الخلاصة :
تناولنا في هذا البحث الحلول الحقيقية والتقريبية للمعادلات التكاملية(1D-2D) باستخدام طريقة(فارييشن اترايتد), وقدمنا بعض الامثلة الخطية والغير خطية . وقدمنا النتائج في جداول.

Abstract

The main objective of this is to study the exact solution and approximate solution (1D-2D) integral equations, by using the variational iteration method, as well as, give some illustrative examples of linear and nonlinear equations .We tabulate,also the exact and approximate solutions.

Keywords: (Variational iterated method, Integral equations).

1. Introduction

In some cases, the analytical solution may be difficult to evaluate, therefore numerical and approximate method seem to be necessary to be used which cover the problem under consideration. The method that will be considered in this work is the variational iteration method (which is abbreviated by VIM) for finding the solution of linear and nonlinear problems. This method is a modification of the general Lagrange multiplier method into an iteration method, which is called the correction functional. Heuristic interpretation of those concepts leads to new comers in the field to start working immediately without the long search and preparation of advanced calculus and calculus of variations. At the same time those concepts coagulation problem with mass loss by Abulwafa and Momani ,[1],[4] .

In this paper, we apply the variational iteration method to solve the(1D-2D) integral already familiar with variational iteration method which will find the most recent new results

$$U(x)= f(x) + \int_0^1 k1(x,s)u(s)\, ds + \int_0^x k_2(x,s)u(s)ds \;, \quad \ldots (1)$$

where f(x) , k_1 and k_2 be continouse functions.

And the forme

$$U(x,y)= g(x,y) + \int_0^x\int_0^y k(x,y,s,t)u(s,t)dsdt \quad \ldots (2)$$

Where g(x,y), k be a continous function

2. Variational Iteration Method [2],[5],[3]

Variational iteration method which was proposed by Ji-Huan 1998 has been recently and intensively studied by several scientists and engineers which is favorably applied to various kinds of linear and nonlinear problems.

To illustrate the basic idea of the VIM, we consider the following general non-linear equation given in an operator form:

$$L(u(x)) + N(u(x)) = g(x), \; x \in [a, b], \quad \ldots(3)$$

where L is a linear operator, N is a nonlinear operator and g(x) is any given function which is called the non-homogeneous term.

Now, rewrite Eq.(2) in a manner similar to Eq.(3) as follows:

$$L(u(x)) + N(u(x)) - g(x) = 0 \quad \ldots(4)$$

and let u_n be the n^{th} approximate solution of Eq. (3), then it follows that:

$$L(u_n(x)) + N(u_n(x)) - g(x) \neq 0 \quad \ldots(5)$$

Then the correction functional for (3), is given by:

$$u_{n+1}(x) = u_n(x) + \int_{x_0}^{x} \lambda(s)\{L(u_n(s) + N(\tilde{u}_n(s)) - g(s)\}ds$$

...(6)

where λ is the general Lagrange multiplier which can be identified optimally via the variational theory, the subscript n denotes the n^{th} approximation of the solution u and $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta \tilde{u}_n = 0$.

To solve eq. (6) by the VIM, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. Then the successive approximation $u_n(x)$, n = 0, 1, …; of the solution u(x) will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $u_0(x)$. The zeroth approximation u_0 may be selected by any function that just satisfies at least the initial and boundary conditions with λ determined, then several approximations $u_n(x)$, n = 0, 1, …; follow immediately, and consequently the exact solution may be arrived since

$$u(x) = \lim_{n \to \infty} u_n(x) \qquad ...(7)$$

In other words, the correction functional for Eq. (2) will give several approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations.

g general non-linear equation given in operator form:

$L(u(x,y)) + N(u(x,y)) = g(x,y), \quad x, y \in [a, b]$ ...(8)

where L is a linear operator, N is a nonlinear operator and g(x,y) is any given function which is called the non-homogeneous term.

Now, rewrite Eq.(8) in a manner similar to eq.(8) as follows:

$L(u(x,y)) + N(u(x,y)) - g(x,y) = 0$ ...(9)

and let u_n be the n^{th} approximate solution of eq. (8), then it follows that:

$L(u_n(x,y)) + N(u_n(x,y)) - g(x,y) \neq 0$ ...(10)

and then the correction functional for (2) is given by:

$$U_{n+1}(x,y)= u_n(x,y)+ \int_0^x\int_0^y \lambda(s,t)[Lu_n(s,t)+N(u_n(s,t))-g(s,t)]dsdt \quad \ldots (11)$$

where λ is the general Lagrange multiplier which can be identified optimally via the variational theory, the subscript n denotes the n^{th} approximation of the solution u and $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta \tilde{u}_n = 0$.

To solve eq. (11) by the VIM, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts and by the developed tabulated method . Then the successive approximation $u_n(x,y)$, n = 0, 1, …; of the solution u(x,y) will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $u_0(x,y)$. The $zero^{th}$ approximation u_0 may be selected by any function that just satisfies at least the initial and boundary conditions with λ determined, then several approximations $u_n(x,y)$, n = 0, 1, …; follow immediately, and consequently the exact solution may be arrived since:

"$U(x,y)= \lim_{x\to\infty} u_n(x,y)$ … (12) "

In other words,starting with appropriate function for $u_0(x,y)$,we can obtain the exact solution or an approximate solution using equation (12).

3. Illustrative Examples

In this section, some examples are given to illustrate the applicability and efficiency of the VIM for solving different types of problems.

Example (1)

Consider the linear integral equation ,

$U(x)= x^2+4x+1/4+(x^2+2x)e^x+{}_0\int^1 k_1(x,s)u(s)ds+ {}_0\int^x k_2(x,s)u(s)ds$ …(13)

With exact solution $u= x^2+2x$, $u(0)=0$,

$K_1=x+s$, $k_2=xe^s$.

Solution:

First, differentiate equation (13) with respect to x , yields to :

$u'(x)=2x +4+(x^2+2x)e^x+e^x(2x+2)+{}_0\int^1 (2s+2s^2)ds+{}_0\int^x e^s(s^2+2s)ds$ …(14)

then , the following correction functionl for eqation (14) may be obtained for all n= 0,1,…

$u_{n+1}=u_n(x)+{}_0\int^x \lambda(t) (L(u_n(t)-N(\tilde{u}_n(t)-g(t))dt$ …(15)

$$u_{n+1}=u_n(x)+\int_0^x \lambda(t)\{u'(x)-2t-4-(t^2+2t)e^t-e^t(2t+2)-{}_0\int^1(2s+2s^2)ds-{}_0\int^x e^s(s^2+2s)ds\}dt \quad …(16)$$

where λ is the general lagrange multiplier .

thus by taking the first variation with respectto the independent variable u_n and noticing that $\partial u_n(0)= 0$,we get

$\partial u_{n+1}(s)=\partial u_n(s)+\partial\ {}_0\int^x \lambda(t)\{u_n'(x)-2t-4-(t^2+2t)e^t-e^t(2t+2)-{}_0\int^1 (2s+2s^2)ds- {}_0\int^x e^s(s^2+2s)ds \}dt,$

where u_n is consid as a vestricted variation, which means $\partial u_n=0$ and consequently

$\partial u_{n+1} = \partial u_n(x)+ \partial_0 \int^x \lambda(t) \{u_n'(t)\}dt$ …(17)

Now by the method of integration by parts , then Equation (17) will be reduced to

$\partial u_{n+1}= \partial u_n(x)+ \lambda(t)\partial u_n(t)|_{t=x} -_0 \int^x \lambda'(t) \partial u_n(t)\, dt$

Hence

$\partial u_{n+1}(x) = 1+\lambda(t)|_{t=x} \partial u_n(x) -_0 \int^x \lambda' \partial u_n(t)\, dt = 0$

As aresult , we have the following stationary conditions

$\lambda'(t) = 0$

with natural boundary condition

$1+ \lambda(t) |_{t=x} = 0,$

Which is easily solved to give the lagrange multiplier $\lambda(t) = -1$. Now , substituting $\lambda(s) = -1$ back in to Equation (17) gives for all $n = 0,1, \ldots$

$U_{n+1}= u_n(x) -_0 \int^x \{ u_n'(t) - 2t -4-(t^2+2t)e^t - (2t+2)e^t -_0 \int^1 (2s+2s^2)\, ds -_0 \int^x e^s(s^2+2s)\, ds\}dt$ … (18)

$U_1= (-0.41)x - x^3/3$

$U_{2}= u_1(x) - {}_0\int^x \{ u_1'(t) - 2t -4-(t^2+2t)e^t - (2t+2)e^t - {}_0\int^1 (2s+2s^2)\, ds- {}_0\int e^s(s^2+2s)ds\}dt$

$U_2= x^2+4x+x(x-2)e^x$

$U_3 = u_2(x) - \int_0^x \{ u_2'(t) - 2t - 4 - (t^2+2t)e^t - (2t+2)e^t - \int_0^1 (2s+2s^2)\ ds - \int_0^x e^s(s^2+2s)\ ds \} dt$

$U_3 = (3x^2 - 2x - 2x^3 - 2x^4 - x^5)\, e^x - 16x$

Table (1)

X	$\lvert u(x) - u_1(x) \rvert$	$\lvert u(x) - u2(x) \rvert$	$\lvert u(x) - u_3(x) \rvert$
0	0	0	0
0.1	0.2513	0.1499	0.4014
0.2	0.5222	0.761	0.2057
0.3	0.822	0.871	0.9331
0.4	1.1453	1.446	1.5195
0.5	1.4966	1.764	1.6331
0.6	1.878	1.662	1.3793
0.7	2.291	2.369	2.834
0.8	2.738	2.664	2.956
0.9	3.222	3.866	3.123

1	3.743	3.282	3.872

In Table (1) , we introduced the exact and approximate solutions for some points.The results show that the rate of error is very small and the approximate solution is very close to the exact one . This means that is our method presents a good agreement between the solutions which is good result.

Example(2)

Consider the nonlinear integral equation ,

$U(x)= 2x/3 - x^3/6 + {}_0\int^1 k_1 (u (t))^2dt + {}_0\int^x k_2 (u (t))^2dt$ …(19)

, $u(0)= 0$, $x \in [0,1]$,

where $k_1= xt$, $k_2=x-t$, $u(x)= x$ is the exact solution of Equation (20)

Solution:

First, differentiate equation (20) with respect to x

$u'(x) = 2/3 - 3x^2/6 + {}_0\int^1 t (u (t))^2dt + {}_0\int^x (u (t))^2dt$ …(21)

then by (VIM) ,

$u_{n+1}(x) = u_n(x) + {}_0\int^x \lambda(s) \{ u_n' (s) -2/3 - 3x^2/6 - {}_0\int^1 t (u_n (t))^2dt - {}_0\int^x (u_n (t))^2dt$
$\}ds$, … (22)

where λ is the general lagrange multiplier.

Thus by taking the first variation with respect to the independent variable u_n and noiting that $\partial u_n(0) = 0$, we get :

$\partial u_{n+1}(t) = \partial u_n(t) + \partial {}_0\int^x \lambda(s) \{ u_n'(s) -2/3 + 3x^2/6 - {}_0\int^1 t (\tilde{u}_n (t))^2dt - {}_0\int^x (\tilde{u}_n (t))^2dt \}ds$ … (23)

Where $\tilde{u}_n$ is considered as a restricted variation , which means

$\partial \tilde{u}_n = 0$

$\partial u_{n+1} = \partial u_n(x) + \partial \int_0^x \lambda(s)\, u_n'(s)\, ds$ … (24)

And by the method of integration by parts , then equation (15) will be reduced to :

$\partial u_{n+1} = \partial u_n(x) + \lambda(s)\, \partial u_n(s)\,|_{s=x} - \int \lambda'(s) \partial u_n(s) ds$

Hence:

$\partial u_{n+1} = 1 + \lambda(s)\,|_{s=x} \partial u_n(x) - \int_0^x \lambda'(s) \partial u_n(s) ds = 0$

As a result , we have

$\lambda'(s) = 0$

with natural boundary condition $1 + \lambda(s)|_{s=x} = 0$

so $\lambda(s) = -1$

now , substituting $\lambda(s) = -1$ back in to equation (13) give for all n= 0 ,1,…

$u_{n+1}(x) = u_n(x) - \int_0^x \{ u_n'(s) - 2/3 - 3x^2/6 - \int_0^1 t\,(u_n(t))^2 dt - \int_0^x (u_n(t))^2 dt \} ds$ … (25)

let the initial approximate solution be

$u_0 = 2x/3 - x^3/6$

then

$u_1(x) = u_0(x) - \int_0^x \{ u_0'(s) - 2/3 - 3x^2/6 - \int_0^1 t\,(u_0(t))^2 dt - \int_0^x (u_0(t))^2 dt \} ds$

$= 0.748x + 0.273\, x^4 + 0.039\, x^6$

$$U_2(x) = u_1(x) - \int_0^x \{ u_1'(s) - 2/3 - 3x^2/6 - \int_0^1 t\,(u_1(t))^2 dt - \int_0^x (u_1(t))^2 dt \} ds$$

$= 0.248\, x^3 - 0.0581\, x + 0.39\, x^6$

$$U_3(x) = u_2(x) - \int_0^x \left\{ u_2'(s) - 2/3 - 3x^2/6 - \int_0^1 t\,(u_2(t))^2 dt - \int_0^x (u_2(t))^2 dt \right\} ds$$

$= 0.6272\, x - 0.062\, x^4 + 0.091 x^6$

The absolute error between the exact and approximate solution of Example (2)

Tab (2)

X	$\lvert u(x)-u_1(x)\rvert$	$\lvert u(x)-u_2(x)\rvert$	$\lvert u(x)-u_3(x)\rvert$
0	0	0	0
0.1	0.0252	0.1055	0.0372
0.2	0.0499	0.3184	0.0743
0.3	0.0733	0.4829	0.111164
0.4	0.094	0.4459	0.148014
0.5	0.109	0.84306	0.18355
0.6	0.117	0.9015	0.21941
0.7	0.1093	0.3572	0.25622
0.8	0.085	0.6027	0.29601
0.9	0.032	0.8817	0.3425
1	0.05	0.302	0.401

In Table (2) , we introduced the exact and approximate solutions for some points.The results show that the rate of error is very small and the approximate solution is very close to the exact one .

Example(3): *Consider the linear integral equation*

$$U(x,y)= g(x,y) + \int_0^x \int_0^y k(x,y,s,t)u(s,t)dsdt \quad \ldots (26)$$

where $g(x,y)= 2+x+y$, $k(x,y.s,t)=xye^{st}$ with the exact solution $u(x,y)=x+y$,and initial condition $u(0,0)=0$.

solution:

first, differentiate equation (26)

$$u_x= 1+ \int_0^x \int_0^y ye^{st}(s+t)\, dsdt \ldots\ldots(27)$$

$$u_{xy}= \int_0^x \int_0^y e^{st}(s+t)dsdt \quad \ldots\ldots(28)$$

then the following correction function for equation (28)

for all n= 0,1,….then by VIM

$$u_{n+1}(x,y)= u_n(x,y)+ \int_0^x \int_0^y \lambda(I,J)\{L(u_n(I,J)+N(u_n(I,J))-g(I,J)\}dIdJ..(30).$$

$$U_{n+1}=u_n(x,y)+ \int_0^x \int_0^y \lambda(I,J)\ \{ u_{xy}- \int_0^x \int_0^y e^{IJ}(I+J)\}dIdJ\ldots(31)$$

Where λ is the general lagrange multiplier,thus by taking the first variation with respect to the independent variable u_n and noticing that $\partial u_n(0,0)=0$, we get

$$\partial u_{n+1}(s,t)=\partial u_n(s,t)+\partial\int_0^x\int_0^y \lambda(I,J)\{ u_{xy} - \int_0^x\int_0^y e^{IJ}(I+J)\}\, dIdJ \ldots(32)$$

Where u_n is consid as vestricted variation,which means $\partial u_{n=0}$

And consequently $\partial u_{n+1} = \partial u_n +\partial\int_0^x\int_0^y \lambda\, u_{xy}(I,J)\, dIdJ$

,by the method of integration by parts,and the Developed tabulated method then equation(32) will be reduced to

$\partial u_{n+1} = \partial u_n+ \lambda(I,J)\, \partial u_n(I,J)/_{I=x,J=y}$ we have $1+\lambda = 0$, so $\lambda = -1$

Then $u_{n+1}= u_n(x,y)-\int_0^x\int_0^y \{u_n xy+\int_0^x\int_0^y e^{IJ}(I+J)dIdJ\}dsdt \ldots(33)$

$U_0=2+x+y$

$U_1=2+x+y-2yx-((y+x)e^{yx}-y-x^2y-xy^2-x)x^2$

$U2=xy-(y+x)x^2e^y+xy(x^2-2x)e^{yx}+(x^2-2x(y+x)xye^x+2x^2y+(y+x)x^2e^{yx}-yx^2-x^3y-x^3y^2-x^3.$

Table (3)

(X,y)	\u(x,y)-u_1(x,y)\	\ u(x,y)-u_2(x,y)\
(0.1,0.1)	1.9886660	1.877889
(0.2,0.2)	1.9288999	1.789900
(0.3,0.3)	1.816178	1.67087
(0.4,0.4)	1.678806	1.566022
(0.5,0.5)	0.50350	0.40360
(0.6,0.6)	0.248899	0.14811991
(0.7,0.7)	0.1900	0.08010
(0.8,0.8)	0.186778	0.198865
(0.9,0.9)	0.076766	0.06678
(1,1)	0.0866778	0.0078876

In Table (3) , we introduced the exact and approximate solutions for some points.The results show that the rate of error is very small and the approximate solution is very close to the exact one . This means that is our method presents a good agreement between the solutions which is good result.

Example(4):

Consider the nonlinear integral equation

$$u(x,y) = g(x,y) + \int_0^x \int_0^y k(x,y,s,t)(u(s,t))^2 dsdt \ldots(34)$$

where $g(x,y)= xy$, $k(x,y,s,t)=t \sin x + yxs$, $u(x,y)= xy - 1$, is the exact solution of Equation (34)

first, differantion Equation(34) with respect to x

$u_x= y+\int_0^x\int_0^y (t\cos x+ys)(st-1)^2 dsdt$ and differation u_x with respect to y

$$u_{xy}= 1+ s\int_0^x\int_0^y (st-1)^2 dsdt$$

then by VIM

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x\int_0^y \lambda(i,j)\{ L(u_n(i,j)+N(u_{\tilde{n}}(i,j)-g(i,j)\}didj$$

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x\int_0^y \lambda(i,j) \{u_{xy}-1\int_0^x\int_0^y s(st-1)^2 dsdt\}didj$$

where λ is the general lagrange multiplier, thus by taking first variation with respect to the independ variable u_n and noticing that

$\partial u_n(0,0)=0$

$$\partial u_{n+1}= \partial u_{n+}\partial\iint\lambda(i,j)\{u_{xy}-1-\int_0^x\int_0^y (i(ij-1)^2\} \; didj \ldots(35)$$

Where u_n is consid as vestricted variation , which means $\partial u_n=0$ and consequently

$$\partial u_{n+1}=\partial u_n+ \partial \int_0^x\int_0^y \lambda(i,j)u_{xy}didj$$

By the method of integration by parts and the Developed tabulated method for evaluating integrals,then Equation(35) will be reduced to

$\partial u_{n+1} = \partial u_n(x,y) + \lambda(i,j)\backslash_{i=x\ ,\ j=y}$

We have

$1+\lambda(i,j)\backslash_{i=x\ ,\ j=y} = 0$, so $\lambda = -1$

Now , substituting $\lambda(i,j) = -1$ for all n=0,1,…

$$U_{n+1} = u_n(x,y) - \int_0^x\int_0^y \{u_{xy} - 1 - \int_0^x\int_0^y s(st-1)^2 dsdt\} didj$$

$U_0 = xy$

$U_1 = 1\backslash 12\ x^4y^5 - 1\backslash 3x^3y^4 + 1\backslash 2y^3x^2 + xy.$

$U_2 = 18\backslash 12\ x^4y^5 + 20\backslash 39\ x^3y^4 - 5\backslash 2\ y^3x^2 + 1\backslash 2y^2x^3 + xy.$

Table (4)

(x,y)	$\backslash u(x,y)-u_1(x,y)\backslash$	$\backslash u(x,y)-u_2(x,y)\backslash$
(0,0)	1	1
(0.1,0.1)	1.30000496	0.9999801815
(0.2,0.2)	1.249755	0.999383808
(0.3,0.3)	1.0023588	0.9055631845
(0.4,0.4)	1.0045962	0.98286233
(0.5,0.5)	0.985355	0.95475
(0.6,0.6)	0.9509882	0.909985392
(0.7,0.7)	0.888277	0.8726281505
(0.8,0.8)	0.9337655	0.923453952
(0.9,0.9)	0.9226770	0.97099658

In Table (4),we present some points and we calculated the difference between the excat and approximate solutions by using the varitional iterated method.The table show that the error rate is reducing to be more smaller . This means that the solution is going to be close to the excat solution.

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