

Certain Types of Complex Lie Group Action

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Abstract

The main aim in this paper is to look for a novel action with new properties on *Complex Lie Group* from the *Lemma of Schure* , the Literature are concerned with studying the action of *Lie algebra* of two representations , one is usual and the other is the dual, while our interest in this work is focused on some actions on complex Lie group[10] . Let G be a matrix complex *Lie* group , and π is representation of G . In this study we will present and analytic the concepts of action of complex *Lie* group on *Hom – space*. We recall the definition of tensor product of two representations of *Lie* group and construct the definition of action of *Lie* group on *Hom – space*, then by using the equivalent relation $Hom(w_2, w_1) \cong w_2^* \otimes w_1$ between *Hom* and *Tensor product* , we get a new action : *Action – complex Lie Group on tensor product*. The two actions are forming smooth representation of G [8], [9]. This we have new action which called *triple action of Complex Lie Group G* denoted by *TAC – complex Lie group* which acting on $Hom(Hom((w_5 \oplus w_4), w_3^*), Hom(w_2 \oplus w_1, w^*))$. This *TAC* is smooth representation of G . The theoretical Justifications are developed and prove supported by some concluding remarks and illustrations.

Key words : Hom – Space , Tensor Product , Action of Lie Group , Complex Lie Group .

Mathematics subject classification: 64S40

1- Introduction

A complex Lie group is a finite dimensional analytic manifold G together with a group structure on G , Such that the multiplication $G \times G \rightarrow G$ and attaching of an inverse $g \rightarrow g^{-1} : G \rightarrow G$ is analytic map [4],[6].

A matrix Lie group is any subgroup G of $GL(n, \mathbb{C})$ with the following property [7] . If A_m converges to some Matrix A , then $A \in G$ or A is not invertible [5]. *The Schur's lemma* introduced the concepts of Lie algebra on the space of Linear maps from W_2 into W_1 , which denoted by $Hom(W_2, W_1)$ [1],[3]. Also introduced the concepts of action on Hom – space of two representations of *Lie algebra* [1]. Also the main work here is to give a representation of complex Lie group by intertwine these actions (representations) and to give representation by intertwine dual of these actions (representations) and Then generalizing them.

2- The TcoA of complex Lie Groups on Hom - Space

In [2- P327], *the Schur's lemma* introduced the concepts of the action of *Lie algebra* on *Hom space* of Two representations of Lie algebra.

Lemma (2.1) [2]:

Suppose that π_1 and π_2 are two representations of *Lie algebra* g action on finite dimensional space W_1 and W_2 respectively . Define an Co-action of g on $Hom_k(W_2, W_1)$, $\pi : g \rightarrow gl(Hom_k(W_2, W_1))$ for all $x \in g$, $F \in Hom_k(W_2, W_1)$, $\pi(x)F = F\pi_2(x) - \pi_1(x)F$ and $Hom(W_2, W_1) = W_2^* \otimes W_1$ as equivalence of representations.

Lemma (2.2):

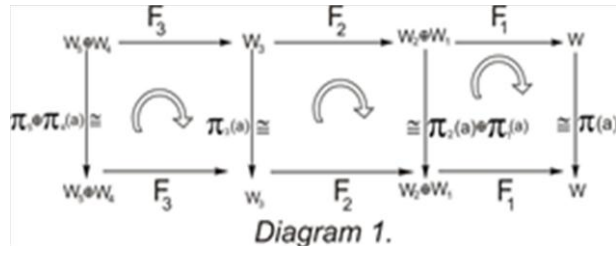
Put $Hom(Hom(W_5 \oplus W_4), W_3^*)$, $(Hom(W_2 \oplus W_1, W^*))$ the K – vector – space of all Linear maps $(Hom(W_5 \oplus W_4), W_3^*)$ onto $(Hom(W_2 \oplus W_1, W^*))$.

Define $\pi : G \rightarrow GL(Hom(Hom(W_5 \oplus W_4), W_3^*), (Hom(W_2 \oplus W_1, W^*)))$,
 by $\pi(a) = \pi^*(a) \circ F_1 \circ (\pi_2(a) \oplus \pi_1(a))^{-1} \circ F_2 \circ (\pi_3(a) \circ F_3 \circ (\pi_5(a) \oplus \pi_4(a))^{-1})$,
 for all $a \in G$, $F_1 \in Hom(W_2^* \oplus W_1, W)$

$F_2 \in Hom(W_5 \oplus W_4, W_3^*)$

$F_3 \in Hom(Hom(W_5 \oplus W_4, W_3^*), Hom(W_2 \oplus W_1, W^*))$.

$$\pi(a)_v = \pi(a) \circ F_1 \circ (\pi_2(a) \oplus \pi_1(a))^{-1} \circ F_2 \circ (\pi_3(a) \circ F_3 \circ (\pi_5(a) \oplus \pi_4(a))^{-1})_v$$



For all $a \in G, v \in (W_5 \oplus W_4)$
 $\pi(a)_v = \pi(a) \circ F_1 \circ (\pi_2(a)^{-1} \oplus \pi_1(a)^{-1}) \circ F_2 \circ (\pi_3(a) \circ F_3 \circ (\pi_5(a)^{-1} \oplus \pi_4(a)^{-1}))_v$

For all $a \in G, v \in (W_5 \oplus W_4)$.
 Where the arrow that makes the diagram 1 commutative π is *homomorphism* of groups

G into $GL((Hom(W_5 \oplus W_4), W_3^*), (Hom(W_2 \oplus W_1, W^*)))$.

Let $\pi_i : G \rightarrow GL(W_i)$, and $\pi_i^* : G \rightarrow GL(W_i^*)$, for $i = 1, 2, 3, 4, 5$.

The *TAS* of complex Lie group G on $Hom_k(Hom((W_5 \oplus W_4), W_3^*), Hom(W_2 \oplus W_1, W^*))$

is given by a representation π such that $\pi(a) = \pi(a) \circ F_1 \circ (\pi_2(a)^{-1} \oplus \pi_1(a)^{-1}) \circ F_2 \circ (\pi_3(a) \circ F_3 \circ (\pi_5(a)^{-1} \oplus \pi_4(a)^{-1}))$,

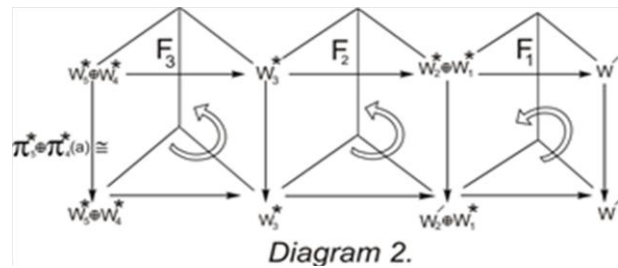
For all $a \in G$.
 Then the *TAS* of complex Lie group G on $Hom_k(Hom((W_5 \oplus W_4), W_3^*), Hom(W_2 \oplus W_1, W^*))$ is also given by representation π^* such that

$$\pi^*(a) = \pi^*(a) \circ F_1 \circ (\pi_2^*(a)^{-1} \oplus \pi_1^*(a)^{-1}) \circ F_2 \circ (\pi_3^*(a) \circ F_3 \circ (\pi_5^*(a)^{-1} \oplus \pi_4^*(a)^{-1}))$$

Proof of Lemma (2.2):

Let *TCoA* of complex Lie group G on $Hom(Hom(W_5 \oplus W_4), W_3^*), Hom(W_2 \oplus W_1, W)$ is induced by the representation

$$\pi : G \rightarrow GL(Hom(Hom(W_5 \oplus W_4, W_3^*), Hom(W_2 \oplus W_1, W^*)))$$



such that $\pi(a) = \pi_1(a) \circ F_1 \circ (\pi_2(a)^{-1} \oplus \pi_1(a)^{-1}) \circ F_2 \circ (\pi_3(a) \circ F_3 \circ \pi_5(a)^{-1} \oplus \pi_4(a)^{-1})$

For all $a \in G$. Thus π^* is a representation from G to $Hom - space$

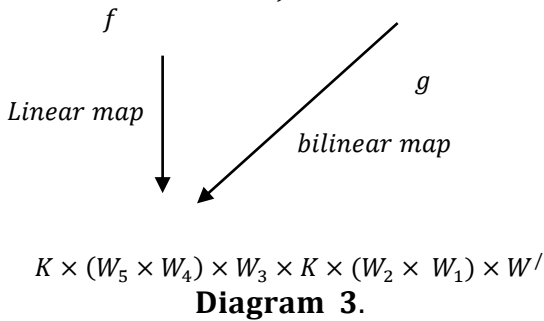
This arrow makes the diagram 2 commutative .

Remark (2.4):

Since $Hom(Hom(W_5 \oplus W_4, W_3^*), Hom(W_2 \oplus W_1, W^*)) \cong ((W_5 \oplus W_4, W_3^*) \otimes ((W_2 \oplus W_1)^* \otimes W^*)) \cong ((W_5 \oplus W_4) \otimes W_3) \otimes ((W_2 \oplus W_1) \otimes W^*)$

So we construct an action of G on the product , Let $\pi(G) \rightarrow GL(((W_5 \oplus W_4) \otimes W_3) \otimes ((W_2 \oplus W_1) \otimes W^*))$, then π forms a representation of G acting on vector space

$$((W_5 \oplus W_4) \otimes W_3) \otimes ((W_2 \oplus W_1) \otimes W^*) \xrightarrow{\text{canianical map}} (W_5^* \times W_4^*) \times W_3 \times (W_2^* \times W_1^*) \times W'$$

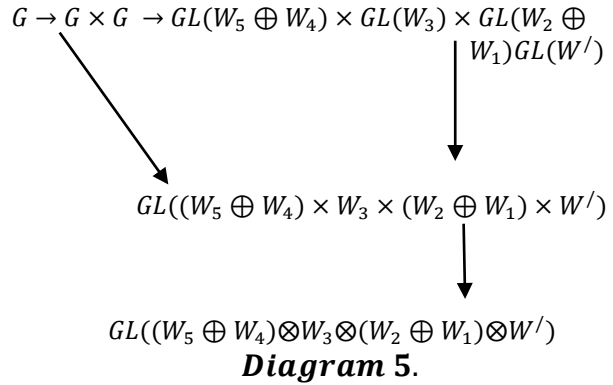
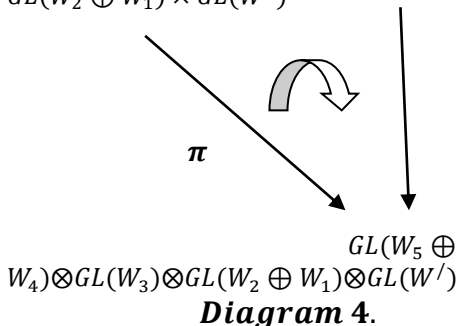


$$f((W_5^* \times W_4^*) \times W_3 \times (W_2^* \times W_1^*) \times W') = w' \circ (w_2^* \oplus w_1^*) \circ w_3 \circ (w_5^* \times w_4^*)$$

$$g((W_5^* \oplus W_4^*) \otimes W_3 \otimes (W_2^* \oplus W_1^*) \otimes W') = w' \circ (w_2^* \oplus w_1^*) \circ w_3 \circ (w_5^* \times w_4^*)$$

For all $w_5 \in W_5, w_5 \in W_5, w_4 \in W_4, w_3 \in W_3, w_2 \in W_2, w_1 \in W_1, w' \in W'$.

$$G \xrightarrow{\Delta} G \times G \xrightarrow{\pi_3 \pi_2 \pi_1} GL(W_5 \oplus W_4) \times GL(W_3) \times GL(W_2 \oplus W_1) \times GL(W')$$



That π is a representation of G acting on $GL((W_5 \oplus W_4) \otimes W_3 \otimes (W_2 \oplus W_1) \otimes W')$ where $\pi_i, i = 1,2,3,4,5$ are five representations of G acting on $W_i, i = 1,2,3,4,5$ respectively, thus :

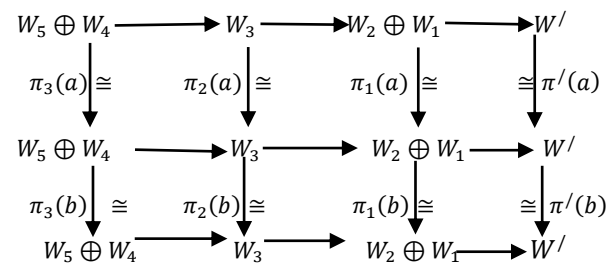
$$\pi(ab) = \pi'(ab) \circ W' \otimes (W_2 \oplus W_1) \otimes W_3 \otimes (W_5 \oplus W_4)$$

$$\pi'(ab) \otimes \pi_1'(ab) \circ (W')_{(v)} \circ \pi_2(ab)^{-1} (W_2 \oplus W_1)_{(v)} \circ \pi_3(ab) W_3(v) \circ \pi_4(ab)^{-1} (W_5 \oplus W_4)_{(v)} = \pi_4(ab)^{-1} \oplus \pi_4(ab) \otimes W_3(ab)^{-1} \otimes \pi_2(ab) \otimes \pi_2(ab) \otimes \pi_1(ab)^{-1} \otimes \pi'(ab)$$

$$\pi(a) \circ \pi(b) = \pi(b)(\pi(a)) = \pi(b)(\pi'(a) \circ (W')_{(v)} \circ \pi_1(a) (W_2 \oplus W_1)_{(v)} \circ \pi_2(a) W_3(v) \circ \pi_3(a) (W_5 \oplus W_4)_{(v)} = \pi'(b) (\pi'(a) \circ (W')_{(v)} \circ \pi_1(b) \pi_1(a) (W_2 \oplus W_1)_{(v)} \circ \pi_2(b) \pi_2(a) W_3(v) \circ \pi_3(b) \pi_3(a) (W_5 \oplus W_4)_{(v)} = \pi'(ab) \circ W' \otimes (W_2 \oplus W_1) \otimes W_3 \otimes (W_5 \oplus W_4)$$

$$\pi(ab) = \pi(a) \circ \pi(b)$$

π is a group homomorphism of G on $GL((W_5 \oplus W_4) \otimes W_3 \otimes (W_2 \oplus W_1) \otimes W')$.



3- The TCoA of Complex Lie Groups on Tensor Product

We have been introduced the triple Co-action of complex Lie groups by the tensor product of the five representations, which are TCoA-complex Lie groups on tensor product

$(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))))$ and constructed this definition. Depending on what has been mentioned above, Π is called Triple Co-Action of complex Lie groups on tensor product denoted by (TCOA-complex Lie groups).

Example (3.1):

Let $\Pi_1: \mathbb{R} \longrightarrow GL(2, \mathbb{C})$ such that $\Pi_1(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_{2 \times 2}$, for all $a \in \mathbb{R}$, $\Pi_2: \mathbb{R} \longrightarrow$

$GL(2, \mathbb{C})$ such that $\Pi_2(a) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}_{2 \times 2}$, for all $a \in$

\mathbb{R} , $\Pi_3: \mathbb{R} \longrightarrow GL(2, \mathbb{C})$ such that $\Pi_3(a) = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}_{2 \times 2}$, for all $a \in \mathbb{R}$, $\Pi_4: \mathbb{R} \longrightarrow$

$GL(2, \mathbb{C})$ such that $\Pi_4(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_{2 \times 2}$, for all $a \in$

\mathbb{R} and $\Pi_5: \mathbb{R} \longrightarrow GL(3, \mathbb{C})$ such that

$$\Pi_5(a) = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{3 \times 3}, \text{ for all } a \in \mathbb{R}.$$

The representation Π of $GL(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))))$ is:

$\Pi: G \longrightarrow GL(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1)))) \cong GL(M(8 \times 3), \mathbb{C})$, such that

$\Pi(a) = (((\Pi_1(a) \otimes \Pi_3^*(a)^{-1} \oplus \Pi_2(a) \otimes \Pi_3^*(a)^{-1}) \otimes \Pi_4(a)) \otimes \Pi_5^*(a)^{-1})$, where Π^* is dual representation

$$= \left(\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} \oplus \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} \right) \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \otimes \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}$$

$$= \left(\left(\begin{pmatrix} 1 & 0 & -2 & 0 \\ 1 & 1 & -2 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}_{4 \times 4} \oplus \begin{pmatrix} 1 & 2 & -2 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}_{4 \times 4} \right) \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_{2 \times 2} \right) \otimes \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}_{3 \times 3}$$

$$= \left[\begin{pmatrix} 2 & 2 & -4 & -4 \\ 1 & 2 & -2 & -4 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 4 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_{2 \times 2} \right] \otimes \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}_{3 \times 3}$$

$$= \begin{pmatrix} 2 & 2 & -4 & -4 & 0 & 0 & 0 & 0 \\ 1 & 2 & -2 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 & 0 & 0 \\ 2 & 2 & -4 & -4 & 2 & 2 & -4 & -4 \\ 1 & 2 & -2 & -4 & 1 & 2 & -2 & -4 \\ 0 & 0 & 4 & 4 & 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 4 & 0 & 0 & 2 & 4 \end{pmatrix}_{8 \times 8} \otimes \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}_{3 \times 3}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -4 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2 & -4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -4 & -4 & 2 & 2 & -4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2 & -4 & 1 & 2 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 0 & 2 & 4 \\ -1 & -1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & 2 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 2 & -1 & -1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & 2 & -1 & -1 & 2 & 2 \\ -\frac{1}{2} & -1 & 1 & 2 & -\frac{1}{2} & -1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -1 & 1 & 2 & -\frac{1}{2} & -1 & 1 & 2 \\ 0 & 0 & -2 & -2 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 0 & 0 & -2 & -2 \\ 0 & 0 & -1 & -2 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & -1 & -2 \\ -1 & -1 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & -4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & -2 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -1 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & -2 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 2 & -1 & -1 & 2 & 2 & 2 & 2 & -4 & -4 & 2 & 2 & -4 & -4 & 1 & 1 & -2 & -2 & 1 & 1 & -2 & -2 & 1 & 1 & -2 & -2 \\ -\frac{1}{2} & -1 & 1 & 2 & -\frac{1}{2} & -1 & 1 & 2 & 1 & 2 & -2 & -4 & 1 & 2 & -2 & 0 & -3 & 3 & 0 & -\frac{3}{2} & \frac{3}{2} & 0 & \frac{3}{2} & -\frac{3}{2} & \frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 6 & 0 & 0 & -3 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 & 0 & -3 & 3 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & -\frac{3}{2} & -\frac{3}{2} \end{pmatrix}_{24 \times 24}$$

Proposition (3.2):

Let $\Pi_i: G \longrightarrow GL(W_i)$, $\Pi_i^*: G \longrightarrow GL(W_i^*)$ for $i = 1, 2, 3, 4, 5$ and the TCOA-complex Lie groups of G on $(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))))$ is given by a representation Π such that $\Pi(a) = [(\Pi_1(a) \circ W_1 \circ W_3^* \circ \Pi_3^*(a^{-1})) \oplus (\Pi_2(a) \circ W_2 \circ W_3^* \circ \Pi_3^*(a^{-1}) \circ \Pi_4(a) \circ W_4 \circ W_5^* \circ \Pi_5^*(a^{-1})]$, for all $a \in G$.

Then the *TCoA*-complex Lie group of G on $(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))))$ is also given by a representation Π^* , such that:

$$\Pi^*(a) = \Pi_5^*(a)^{-1} \circ W_5 \circ W_3^* \circ (\Pi_4^*(a) \circ (\Pi_3^*(a)^{-1} \circ W_3 \circ W_1^* \circ \Pi_1^*(a)))$$

for all $a \in G$.

Proof: Let *TCoA*-complex Lie group G on $(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))))$ is induced by the representation $\Pi: G \rightarrow GL((W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))))$ such that

$$\Pi(a) = (((\Pi_1(a) \circ W_1 \circ W_3^* \circ \Pi_3(a^{-1})) \oplus (\Pi_2(a) \circ W_2 \circ W_3^* \circ \Pi_3(a^{-1})) \circ \Pi_4(a) \circ W_4 \circ W_5^* \circ \Pi_5(a^{-1})),$$

for all $a \in G$, $W_3' \times W_1' \in W_3 \times W_1$, $\Pi_3 \in (W_3, (W_2 \times W_1))$, $\Pi_4 \in W_3 \times W_4$.

To show that $\Pi^*: G \rightarrow GL(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))))^*$ is a representation, such that

$$\Pi^*(a) = (((\Pi_5^*(a)^{-1} \circ W_5 \circ W_4^* \circ (\Pi_4^*(a) \circ (\Pi_3^*(a)^{-1} \circ W_3 \circ W_2^* \circ \Pi_2^*(a) \oplus \Pi_3^*(a)^{-1} \circ W_3 \circ W_1^* \circ \Pi_1^*(a))))$$

is a representation for all $a \in G$ and $\Pi_4^* \in (W_5^* \otimes W_4)^*$, $\Pi_3^* \in (W_3, (W_2 \otimes W_1))^*$,

$\Pi_2^* \times \Pi_1^* \in (W_2 \otimes W_1)^*$ since

$$\Pi^*(a) = \Pi_5^*(a)^{-1} \circ W_5 \circ W_4^* \circ (\Pi_4^*(a) \circ ((\Pi_3^*(a)^{-1} \circ W_3 \circ W_2^* \circ \Pi_2^*(a) \oplus \Pi_3^*(a)^{-1} \circ W_3 \circ W_1^* \circ \Pi_1^*(a)))$$

For all $a \in G$, $\Pi_4^*: W_4^* \rightarrow W_5^*$ and

$$\Pi^*(ab) = (\Pi(ab))^* = (\Pi(b) \circ \Pi(a))^* =$$

$\Pi^*(a) \circ \Pi^*(b)$. Thus Π^* is a representation from G (Π^* is a group homomorphism of G)

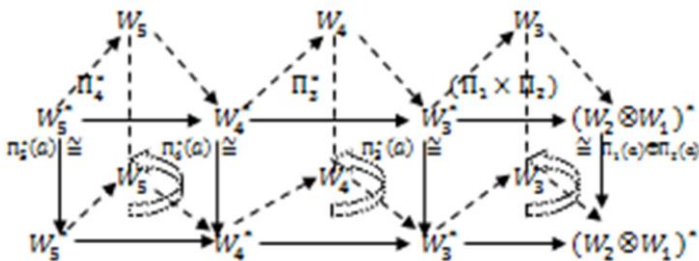


Diagram 7.

This arrow makes the diagram 7 commutative.

Proposition (3.3):

Let W_i for $i = 1, 2, 3, 4, 5$ are finite vector spaces, W_i^* is the dual of vectors W_i , for $i = 1, 2, 3, 4, 5$ then the following assertions are equivalent.

- (1) $[(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))))]^*$.
- (2) $((W_3^* \otimes W_2)^* \oplus (W_3^* \otimes W_1)^*) \otimes W_4^* \otimes W_5^{**}$.
- (3) $(W_2^* \otimes W_3) \oplus (W_1^* \otimes W_3) \otimes W_4^* \otimes W_5$.
- (4) $((W_2^* \otimes W_1^*) \otimes W_3) \otimes W_4^* \otimes W_5$.
- (5) $((W_2^* \otimes W_1^*) \otimes (W_3, K) \otimes W_4^*) \otimes W_5$.
- (6) $((W_2^* \otimes W_1^*) \otimes W_3) \otimes W_4^* \otimes (W_5^{**}, K)$.
- (7) $((W_1 \otimes W_2)^* \otimes W_4^*) \otimes (W_3, K) \otimes W_4^* \otimes W_5$.
- (8) $[(W_5 \otimes (W_4 \otimes (W_3 \otimes W_2) \oplus (W_3 \otimes W_1)))]^{***}$

$$= \begin{cases} (W_5 \otimes (W_4 \otimes (W_3 \otimes W_2) \oplus (W_3 \otimes W_1))) & \text{if } n \text{ is an even number} \\ (W_3 \otimes W_2)^* \otimes (W_3 \otimes W_1)^* \otimes W_4^* \otimes W_5 & \text{if } n \text{ is an odd number} \end{cases}$$

Proof:

(1) \cong (2) To show $[(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))))]^* \cong$

$$(((W_3^* \otimes W_2)^* \oplus (W_3^* \otimes W_1)^*) \otimes W_4^* \otimes W_5^{**}).$$

Let $\Pi_4^* \in (W_5^* \otimes (W_4))^*$, $\Pi_3^* \in (W_3 \otimes (W_2 \otimes W_1))^*$,

$\Pi_2^* \times \Pi_1^* \in (W_3^* \otimes (W_2 \otimes W_1))^*$, $\Pi_4^* \in (W_4^* \otimes W_5)$,

$\Pi_3^* \in (W_3, W_4^*)$, $\Pi_1^* \times \Pi_2^* = (\Pi_2^* \times \Pi_1^*)^* \in ((W_1^* \otimes W_2^*) \otimes W_3)$

and there exists an intertwining map

$$\psi: (W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1)))) \rightarrow ((W_3^* \otimes W_2)^* \oplus (W_3^* \otimes W_1)^*) \otimes W_4^* \otimes W_5^{**},$$

such that

$$\psi(\Pi^*(a)(v)) = \Pi^*(a)\psi(v), \text{ for all } v \in W_1^* \times W_2^*$$

and ψ is an invertible map.

(1) \cong (3) To show $(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus$

$(W_3^* \otimes W_1)))^* \cong$

$((W_2^* \otimes W_3^*) \oplus (W_1^* \otimes W_3^*) \otimes W_4^*) \otimes W_5$, since

$$(W_3^*, W_2)^* \cong (W_2^*, W_3^{**}), \quad (W_3^*, W_1)^* \cong$$

$$(W_1^*, W_3^{**}), \quad W_5^{**} \cong W_5 \text{ and } W_3^{**} \cong W_3.$$

By the same methods, we have the other parts. ■

Example (3.4):

Let $\Pi_i, i = 1,2,3,4, \Pi_i: S^1 \longrightarrow SO(2) \subset GL(2, \mathbb{C})$ and $\Pi_5: S^1 \longrightarrow O(3) \subset GL(3, \mathbb{C})$, where $G = S^1, (n = 2, m = 3)$ and $W_i, i = 1,2,3,4$ are the \mathbb{C} -vector spaces of dimensional 2 and W_5 is the \mathbb{C} -vector space of dimensional 3 such that,

$$\Pi_1(e^{i\theta}) = \Pi_2(e^{i\theta}) = \Pi_3(e^{i\theta}) = \Pi_4(e^{i\theta}) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$, 0 \leq \theta \leq 2\pi, i^2 = -1, \Pi_5(e^{i\theta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}, 0$$

$\leq \theta \leq 2\pi$. The *TCOA*-complex Lie group G on $(W_5^*$

$\otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1)))$ is a representation:

$\Pi: S^1 \longrightarrow GL(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1)))$ such that

$$\Pi^*(a) = (\Pi_5(a)^{-1} \circ W_5 \circ W_4 \circ (\Pi_4(a) \circ ((\Pi_3(a)^{-1} \circ W_3 \circ W_2 \circ \Pi_2(a) \oplus (\Pi_1(a)^{-1} \circ W_1 \circ W_2 \circ \Pi_2(a))))$$

$$\Pi^*(e^{i\theta}) = \Pi_5(e^{-i\theta}) \otimes (\Pi_4^*(e^{i\theta}) \otimes ((\Pi_3(e^{-i\theta}) \otimes \Pi_2^*(e^{i\theta})) \oplus (\Pi_1(e^{-i\theta}) \otimes \Pi_2^*(e^{i\theta})))$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \otimes \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \otimes \left(\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \otimes \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \right) \oplus$$

$$\left(\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \otimes \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \right)$$

$$= \begin{pmatrix} \cos \theta & 0 & 0 & -\sin \theta & 0 & 0 \\ 0 & \cos^2 \theta & -\sin \theta \cos \theta & 0 & -\sin \theta \cos \theta & \sin^2 \theta \\ 0 & \sin \theta \cos \theta & \cos^2 \theta & 0 & -\sin^2 \theta & -\sin \theta \cos \theta \\ \sin \theta & 0 & 0 & \cos \theta & 0 & 0 \\ 0 & \sin \theta \cos \theta & -\sin^2 \theta & 0 & \cos^2 \theta & -\sin \theta \cos \theta \\ 0 & \sin^2 \theta & \sin \theta \cos \theta & 0 & \sin \theta \cos \theta & \cos^2 \theta \end{pmatrix} \otimes$$

$$\left[\begin{pmatrix} \cos^2 \theta & \sin \theta \cos \theta & -\sin \theta \cos \theta & -\sin^2 \theta \\ -\sin \theta \cos \theta & \cos^2 \theta & \sin^2 \theta & -\sin \theta \cos \theta \\ \sin \theta \cos \theta & \sin^2 \theta & \cos^2 \theta & \sin \theta \cos \theta \\ -\sin^2 \theta & \sin \theta \cos \theta & -\sin \theta \cos \theta & \cos^2 \theta \end{pmatrix} \oplus$$

$$\left[\begin{pmatrix} \cos^2 \theta & \sin \theta \cos \theta & -\sin \theta \cos \theta & -\sin^2 \theta \\ -\sin \theta \cos \theta & \cos^2 \theta & \sin^2 \theta & -\sin \theta \cos \theta \\ \sin \theta \cos \theta & \sin^2 \theta & \cos^2 \theta & \sin \theta \cos \theta \\ -\sin^2 \theta & \sin \theta \cos \theta & -\sin \theta \cos \theta & \cos^2 \theta \end{pmatrix} \right]_{4 \times 4}$$

$$= \begin{pmatrix} \cos \theta & 0 & 0 & -\sin \theta & 0 & 0 \\ 0 & \cos^2 \theta & -\sin \theta \cos \theta & 0 & -\sin \theta \cos \theta & \sin^2 \theta \\ 0 & \sin \theta \cos \theta & \cos^2 \theta & 0 & -\sin^2 \theta & -\sin \theta \cos \theta \\ \sin \theta & 0 & 0 & \cos \theta & 0 & 0 \\ 0 & \sin \theta \cos \theta & -\sin^2 \theta & 0 & \cos^2 \theta & -\sin \theta \cos \theta \\ 0 & \sin^2 \theta & \sin \theta \cos \theta & 0 & \sin \theta \cos \theta & \cos^2 \theta \end{pmatrix}_{6 \times 6} \otimes$$

$$\begin{pmatrix} 2 \cos^2 \theta & 2 \sin \theta \cos \theta & -2 \sin \theta \cos \theta & -2 \sin^2 \theta \\ -2 \sin \theta \cos \theta & 2 \cos^2 \theta & 2 \sin^2 \theta & -2 \sin \theta \cos \theta \\ 2 \sin \theta \cos \theta & 2 \sin^2 \theta & 2 \cos^2 \theta & 2 \sin \theta \cos \theta \\ -2 \sin \theta \cos \theta & 2 \sin^2 \theta & -2 \sin \theta \cos \theta & 2 \cos^2 \theta \end{pmatrix}_{4 \times 4}$$

$$\begin{pmatrix} 0 & 2 \sin \theta \cos^2 \theta & 0 & 0 & -2 \sin^2 \theta \cos \theta & 0 & 0 \\ 0 & 2 \cos^4 \theta & -2 \sin \theta \cos^3 \theta & 0 & -2 \sin^2 \theta \cos^2 \theta & 2 \sin^3 \theta \cos \theta & 0 \\ 0 & 2 \sin \theta \cos^3 \theta & 2 \cos^4 \theta & 0 & -2 \sin^2 \theta \cos^2 \theta & -2 \sin^3 \theta \cos \theta & 0 \\ 2 \sin \theta \cos^2 \theta & 0 & 0 & 2 \cos^4 \theta & 0 & 0 & 2 \sin^2 \theta \cos \theta \\ 0 & 2 \sin \theta \cos^2 \theta & -2 \sin^2 \theta \cos^2 \theta & 0 & 2 \cos^4 \theta & -2 \sin^2 \theta \cos \theta & 0 \\ 0 & 2 \sin^2 \theta \cos^2 \theta & 2 \sin \theta \cos^3 \theta & 0 & 2 \sin \theta \cos^2 \theta & 2 \cos^4 \theta & 0 \\ -2 \sin \theta \cos^2 \theta & 0 & 0 & 2 \sin^2 \theta \cos \theta & 0 & 0 & 2 \cos^4 \theta \\ 0 & -2 \sin \theta \cos^3 \theta & 2 \sin^2 \theta \cos^2 \theta & 0 & 2 \sin^2 \theta \cos^2 \theta & -2 \sin^3 \theta \cos \theta & 0 \\ 0 & -2 \sin^2 \theta \cos^2 \theta & -2 \sin \theta \cos^3 \theta & 0 & -2 \sin^2 \theta \cos \theta & 2 \cos^4 \theta & 0 \\ -2 \sin \theta \cos^2 \theta & 0 & 0 & -2 \sin \theta \cos^2 \theta & 0 & 0 & 2 \cos^4 \theta \\ 0 & -2 \sin^2 \theta \cos \theta & 2 \sin^3 \theta \cos & 0 & -2 \sin \theta \cos^2 \theta & 2 \sin^2 \theta \cos \theta & 0 \\ 0 & -2 \sin^2 \theta \cos \theta & -2 \sin^2 \theta \cos^2 \theta & 0 & -2 \sin^2 \theta \cos \theta & 2 \cos^4 \theta & -2 \sin \theta \cos^2 \theta \\ 2 \sin \theta \cos^2 \theta & 0 & 0 & -2 \sin^2 \theta \cos \theta & 0 & 0 & 2 \sin^2 \theta \cos \theta \\ 0 & -2 \sin \theta \cos^3 \theta & -2 \sin^2 \theta \cos^2 \theta & 0 & -2 \sin^2 \theta \cos \theta & 2 \sin^3 \theta \cos \theta & 0 \\ 0 & 2 \sin^2 \theta \cos^2 \theta & 2 \sin \theta \cos^3 \theta & 0 & -2 \sin^2 \theta \cos \theta & -2 \sin^2 \theta \cos \theta & 0 \\ 2 \sin^2 \theta \cos \theta & 0 & 0 & 2 \sin \theta \cos^2 \theta & 0 & 0 & -2 \sin^2 \theta \cos \theta \\ 0 & 2 \sin^2 \theta \cos^2 \theta & -2 \sin^2 \theta \cos \theta & 0 & 2 \sin \theta \cos^2 \theta & -2 \sin^2 \theta \cos \theta & 0 \\ 0 & 2 \sin^2 \theta \cos \theta & 2 \sin^2 \theta \cos^2 \theta & 0 & 2 \sin^2 \theta \cos \theta & 2 \sin^2 \theta \cos \theta & 0 \\ 2 \sin^2 \theta \cos \theta & 0 & 0 & 2 \sin^2 \theta & 0 & 0 & -2 \sin^2 \theta \cos \theta \\ 0 & -2 \sin^2 \theta \cos^2 \theta & 2 \sin^3 \theta \cos \theta & 0 & 2 \sin^2 \theta \cos^2 \theta & -2 \sin^2 \theta \cos \theta & 0 \\ 0 & -2 \sin^2 \theta \cos \theta & -2 \sin^2 \theta \cos^2 \theta & 0 & 2 \sin^2 \theta \cos \theta & 2 \sin \theta \cos^2 \theta & 0 \\ -2 \sin^2 \theta & 0 & 0 & -2 \sin^2 \theta \cos \theta & 0 & 0 & 2 \sin \theta \cos^2 \theta \\ 0 & -2 \sin^2 \theta \cos \theta & 2 \sin^4 \theta & 0 & -2 \sin^2 \theta \cos^2 \theta & 2 \sin^3 \theta \cos \theta & 0 \\ 0 & -2 \sin^2 \theta & -2 \sin^2 \theta \cos \theta & 0 & -2 \sin^2 \theta \cos \theta & 2 \sin^2 \theta \cos^2 \theta & 0 \end{pmatrix}$$

$-2\sin^2\theta\cos^2\theta$	0	0	$2\sin^2\theta\cos^2\theta$	0	0	$-2\sin^2\theta\cos\theta$	0	0	$2\sin^2\theta$	0	0
0	$-2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos^2\theta$	0	$2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos\theta$	0	$-2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos\theta$	0	$2\sin^2\theta\cos\theta$	$-2\sin^4\theta$
0	$-2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$	0	$-2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos^2\theta$	0	$-2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$	0	$2\sin^4\theta$	$2\sin^2\theta\cos\theta$
$-2\sin^2\theta\cos\theta$	0	0	$-2\sin^2\theta\cos^2\theta$	0	0	$-2\sin^2\theta$	0	0	$-2\sin^2\theta\cos\theta$	0	0
0	$-2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos\theta$	0	$-2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos^2\theta$	0	$-2\sin^2\theta\cos\theta$	$2\sin^4\theta$	0	$-2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos\theta$
0	$-2\sin^2\theta\cos\theta$	$-2\sin^2\theta\cos^2\theta$	0	$-2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$	0	$-2\sin^4\theta$	$-2\sin^2\theta\cos\theta$	0	$-2\sin^2\theta\cos\theta$	$-2\sin^2\theta\cos^2\theta$
$2\sin^2\theta\cos\theta$	0	0	$-2\sin^2\theta$	0	0	$-2\sin^2\theta\cos^2\theta$	0	0	$2\sin^2\theta\cos\theta$	0	0
0	$2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos\theta$	0	$-2\sin^2\theta\cos\theta$	$2\sin^4\theta$	0	$-2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos^2\theta$	0	$-2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos\theta$
0	$2\sin^2\theta\cos\theta$	$2\sin^2\theta\cos^2\theta$	0	$-2\sin^4\theta$	$-2\sin^2\theta\cos\theta$	0	$-2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$	0	$-2\sin^2\theta\cos\theta$	$2\sin^2\theta\cos^2\theta$
$2\sin^2\theta$	0	0	$2\sin^2\theta\cos\theta$	0	0	$-2\sin^2\theta\cos\theta$	0	0	$-2\sin^2\theta\cos^2\theta$	0	0
0	$2\sin^2\theta\cos\theta$	$-2\sin^4\theta$	0	$2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos\theta$	0	$-2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos\theta$	0	$-2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos^2\theta$
0	$2\sin^4\theta$	$2\sin^2\theta\cos\theta$	0	$2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos\theta$	0	$-2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$	0	$-2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$
$2\cos^2\theta$	0	0	$-2\sin^2\theta\cos^2\theta$	0	0	$-2\sin^2\theta\cos^2\theta$	0	0	$-2\sin^2\theta\cos\theta$	0	0
0	$2\cos^4\theta$	$-2\sin^2\theta\cos^2\theta$	0	$-2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos^2\theta$	0	$-2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$	0	$-2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos\theta$
0	$2\sin^2\theta\cos^2\theta$	$2\cos^4\theta$	0	$-2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$	0	$2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$	0	$-2\sin^2\theta\cos\theta$	$-2\sin^2\theta\cos^2\theta$
$2\sin^2\theta\cos^2\theta$	0	0	$2\cos^2\theta$	0	0	$2\sin^2\theta\cos\theta$	0	0	$2\sin^2\theta\cos^2\theta$	0	0
0	$2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$	0	$2\cos^4\theta$	$-2\sin^2\theta\cos^2\theta$	0	$2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$	0	$2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$
0	$2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos^2\theta$	0	$2\sin^2\theta\cos^2\theta$	$2\cos^4\theta$	0	$2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$	0	$2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos^2\theta$
$-2\sin^2\theta\cos^2\theta$	0	0	$2\sin^2\theta\cos\theta$	0	0	$2\cos^2\theta$	0	0	$-2\sin^2\theta\cos^2\theta$	0	0
0	$-2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos^2\theta$	0	$2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos\theta$	0	$2\cos^2\theta$	0	$-2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$
0	$-2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$	0	$2\sin^2\theta\cos\theta$	$2\sin^2\theta\cos^2\theta$	0	$2\sin^2\theta\cos^2\theta$	$2\cos^4\theta$	0	$-2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$
$-2\sin^2\theta\cos\theta$	0	0	$-2\sin^2\theta\cos^2\theta$	0	0	$2\sin^2\theta\cos^2\theta$	0	0	$2\cos^2\theta$	0	0
0	$-2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos\theta$	0	$-2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos^2\theta$	0	$2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$	0	$2\cos^4\theta$	$-2\sin^2\theta\cos^2\theta$
0	$-2\sin^2\theta\cos\theta$	$-2\sin^2\theta\cos^2\theta$	0	$-2\sin^2\theta\cos^2\theta$	$-2\sin^2\theta\cos^2\theta$	0	$2\sin^2\theta\cos^2\theta$	$2\sin^2\theta\cos^2\theta$	0	$2\sin^2\theta\cos^2\theta$	$2\cos^4\theta$

Proposition (3. 5):

Let $\Pi_i, i = 1,2,3,4,5$ be representations of G acting on K -finite dimensional vector spaces $W_i, i = 1,2,3,4,5$ respectively, then the $TC\phi A$ -reductive Lie group of G on $\text{Hom}_K(W_5, \text{Hom}(W_4^*, \text{Hom}((W_3, W_2) \oplus \text{Hom}(W_3, W_1))))$ is equivalent to the representation $\Pi_5^* \otimes (\Pi_4 \otimes ((\Pi_3^* \otimes \Pi_2) \oplus (\Pi_3^* \otimes \Pi_1)))$ of G on $\text{GL}(W_5^* \otimes (W_4 \otimes ((W_3^*, W_2) \oplus (W_3^*, W_1))))$.

Proof: To show that:

$$\begin{aligned} \psi: (W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1)))) &\longrightarrow \\ \text{Hom}_K(W_5, \text{Hom}(W_4^*, \text{Hom}((W_3, W_2) \oplus & \\ \text{Hom}(W_3, W_1)))) &\text{ is bilinear map, defined by} \\ \psi(W_5^*, w_1) = F &\text{ for all } W_5^* \in W_5 \text{ and } w_1 \in W_1, \\ \text{where } F: W_5 &\longrightarrow W_1 \text{ is a linear map defined by } F(v) \\ = W_5^*(v)w_1, &\text{ for all } W_5^*, W_5^* \in W_5^*, v \in W_5, \\ \alpha, \beta \in K, w_1 \in W_1 & \\ \psi(\alpha W_5^* + \beta W_5^*, w_1) &= (\alpha W_5^* + \beta W_5^*(v))w_1 \\ &= \alpha W_5^*(v)w_1 + \beta W_5^*(v)w_1 \\ &= \alpha\psi(W_5^*, w_1) + \beta\psi(W_5^*, w_1) \end{aligned}$$

Other for all $w_1, W_1' \in W_1$ and $W_5^* \in W_5^*$
 $\psi(\alpha w_1 + \beta W_1') = (W_5^*(v)(\alpha w_1 + \beta W_1'))$

$$\begin{aligned} &= W_5^*(v)(\alpha w_1) + W_5^*(v)(\beta W_1') \\ &= \alpha W_5^*(v)w_1 + \beta W_5^*(v)W_1' \end{aligned}$$

$$\psi(W_5^*, \alpha w_1 + \beta W_1') = \alpha\psi(W_5^*, w_1) + \beta\psi(W_5^*, W_1').$$

So $\psi: W_5^* \times (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))) \longrightarrow$

$\text{Hom}_K(W_5, \text{Hom}(W_4^*, \text{Hom}((W_3, W_2) \oplus \text{Hom}(W_3, W_1))) \oplus \text{Hom}(W_3, W_1))$ is a bilinear map, thus by using the tensor product and universal property of this tensor product, we get a unique linear map ϕ .

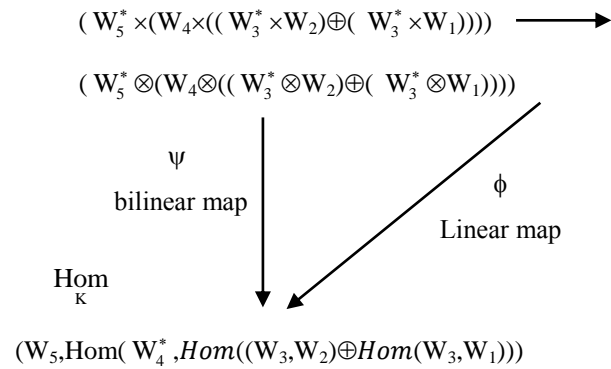


Diagram 8.

So by universal property of tensor product $W_5^* \times (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1)))$ there exists a unique linear map $\phi: W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))) \longrightarrow \text{Hom}_K(W_5, \text{Hom}(W_4^*, \text{Hom}((W_3, W_2) \oplus \text{Hom}(W_3, W_1))))$. This makes the above diagram commutative:

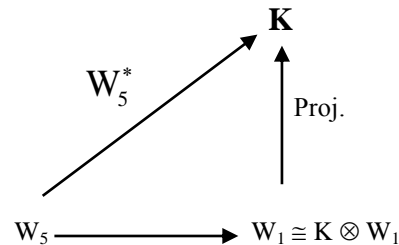


Diagram 9.

Consider the composition of linear maps where $W_5^*(v)$ is defined as follows:
 $F(v) = w_1, \exists! k \in K$, such that $w_1 \longrightarrow (k, w_1)$ since all maps are linear and k is unique, put $W_5^*(v) = k$ related to w_1 .

Define $\zeta: \text{Hom}_K(W_5, \text{Hom}(W_4^*, \text{Hom}((W_3, W_2) \oplus \text{Hom}(W_3, W_1)))) \longrightarrow W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1)))$ by $\zeta(F') = W_5^*(v)w_1$.

Define $W_5^*: W_5 \longrightarrow K$ by $W_5^*(v) = k$, where k is given by $\zeta(F'(v)) = (k, F'(v))$

We can show that W_5^* is linear put $F'(v) = w_1$, for all $F' \in \text{Hom}_K(W_5, \text{Hom}(W_4^*, \text{Hom}((W_3, W_2) \oplus \text{Hom}(W_3, W_1))))$, $w_1 \in W_1$ and $w_5^* \in W_5$ and is related to W_1 .

$$\begin{aligned} F'(\alpha v_1 + \beta v_2) &= \alpha F'(v_1) + \beta F'(v_2) \\ &= \alpha k_1 + \beta k_2 \\ &= \alpha W_5^*(v_1) + \beta W_5^*(v_2), \text{ for all } W_5^* \in W_5 \end{aligned}$$

Where: $W_5^*(v_1) = k_1 \Rightarrow W_5^*(\alpha v_1) = \alpha k_1$,

$$W_5^*(\alpha v_1 + \beta v_2) = (\alpha k_1 + \beta k_2)$$

$$W_5^*(v_2) = k_2 \Rightarrow W_5^*(\beta v_2) = \beta k_2,$$

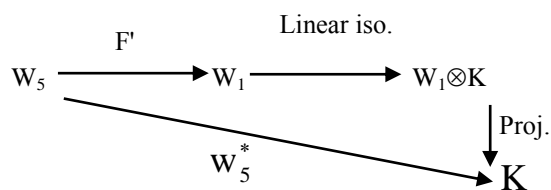


Diagram 10.

Clear F' is a linear and $\zeta^{-1} = \phi$, thus ζ is linear map. Related between the $TCOA$ of reductive Lie groups of G on $\text{Hom}_K(W_5, \text{Hom}(W_4^*, \text{Hom}((W_3, W_2) \oplus \text{Hom}(W_3, W_1))))$ and $TCOA$ of reductive Lie groups of G on $W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1)))$ up to the representation given above:

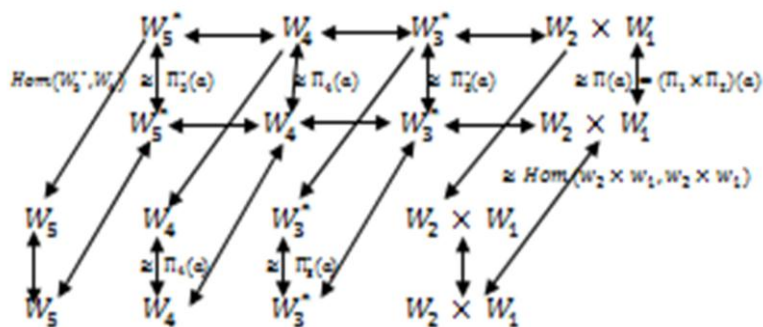


Diagram 11.

References

- [1] Hussian, M.J. “ Local Property of the Lie Algebra and the Exponential Map”, M.Sc. Thesis, college of Education, Al – Mustansiriya University, 2010.
- [2] Jacobson, N., “Basic Algebra II”, Second Edition, Yale University, 2012.
- [3] Kac, V., “Introduction to Lie Algebras”, Vol. 18.745, Patrick Lam, 2004.
- [4] Kirillov, A.Jr., “Introduction to Lie Groups and Lie Algebras”, Vol.11794, Sunt at stony Brook, 2004.
- [5] Knapp, A.W., “Lie Groups Beyond an Introduction”, Progress in Mathematics 140, Birkhauser Boston Inc. Boston, MA, 2002.
- [6] Lee, D.H., “The Structure of Complex Lie Groups”, New York, ISBN, May, 2001.
- [7] Majeed, T.H., “The Universal Property of Tensor Product for Representation of Lie Groups”, Ph.D. Thesis, college of Education, Al-Mustansiriya University, 2012.
- [8] Nadjafikhah, M. and Chamazkoti, R., “The Special Linear Representations of Compact Lie Groups”, Mathematics Science, Vol.4, No.3, Iran, 2010.
- [9] Onishchik, A.L., “Lie Algebra, Reductive”, Hazewinkel, Michiel, Encyclopedia of Mathematics, Springer, ISBN 978-1-55608-010-4, 2001.
- [10] Popov, V.L., “Reductive Group”, Hazewinkel, Michiel, Encyclopedia of Mathematics, Springer, ISBN 968-1-55608-010-4, 2001.

أنواع معينة لفعال زمري المركبة

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المستخلص :

الهدف الرئيسي من البحث هو الحصول على فعل بصفات جديدة في زمري المركبة من بديهية *Schure* التي درست وركزت على فعل جبر لي لتمثيلين احدهما عادي والاخر ثنائي ، والشئ المهم والممتع في العمل هو التركيز على بعض الأفعال لزمرة لي المركبة. في هذه الدراسة قمنا بتحليل مفاهيم من فعل زمري المركبة على فضاءات *Hom* وتعريف الضرب التنسوري لتمثيلات اثنتين في زمري لي وركزنا على فعل زمري لي على فضاءات *Hom* ، باستخدام التكافؤ

$$Hom(w_2, w_1) \cong w_2^* \otimes w_1$$

بين فضاءات *Hom* والضرب التنسوري للحصول على فعل زمري لي المركبة على الضرب التنسوري. الفعل الثاني هو بصيغة تمثيلات لمساء للمجموعة *G* . هذا الفعل هو فعل ثلاثي لزمري لي المركبة *G* ويرمز لها (فعل زمري لي المركبة *TAC*) على

$$Hom(Hom((w_5 \oplus w_4), w_3^*), Hom(w_2 \oplus w_1, w^*)).$$

وهذا *TAC* هو تمثيلات لمساء في *G* ، أن النظريات المقدمة في هذا البحث انشأت وبرهنت وجهزت ببعض النتائج كملاحظات ورسوم.