Maximum Likelihood and Moment Estimation for the Exponentiated Pareto Distribution Based on Fuzzy Data

تقديرات الإمكان الأعظم والعزوم لتوزيع باريتو الاسي بالاعتماد على بيانات ضبابية

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Abstract:
Maximum likelihood and moment methods of estimation are used for estimating the shape parameter β reliability and hazard functions of exponentiated pareto distribution based on fuzzy data. Using the Monte-Carlo simulation for comparison the moment and maximum likelihood estimators of shape parameter , reliability and hazard functions according to the MSE values with four different cases, initial values of shape parameter and three sample sizes.
Keywords: exponentiated pareto distribution; maximum likelihood estimators; reliability function; moment method; mean squared errors; fuzzy numbers.

المستخلص:
طرق تقدير الإمكان الاعظم والعزوم استخدمت لتقدير معلمة الشكل β ودوال المعولية والخطورة لتوزيع باريتو الاسي بالاستناد الى البيانات الضبابية .وتم استخدام دراسة محاكاة مونتي-كارلو لمقارنة تقديرات الامكان الاعظم والعزوم لمعلمة الشكل ودوال المعولية والخطورة وفقا لقيم متوسط مربعات الخطا لاربع حالات مختلفة وقيم ابتدائية لمعلمة القياس لثلاثة من حجوم العينه.

1. Introduction: In non -Bayesian estimation procedures usually assumed that available data are accurate real numbers. However although, some collected data might be inaccurate and are shown in the form of fuzzy numbers.

Gupta et al. (1998)[1] was introduced the Exponentiated Pareto Distribution, EPD as a survival, time model. The EPD can have decreasing and upside-down both tub shaped failure averages depending the shape parameter.

A random variable Y is said to have a two-parameter exponentiated pareto distribution if it has the following probability density (PDF)[2],

$$f_Y(y;\beta,\lambda)=\beta\lambda\left[1-(1+y)^{-\lambda}\right]^{(\beta-1)}(1+y)^{-(\lambda+1)};y>0,\beta,\lambda>0 \quad ...(1)$$

where β and λ are the parameters. The cumulative distribution function CDF, reliability and hazard functions are given respectively by:

$$F_Y(y;\beta,\lambda)=\left[1-(1+y)^{-\lambda}\right]^{\beta};y\geq 0,\beta,\lambda>0 \quad ...(2)$$

$$R(y)=1-\left[1-(1+y)^{-\lambda}\right]^{\beta};y\geq 0,\beta,\lambda>0 \quad ...(3)$$

and

$$h(y)=\frac{\beta\lambda\left[1-(1+y)^{-\lambda}\right]^{(\beta-1)}(1+y)^{-(\lambda+1)}}{1-[1-(1+y)^{-\lambda}]^{\beta}};y\geq 0,\beta,\lambda>0 \quad ...(4)$$

2. Definitions

Def.(1) [3]: A fuzzy number $\tilde{x} = (x_1, x_2, x_3)$ with the following membership function is called the triangular fuzzy number

$$\mu_{\tilde{x}}(x) = \begin{cases} \dfrac{x - x_1}{x_2 - x_1} & ; \quad x_1 \le x \le x_2 \\ \dfrac{x_3 - x}{x_3 - x_2} & ; \quad x_2 \le x \le x_3 \\ 0 & ; \quad otherwise \end{cases} \qquad \dots(5)$$

Def.(2) [4]: Let $(\Omega, \mathcal{F}, P)$ be a probability space ,where Ω is the $sample\ space$, $\mathcal{F}$ is the event space, and $P$is probability function then.

$$P(\tilde{\mathcal{F}}) = \int_{\Omega} \mu_{\tilde{\mathcal{F}}}(x)\, dP = \int_{\mathbb{R}^n} \mu_{\tilde{\mathcal{F}}}(x)\, f(x)\, dx \qquad \dots(6)$$

where $dP(x) = f(x)dx$

Now, let P is the probability distribution of a continuous random variable Z with p.d.f. $\psi(z)$.The conditional density of a crisp subset Z given $\tilde{\mathcal{F}}$ is given by:

$$\psi(z|\tilde{\mathcal{F}}) = \frac{\mu_{\tilde{\mathcal{F}}}(z)\, \psi(z)}{\int \mu_{\tilde{\mathcal{F}}}\,(u)\psi(u)du} \qquad \dots(7)$$

Def. (3) [5]: Given a random experiment$(A, \mathcal{F}, P)$, then a fuzzy information system (FIS) $\tilde{S}$ associated with the experiment E is a fuzzy partition $\mathring{A} = \{\tilde{x}_1, \dots, \tilde{x}_k\}$ of $A = \mathbb{R}^n$,i.e., a set of **I** fuzzy events on A satisfying orthogonality condition,

$$\sum_{i=1}^{\boldsymbol{I}} \mu_{\tilde{x}_i}(x) = 1 \quad ; \ for\ all\ \ x \in A$$

Where $\mu_{\tilde{x}_i}$ denotes the grade of membership of $\tilde{x}_i$.

3. Estimation

In this section we first estimate the shape parameter and the second estimate its reliability and hazard functions respectively using (1) Maximum likelihood Method and (2)Moment Method as follows .

First: shape Parameter $\boldsymbol{\beta}$

1) Maximum likelihood Method

Maximum likelihood estimator (MLE) is this value of the parameter that maximizes the natural log-likelihood function. Let $\underline{y}$ be a random vector of a random sample ,which contain i.i.d. random vectors of size n from EPD with pdf given by equation (1). And suppose that $\underline{y}$ is not observed exactly (precisely) and only partial information is available in the form of a fuzzy subset $\underline{\tilde{y}}$ with the measurable membership function $\mu_{\underline{\tilde{y}}}(x)$ (see Pak2013 [3]).

The observed-data likelihood function and its natural log for the EPD using the expression(6)) can be obtained ,respectively as:

$$L\left(\beta, \lambda|\underline{\tilde{y}}\right) = \prod_{i=1}^{n} \int f(y; \beta, \lambda)\, \mu_{\tilde{y}_i}(y)\, dy$$

$$\Rightarrow L\left(\beta, \lambda|\underline{\tilde{y}}\right) = \prod_{i=1}^{n} \int \beta\lambda\left[1 - (1 + y)^{-\lambda}\right]^{(\beta - 1)} (1 + y)^{-(\lambda + 1)}\ \mu_{\tilde{y}_i}(y)dy \quad \dots(8)$$

$$\ell\left(\beta,\lambda|\underline{\tilde{y}}\right) = \ln L\left(\beta,\lambda|\underline{\tilde{y}}\right)$$

$$= n\ln\beta + n\ln\lambda + \sum_{i=1}^{n}\ln\int\left[1-(1+y)^{-\lambda}\right]^{(\beta-1)}(1+y)^{-(\lambda+1)}\,\mu_{\tilde{y}_i}(y)dy \quad \dots(9)$$

Now, assuming that the parameter β is unknown and λ is known.

To maximize function (9), differentiating this equation (9), partially with respect to β, and then set the resulting equal to 0.

$$\frac{\partial\ell\left(\beta,\lambda|\underline{\tilde{y}}\right)}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\frac{\int\left[1-(1+y)^{-\lambda}\right]^{(\beta-1)}(1+y)^{-(\lambda+1)}\ln\left[1-(1+y)^{-\lambda}\right]\,\mu_{\tilde{y}_i}(y)dy}{\int\,\left[1-(1+y)^{-\lambda}\right]^{(\beta-1)}(1+y)^{-(\lambda+1)}\,\mu_{\tilde{y}_i}(y)dy}$$
$$= 0 \quad \dots(10)$$

Since there is no closed form of the equation (10), then, Newton Raphson iterative techniques can be used to obtain the solution [3].

As we knew that the steps of the Newton Raphson algorithm are as follows:

Step(1). Let $\hat{\beta}^{(i)}$ be the parameter from the i^{th} step.
Step(2). At the $(i+1)^{th}$ step , $\hat{\beta}^{(i+1)}$is obtained as :

$$\hat{\beta}^{(i+1)} = \hat{\beta}^{(i)} - \frac{\left.\frac{\partial\ell\left(\beta,\lambda;\underline{\tilde{y}}\right)}{\partial\beta}\right|_i}{\left.\frac{\partial^2\ell\left(\beta,\lambda|\underline{\tilde{y}}\right)}{\partial\beta^2}\right|_i} \quad \dots(11)$$

Where,

$$\frac{\partial^2\ell\left(\beta,\lambda|\underline{\tilde{y}}\right)}{\partial\beta^2} = -\frac{n}{\beta^2} + \sum_{i=1}^{n}\frac{\int\left[1-(1+y)^{-\lambda}\right]^{(\beta-1)}(1+y)^{-(\lambda+1)}\ln\left[1-(1+y)^{-\lambda}\right]\,\mu_{\tilde{y}_i}(y)dy}{\int\,\left[1-(1+y)^{-\lambda}\right]^{(\beta-1)}(1+y)^{-(\lambda+1)}\,\mu_{\tilde{y}_i}(y)dy}$$
$$-\sum_{i=1}^{n}\left(\frac{\int\left[1-(1+y)^{-\lambda}\right]^{(\beta-1)}(1+y)^{-(\lambda+1)}\ln\left[1-(1+y)^{-\lambda}\right]\,\mu_{\tilde{y}_i}(y)dy}{\int\,\left[1-(1+y)^{-\lambda}\right]^{(\beta-1)}(1+y)^{-(\lambda+1)}\,\mu_{\tilde{y}_i}(y)dy}\right)^2 \quad \dots(12)$$

3. Repeat step (2) until the convergence occurs, i.e.∃ pre-fixed $\epsilon > 0 \ni \left|\hat{\beta}^{(i+1)} - \hat{\beta}^{(i)}\right| < \epsilon$

Now, $\hat{\beta}_{ML}$ refereed as the maximum likelihood estimates of β via NR algorithm.

2) Moment Method

The moment estimate for β of the EPD can be found by the following equation which is obtained by equating the first population moment to the corresponding sample moments, that is:

$$A_1(\beta) - 1 = \frac{1}{n}\sum_{i=1}^{n}E_{\beta,\lambda}(Y|\tilde{y}_i) \quad \dots(13)$$

Where

$$A_1(\beta) = \beta\,\mathrm{B}\left(\beta, 1-\frac{1}{\lambda}\right), \lambda > 1$$

and , $\mathrm{B}(n,m)$ is beta function defined as:

$\mathrm{B}(n,m)=\int_0^1 y^{n-1}(1-y)^{m-1}\,dy$

$$\Rightarrow \hat{\beta}_{mo}=\frac{\frac{1}{n}\sum_{i=1}^{n}E_{\beta,\lambda}(Y|\tilde{y}_i)+1}{\mathrm{B}\left(\beta,1-\frac{1}{\lambda}\right)} \qquad \ldots(14)$$

Note that ,the direct for of the solution to equation (13) could not be obtained .so, using an iterative numerical process as described below we can obtain the β estimate.

1. Given initial value of the shape parameter β say $\beta^{(0)}$ and set the iteration i=0,

2. At $(i+1)^{th}$ iteration, using the expression (7) to compute $E_{\beta,\lambda}(Y|\tilde{y}_i)$,

$$E_{\beta,\lambda}(Y|\tilde{y}_i)=\frac{\int y\left[1-(1+y)^{-\lambda}\right]^{(\beta^{(i)}-1)}(1+y)^{-(\lambda+1)}\;\mu_{\tilde{y}_i}(y)dy}{\int\left[1-(1+y)^{-\lambda}\right]^{(\beta^{(i)}-1)}(1+y)^{-(\lambda+1)}\;\mu_{\tilde{y}_i}(y)dy}$$

3. Obtain the solution $\beta^{(i+1)}$ from the equation (13),

4. Setting $i=i+1$, repeat 2 and 3 until convergence occurs,

$\hat{\beta}_{mo}$ referred as moment estimate of β via moment method

Second: Reliability and Hazard functions

1) Maximum likelihood Method

Depending on the invariant property of maximum likelihood estimator of the reliability and hazard functions of EPD, denoted by $\hat{R}_{ML}(t)$, $\hat{h}_{ML}(t)$, can be obtained by replacing β in (3) and (4) by this maximum likelihood estimate as:

$$\hat{R}_{ML}(t)=1-\left[1-(1+t)^{-\lambda}\right]^{\hat{\beta}_{ML}};t\geq 0 \qquad \ldots(15)$$

And,

$$\hat{h}_{ML}(t)=\frac{\hat{\beta}_{ML}\,\lambda\left[1-(1+t)^{-\lambda}\right]^{(\hat{\beta}_{ML}-1)}(1+t)^{-(\lambda+1)}}{1-\left[1-(1+t)^{-\lambda}\right]^{\hat{\beta}_{ML}}};t\geq 0 \qquad \ldots(16)$$

2) Moment Method

Depending on the moment estimate of the shape parameter β, the approximated moment estimaters of the reliability and hazard functions of EPD at time t denoted by $\hat{R}_{Mo}(t)$, $\hat{h}_{Mo}(t)$, can be obtained by replacing β in equations (3) and (4) by this moment estimate as:

$$\hat{R}_{Mo}(t)=1-\left[1-(1+t)^{-\lambda}\right]^{\hat{\beta}_{Mo}};t\geq 0 \qquad \ldots(17)$$

And,

$$\hat{h}_{Mo}(t)=\frac{\hat{\beta}_{Mo}\,\lambda\left[1-(1+t)^{-\lambda}\right]^{(\hat{\beta}_{Mo}-1)}(1+t)^{-(\lambda+1)}}{1-\left[1-(1+t)^{-\lambda}\right]^{\hat{\beta}_{Mo}}};t\geq 0 \qquad \ldots(18)$$

4. Simulation Study

In trying to illustrate and compare the estimators obtained in above, using MATLAB (R2010b) program. we generated (100) samples of size n = 10, 20, 40,80 and 100 to represent small, moderate and large sample sizes from the EPD,with four values of $\beta(\beta = 0.5, 1,2,3)$ when $\lambda = 2$. The mean is used to be the initial value required for proceeding algorithms.Then, each observation of $\underline{y}$ was made fuzzied based on an appropriate selected membership function as in FIS shown in figure (1).

$$\mu_{\tilde{y}_1}(y) = \begin{cases} 1 & ; \quad y \le 0.05 \\ \dfrac{0.25 - y}{0.2} & ; \ 0.05 \le y \le 0.25 \\ 0 & ; \quad otherwise \end{cases}$$

$$\mu_{\tilde{y}_5}(y) = \begin{cases} \dfrac{y - 0.75}{0.25} & ; \ 0.75 \le y \le 1 \\ \dfrac{1.5 - y}{0.5} & ; \ 1 \le y \le 1.5 \\ 0 & ; \quad otherwise \end{cases}$$

$$\mu_{\tilde{y}_2}(y) = \begin{cases} \dfrac{y - 0.05}{0.2} & ; \ 0.05 \le y \le 0.25 \\ \dfrac{0.5 - y}{0.25} & ; \ 0.25 \le y \le 0.5 \\ 0 & ; \quad otherwise \end{cases}$$

$$\mu_{\tilde{y}_6}(y) = \begin{cases} \dfrac{y - 1}{0.5} & ; \ 1 \le y \le 1.5 \\ \dfrac{2 - y}{0.5} & ; \ 1.5 \le y \le 2 \\ 0 & ; \quad otherwise \end{cases}$$

$$\mu_{\tilde{y}_3}(y) = \begin{cases} \dfrac{y - 0.25}{0.25} & ; \ 0.25 \le y \le 0.5 \\ \dfrac{0.75 - y}{0.25} & ; \ 0.5 \le y \le 0.75 \\ 0 & ; \quad otherwise \end{cases}$$

$$\mu_{\tilde{y}_7}(y) = \begin{cases} \dfrac{y - 1.5}{0.5} & ; \ 1.5 \le y \le 2 \\ 3 - y & ; \quad 2 \le y \le 3 \\ 0 & ; \quad otherwise \end{cases}$$

$$\mu_{\tilde{y}_4}(y) = \begin{cases} \dfrac{y - 0.5}{0.25} & ; \ 0.5 \le y \le 0.75 \\ \dfrac{1 - y}{0.25} & ; \ 0.75 \le y \le 1 \\ 0 & ; \quad otherwise \end{cases}$$

$$\mu_{\tilde{y}_8}(y) = \begin{cases} y - 2 & ; \ 2 \le y \le 3 \\ 1 & ; \quad y \ge 3 \\ 0 & ; \quad otherwise \end{cases}$$

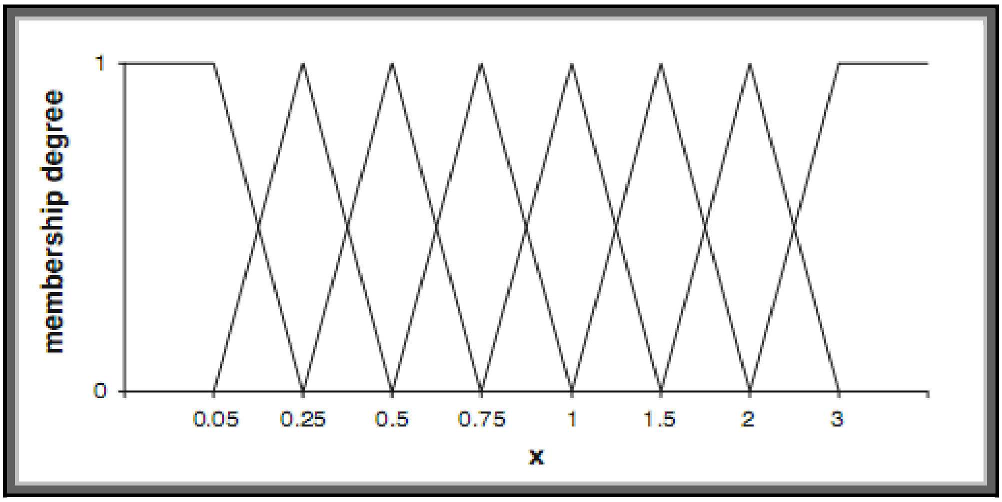

Figure (1): FIS used to Encode the Simulated Data[3][6]

The obtained maximum likelihood and moment estimates of, $R(t)$and $h(t)$ with time **t** were compared based on average values from Mean Square Error (MSE), where:

$$MSE(\hat{\beta}) = \frac{\sum_{j=1}^{L}(\hat{\beta}_j - \beta)^2}{L} \quad \dots(19)$$

$$MSE(\hat{R}(t)) = \frac{\sum_{j=1}^{L}(\hat{R}_j(t) - R(t))^2}{L} \quad \dots(20)$$

$$MSE(\hat{h}(t)) = \frac{\sum_{j=1}^{L}(\hat{h}_j(t) - h(t))^2}{L} \quad \dots(21)$$

$\hat{\beta}_j$: is the estimate of β at the j th run.
L: is the number of sample replicated.
$\hat{R}_j(t)$: is the estimates of $R(t)$ at the j^{th} run and t time.
$\hat{h}_j(t)$: is the estimates of $h(t)$ at the j^{th} run and t time.
t=1.
The results are summarized in the following tables 1-3.

5. Simulation Results

- Tables (1,2,3) , show us that the maximum likelihood estimates for β , $R(t)$ and $h(t)$ are better than those moment estimates for all sample sizes and initial values of β expect with $\beta = 1$ which the moment estimates are the better.

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Table (1): MSE values for maximum likelihood and moment estimates of the parameter β of EPD with different sample sizes.

n	β	$\hat{\beta}_{ML}$	$\hat{\beta}_{mo}$
10	0.5	0.0218882	0.0922539
20	0.5	0.0142522	0.0898122
40	0.5	0.0071006	0.0870715
80	0.5	0.0031116	0.0799187
100	0.5	0.0026645	0.0798406
10	1	0.1321844	0.0655304
20	1	0.0489538	0.0300431
40	1	0.0244556	0.0139701
80	1	0.0131505	0.0096323
100	1	0.0126069	0.0078488
10	2	0.3138269	0.4517680
20	2	0.1785433	0.3793244
40	2	0.1356952	0.3693591
80	2	0.1040305	0.3476982
100	2	0.1021026	0.3409203
10	3	0.6996813	1.9069223
20	3	0.4967565	1.7950724
40	3	0.4669464	1.7875392
80	3	0.4304171	1.7804850
100	3	0.4156072	1.7744345

Table (2): MSE values for maximum likelihood and moment estimates of the Reliability function of EPD with different sample sizes.

n	β	$\hat{R}_{ML}(t)$	$\hat{R}_{Mo}(t)$
10	0.5	0.0013385	0.0049877
20	0.5	0.0008454	0.0049607
40	0.5	0.0004223	0.0049072
80	0.5	0.0001886	0.0045516
100	0.5	0.0001616	0.0045471
10	1	0.0055964	0.0028468
20	1	0.0022631	0.0006691
40	1	0.0011814	0.0006202
80	1	0.0006313	0.0004287
100	1	0.0006035	0.0003509
10	2	0.0091329	0.0149349
20	2	0.0052512	0.0121409
40	2	0.0041008	0.0116758
80	2	0.0030708	0.0108702
100	2	0.0030077	0.0106364
10	3	0.0137271	0.0434820
20	3	0.0095524	0.0400790
40	3	0.0087346	0.0394727
80	3	0.0078842	0.0391142
100	3	0.0075650	0.0389303

Table (3): MSE values for maximum likelihood and moment estimates of the Hazared function of EPD with different sample sizes.

n	β	$\hat{h}_{ML}(t)$	$\hat{h}_{Mo}(t)$
10	0.5	0.0005477	0.0022094
20	0.5	0.0003530	0.0021683
40	0.5	0.0001761	0.0021183
80	0.5	0.0000777	0.0019518
100	0.5	0.0000665	0.0019499
10	1	0.0028829	0.0014481
20	1	0.0011071	0.0013295
40	1	0.0005627	0.0003118
80	1	0.0003020	0.0002152
100	1	0.0002892	0.0001757
10	2	0.0059172	0.0090088
20	2	0.0033959	0.0074743
40	2	0.0026160	0.0072450
80	2	0.0019880	0.0067923
100	2	0.0019497	0.0066548
10	3	0.0112512	0.0326944
20	3	0.0079510	0.0305371
40	3	0.0074061	0.0302919
80	3	0.0067727	0.0301173
100	3	0.0065232	0.0300005