A Comparison Between the Bayesian and the Classical Estimators of Weibull Distribution

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ABSTRACT

In this paper ,we study estimation of two parameters of Weibull distribution .Methods of estimation used are maximum likelihood estimator (MLE) and Bayes. We compared the numerical results by simulation in MATLAB program. The comparison show that the Bayes estimator gives the best results (less error) .

KEYWORDS

Weibull distribution , Maximum likelihood estimator , Bayesian estimator .

1.INTRODUCTION

The main branch of statistical inference is an estimation.There are some procedures of estimation . Some of these procedures which depend on a number of samples are called the classical methods like a maximum likelihood estimators , the other procedures depend on a prior information are called the Bayesian methods like Bayes estimators .

In this paper , we discuss the maximum likelihood estimation and Bayesian estimation and , we compared between them in a simulation study by using MATLAB program .

Bhattacharya &Bhattacharjee[4] presented some methods for estimating Weibull parameters , shape and scale parameter . They discuss the linear least square methods (LLSM) and maximum likelihood estimator (MLE).

Al – Hilaly[5](2004) studied the comparison among some different methods to estimate the three parameters of Weibull distribution and reliability function .

Kaminskiy&Vasilisy[6] (2005) studied a simple procedure for Bayesian estimation of the two – parameter . They suggested that the prior information can be presented in the form of the interval assessment of the reliability function .

The procedure allowed constructing the continuous joint prior distribution of Weibull parameters as well as the posterior estimates of the mean and standard deviation of the CDF .

Preda , Panaitescu& Constantinescu [7](2010) studied the Bayesian estimation by three methods and compared it with the maximum likelihood estimation for a modified – Weibull distribution by using Lindley's approximation .

Ahmed & Ibrahim (2011) used Jeffery prior information to Bayesian survival estimation parameter for Weibull distribution with censored data .

2. NOTATIONS

X random sample

α shape parameter of Weibull distribution

β scale parameter of Weibull distribution

p.d.f. probability density function

LF likelihood function

MLE maximum likelihood estimator

$\hat{\beta}$ estimator of parameter β by MLE

$\hat{\alpha}$ estimator of parameter α by MLE

$L(x_i , \alpha ,\beta)$ joint likelihood function

$g(\beta)$ prior distribution of β

$h(\alpha)$ prior distribution of α

$\Pi(\beta ,\alpha | x)$ marginal distribution

$g(\alpha , \beta , |x_i)$ posterior distribution

a , c constant chosen such that a , c > 1

$\hat{\alpha}_B$ estimator of α by Bayesian

$\hat{\beta}_B$ estimator of β by Bayesian

3. WEIBULL DISTRIBUTION [1]

For more than half a century the Weibull distribution has attracted the attention of statisticians working on theory and methods as well as on various fields of applied statistics. So , one of the most useful probability distributions in failure and test of time is the Weibull distribution . Consider the Weibull probability density function which is given by [1] :

$$f(x ,\alpha,\beta) = \left(\frac{\alpha}{\beta}\right) x^{\alpha-1} \; e\left(-(\frac{x^{\alpha}}{\beta})\right) \ldots (1)$$

4 . MAXIMUM LIKELIHOOD ESTIMARORS (MLE) [2] [3] [4][9]

The method of maximum likelihood (Harter and Moore (1965a), Harter andMoore (1965b), and Cohen (1965)) is a commonly used procedure because it hasvery desirable properties.Let x_1 , x_2 , x_3 , ... , x_n be a random variables of size n , we assumed that the likelihood function (LF) of the probability density function of Weibulldistribution is :

$$L(x_1,\ldots,x_n ,\alpha ,\beta) = \Pi_{i=1}^{n}\left(\frac{\alpha}{\beta}\right) x_i^{\alpha-1} \exp\left(\frac{-x_i^{\alpha}}{\beta}\right) \ldots (2)$$

To find the maximum likelihood estimators (MLEs) for two parameters α , β , we take the natural logarithm for both sides to equation (2) , and differentiating with respect to α , β , and equating to zero , so we get :

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln x_i - \frac{1}{\beta}\sum_{i=1}^{n} x_i^{\alpha} \ln x_i \quad \ldots (3)$$

$$\frac{\partial \ln L}{\partial \hat{\beta}} = \frac{-n}{\hat{\beta}} + \frac{1}{\hat{\beta}^2}\sum_{i=1}^{n} x_i^{\hat{\alpha}} = 0 \quad \ldots (4)$$

Now , we simplifying the above equation to get :

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i^{\hat{\alpha}}}{n} \ldots \quad (5)$$

So , to separate α and β is very difficult since the equation is non linear , so , we will use the numerical analysis by Newten – Raphson method , as follows [4] [9]:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(\hat{\beta}) = \frac{\sum_{i=1}^{n} \ln x_i}{n} \ldots \quad (6)$$

By consider $f(x_n)$ is $f(\hat{\alpha})$

$$f(\hat{\alpha}) = \frac{n}{\hat{\alpha}} + \sum_{i=1}^{n} \ln x_i - \frac{1}{\hat{\hat{\beta}}} \sum_{i=1}^{n} x_i^{\hat{\alpha}} \ln x_i = 0$$

And ,$f'(x_n)$ is $f'(\hat{\alpha})$

So,

$$f'(\hat{\alpha}) = \frac{\partial f(\hat{\alpha})}{\partial \hat{\alpha}}$$

$$= \frac{\sum_{i=1}^{n} x_i^{\hat{\alpha}} \sum_{i=1}^{n} x_i^{\hat{\alpha}} (\ln x_i)^2 - \left(\sum_{i=1}^{n} x_i^{\hat{\alpha}} (\ln x_i)\right)^2}{\left(\sum_{i=1}^{n} x_i^{\hat{\alpha}}\right)^2} + \frac{1}{\hat{\alpha}^2}$$

$$\therefore \hat{\alpha}_1 = 1 + \frac{n^2 + \sum_{i=1}^{n} ln x_i - n \sum_{i=1}^{n} ln x_i}{n} \quad \ldots (7)$$

5. BAYESIAN ESTIMATORS

Let $x_1, x_2, x_3, \ldots, x_n$, be a random sample of size n with distribution function $F(x, \alpha, \beta)$ and the probability density function $f(x, \alpha, \beta)$, there are several steps to calculate the Bayes estimators of the Weibull distribution with two parameters α and β , . So to do this we must know the prior distribution and posterior distribution as follows :

$$posterior\ distribution = \frac{prioi\ distribution\ x\ likelihood\ function}{marginal\ distribution}$$

In other words :

$$g(\alpha, \beta \mid x_i) = \frac{g(\alpha, \beta) \,.\, L(x_i, \alpha, \beta)}{\int_0^{\beta} \int_0^{\infty} f(x_i, \alpha, \beta) d\alpha\, d\beta}$$

Now , consider the case when α and β are unknown parameters , we will use the prior distribution of (β, α) is $g(\beta)\, h(\alpha)$ where

$$g(\beta) \propto \left(\frac{1}{\beta^c}\right), c > 0$$

$$h(\alpha) \propto \frac{1}{a}, 0 < \alpha < a$$

with these prior distributions we obtain the posterior distribution of (β, α) :

$$\pi(\beta, \alpha \mid x) = \frac{k\, \alpha^n}{\beta^{n+c}} \lambda^{\alpha-1} \exp\left(\frac{\sum_{i=1}^{n} x_i^{\alpha}}{\beta}\right) \quad \ldots (8)$$

Where

$$\lambda = \prod_{i=1}^{n} x_i$$

And

$$k = \frac{1}{\int_0^a \int_0^\infty \pi\,(\beta,\alpha \mid x) d\beta\, d\alpha}$$

$$K^{-1} = \int_0^a \int_0^\infty \pi\,(\beta,\alpha \mid x) d\beta\, d\alpha$$

$$= \Gamma(n+c-1) \int_0^a \frac{\alpha^n \lambda^{\alpha-1}}{\sum_{i=1}^n x_i^\alpha} d\alpha$$

Integrating out β in (8) we have the marginal posterior of α :

$$\pi\,(\alpha \mid x) = \frac{\frac{\alpha^n \lambda^{\alpha-1}}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n+c-1}}}{\int_0^a \frac{\alpha^n \lambda^{\alpha-1}}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n+c-1}} d\alpha} \quad \dots (9)$$

Similarly we obtain the marginal posterior of β :

$$\pi\,(\beta \mid x) = \frac{\frac{1}{\beta^{n+c}} \int_0^a \alpha^n \lambda^{\alpha-1} \exp\left(\frac{-\sum_{i=1}^n x_i^\alpha}{\beta}\right) d\alpha}{\Gamma(n+c-1) \int_0^a \frac{\alpha^n \lambda^{\alpha-1}}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n+c-1}} d\alpha}$$

$\dots (10)$

We expect erratic behavior of these posteriors for larger values of (a , boundary of integration) and from (9) and (10) we obtain the corresponding Bayes estimators :

$$\widehat{\alpha}_B = \frac{\int_0^a \alpha^{n+1} \frac{\lambda^{\alpha-1}}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n+c-1}} d\alpha}{\int_0^a \alpha^n \frac{\lambda^{\alpha-1}}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n+c-2}} d\alpha} \quad \dots (11)$$

$$\widehat{\beta}_B = \frac{1}{n+c-2} \frac{\int_0^a \alpha^n \frac{\lambda^{\alpha-1}}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n+c-2}} d\alpha}{\int_0^a \alpha^n \frac{\lambda^{\alpha-1}}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n+c-1}} d} d\alpha \quad \dots (12)$$

6. SIMULATION STUDY

There are many methods of simulation (especially after the rapid development that took place in the use of electronic computers), which provides the time, effort ,cost and achieve analytical solutions .Simulation is the imitation of the operation of a real-world process or system over time. The act of simulating something first requires a model to be developed; this model represents the key characteristics or behaviors of the selected physical or abstract system or process. The model represents the system itself, whereas the simulation represents the operation of the system over time .Computer simulations have become a useful part of mathematical modeling of many natural systems in sciences .So , the simulation is a type of sampling .In this simulation we use the nextflowchart :

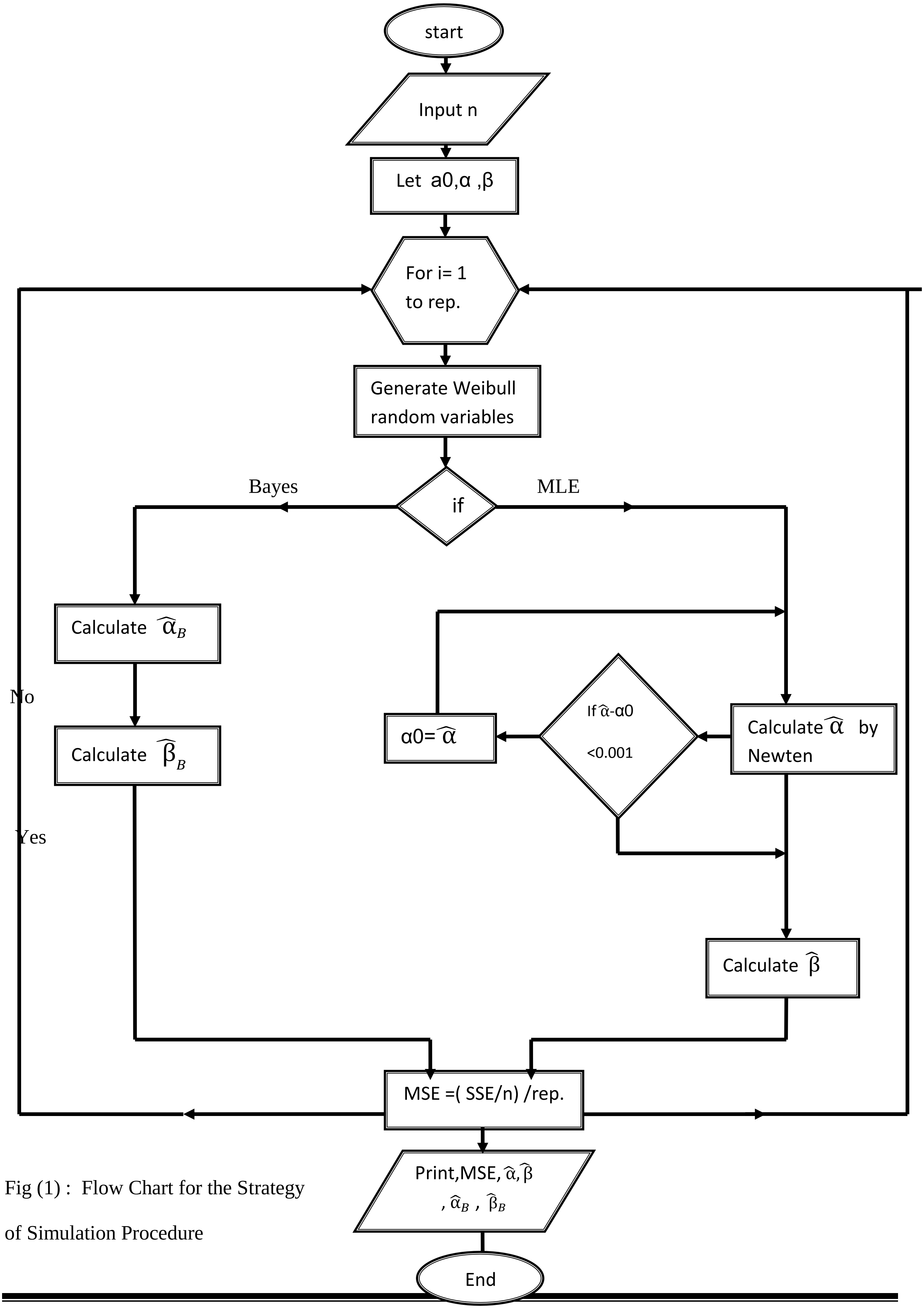


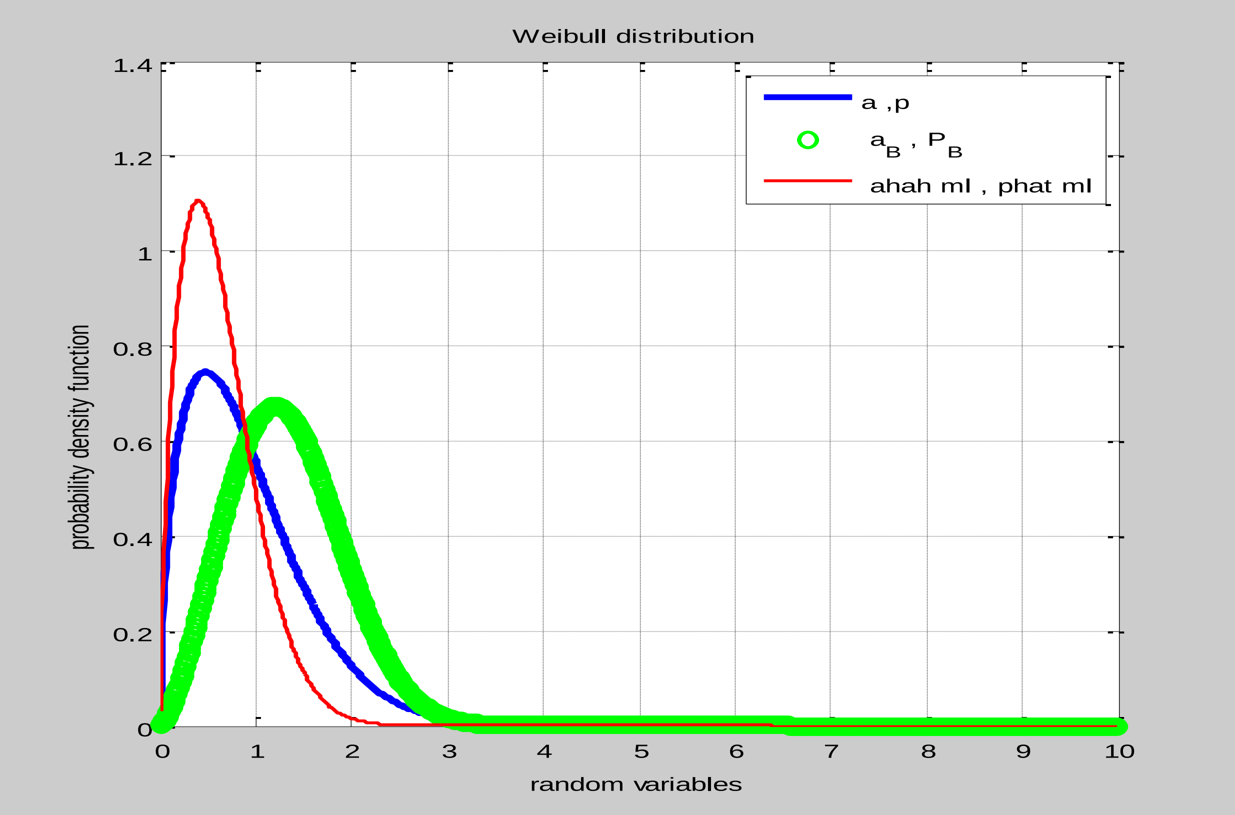
Fig (1) : Flow Chart for the Strategy of Simulation Procedure

Table (1) : Results of Simulation Using MLE

Parameters Values α , β	Sample size	MLE		MSE		
		$\hat{\alpha}$	$\hat{\beta}$	Parameters $\hat{\alpha},\hat{\beta}$	Function	Matrix
$\alpha = 1.5$ $\beta = 1$	10	2.4758	1.4894	6.1295, 2.8710	7.2202e-004	4.0018
	30	2.0773	1.2216	4.3151,1.5809	7.6813e-004	0.3820
	60	1.6446	1.0434	2.7048, 1.1092	9.5760e-004	0.0553
	100	1.6701	1.0508	2.7894, 1.1166	8.1152e-004	0.0348
$\alpha = 1$ $\beta = 0.5$	10	1.6505	0.4744	0.2724 , 0.0291	5.6136e-004	0.1805
	30	1.3849	0.4678	1.9179, 0.2318	6.5619e-004	0.0249
	60	1.0964	0.4880	1.2021, 0.2426	0.0010	0.0054
	100	1.1135	0.4857	1.2398, 0.2385	0.0010	0.0033

Table (2) : Results of Simulation Using Bayesian

Parameters Values α , β	Sample size	MLE		MSE		
		$\hat{\alpha}$	$\hat{\beta}$	Parameters $\hat{\alpha},\hat{\beta}$	Function	Matrix
$\alpha = 1.5$ $\beta = 1$	10	0.1621	0.2973	0.0266,0.0897	0.0013	1.1256e-004
	30	0.1089	0.0870	0.0119,0.0076	0.0013	5.7421e-007
	60	0.0909	0.0412	0.0083,0.0017	0.0015	2.4515e-008
	100	0.0814	0.0253	0.0066 ,0.0006	0.0013	3.0892e-009
$\alpha = 1$ $\beta = 0.5$	10	0.2100	0.2107	0.0045, 0.0045	0.0015	1.4465e-004
	30	0.1262	0.0680	0.0449 , 30.0453	5.4012e-004	7.1266e-010
	60	0.1028	0.0347	0.0106 , 0.0012	6.8811e-004	3.6754e-008
	100	0.0919	0.0224	0.0085 ,0.0005	5.7219e-004	6.4357e-009



Fig(2): Weibullpdf. Plot

The above figure is referring to plot of probability density function of Weibull distribution for data (the random variables are generatedin MATLAB), we see that in the red line .

The blue line show the probability density function after substituted the initial values of parameters (α, β) with the values of maximum likelihood estimators ($\hat{\alpha}$, $\hat{\beta}$) respectively .

Finally , the values of Bayes estimators ($\hat{\alpha}_B, \hat{\beta}_B$) is appeared in the green line .

From this , we see that in Tables (1) (2) the Bayes estimators have MSE less than MSE of MLEs , and from Figure (2) , we see that the curve of Bayes estimators is close of curve of the initial values .

So, the Bayesian estimation is better than the maximum likelihood estimation .

7. CONCLUSIONS

In this paper,we have presented both maximum likelihood and Bayes estimators for estimating the Weibull distribution parameters.we used simulation to compare these methods . It has been shown from the numerical results in Table (1) and Table (2) that the method which gives the best estimate is the Bayesian method, because the results have less mean squares error (MSE), and when the sample size be large then the MSE be less , the best results in the larger sample size since when the sample size be large then the MSE be less.

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