Modules Whose Submodules Are Strongly Stable Relative To An Ideal

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الخلاصة

في هذا العمل، مفهوم مقاسات تامة الاستقرارية بقوة بالنسبة الى مثالي عًرٌض و دًرٌس و هو اقوى من مفهومي مقاسات تامة الاستقرارية و مقاسات تامة الاستقرارية بالنسبة الى مثالي. أعطينا عديد من الخواص و التوصيفات. درسنا حلقة التشاكلات الذاتية لهذا النوع من المقاسات و أعطينا فيصلية لكي تكون حلقة التشاكلات الذاتية تامة الاستقرارية بقوة بالنسبة الى مثالي باستخدام مفاهيم نظرية الفئات. أعطينا توصيف للمقاسات الجزئية بدلالة شرط البواقي و خاصية كل مقاس جزئي متحايد بالنسبة الى مثالي.

ABSTRACT

In this work, the notion of fully strongly stable modules relative to an ideal has been introduced and studied which is stronger than those of fully stable modules and fully stable modules relative to an ideal. Several properties and characterizations have been given. Endomorphism ring of this class of modules has been studied and criteria given, that an endomorphism ring is fully strongly stable relative to an ideal by using categorical concepts.characterization of submodules have been given in terms of some residual condition, and the property that each submodule is idempotent relative to an ideal.

INTRODUCTION

Throughout, R represents an associative ring with identity, unless otherwise stated and M a unitary right R-module. Let M be an R-module , a submodule N of M is called stable if $\alpha(N) \subseteq N$ for each R-homomorphism $\alpha : N \to M$. In case each submodule of M is stable, then M is called fully stable [1]. The relativity manners are usually applicable in mathematics, especially in module theory. Let M be an R-module and A a right ideal of R. A submodule N of M is called stable relative to A if $\alpha(N) \subseteq N + MA$ for each R-homomorphism $\alpha : N \to M$, and M is called fully stable relative to A,if each submodule of M is stable relative to A [2]. It is clear that the class of fully stable modules is contained in that of fully stable module relative to an ideal A. In fact, M is fully stable if and only if it is fully stable relative to the zero ideal.

In this paper, we consider a strong view of fully stable and hence fully stable modules relative to an ideal. Let M be an R-module and A a non-zero right ideal of R. M is called fully strongly stable relative to A if $\alpha(N) \subseteq N \cap MA$ for each submodule N of M and R-homomorphism α of N into M. It is shown that an R-module M is fully strongly stable relative to A if and only if each cyclic submodule of M is strongly stable relative to A. Several properties and characterizations of this class of module were considered. Among others, we proved the following : An R-module M is fully strongly stable relative to A if and only if $\ell_M(r_R(x)) = xR \cap MA$ for each x in M if and only if for each R-homomorphism α of xR into M, there is an element r in R such that α (x) = xr ∈ MA where x in M. We study conditions under which fully strongly stable modules relative to A are equivalent that the double annihilator condition holds for each submodules.

We studied the endomorphism ring of these modules. We show that over commutative ring, fully strongly stable modules relative to an ideal have endomorphism rings with strong view of commutativty. Further, we give the following: Let M be an R-module with endomorphism ring S and M generates ker(β) for each β in S. Then S is a right fully strongly stable ring relative to an ideal A of S if and only if ker(β) ⊆ ker(α) implies that α ∈ βS ∩ SA.

An R-module M is called multiplication if each submodule of M is of the form MA for some ideal A of R [3]. We consider the following residual condition $[r_R(M):r_R(x)] = [xR\cap MA : M]$ where x ∈ M.

We proved that an R-module M is fully strongly stable relative to A if and only if the residual condition holds for each x in M under multiplication modules.

Finally, we introduced the concept of idempotent submodules relative to an ideal and consider modules in which all submodules are idempotent relative to an ideal, and show that an R-module M is fully strongly stable relative to A if and only if each submodule of M is idempotent relative to A where M is a prime module.

FULLY STRONGLY STABLE MODULES RELATIVE TO AN IDEAL.

In this section we introduce a concept which stronger than that of fully stable modules

Definition 2.1: Let M be an R-module and A a non-zero right ideal of R. A submodule N of M is called strongly stable relative to A (simply strongly A-stable). if f(N) ⊆ N ∩ MA for each R-homomorphism f of N into M. M is called fully strongly A-stable, if each submodule of M is strongly A-stable. If R as a right R-module is fully strongly A-stable, then R is called fully strongly A-stable ring.

It is clear that, if a submodule N of an R-module M is strongly A-stable, then it is strongly B-stable for each right ideal B of R containing A. Hence, if N is strongly A-stable submodule of M, then it is strongly R-stable, and this is equivalent to saying that N is stable in M. Thus every fully strongly A-stable R-module is fully stable, while the converse may not be true generally, for example, the Z_6-module Z_6 is fully stable, but it is not fully strongly A-stable for each proper ideal A of Z_6. If M is a fully stable R-module and M= MA for some non-zero right ideal A of R, then M is fully strongly A-stable. If M is projective R-module, then

M = Mtr(M) where tr(M) is the trace of M [4, proposition(2.40)], so if M is a fully stable projective R-module, then M is fully strongly tr(M)-stable. In particular, every fully stable ring R is fully strongly tr(R)-stable. The Z-module Z as well as Q and Q/Z are not fully strongly (mZ)-stable for each non-negative integer m. If $\{N_i \mid i \in I\}$ is a family of strongly A-stable submodules of an R-module M, then so is $\sum_{i\in I} N_i$.

Fully strongly A-stable modules are not closed under submodules. For example, the Z-module Z_{p^∞} is fully strongly (nZ)-stable for each positive integer n, if α: $Z_{p^k} \to Z_{p^\infty}$ is a Z-homomorphism, then α (Z_{p^k}) $\subseteq Z_{p^k} \cap Z_{p^\infty}$. It is well known that Z_{p^∞} is divisible, so $nZ_{p^\infty} = Z_{p^\infty}$ for each positive integer n and hence $\alpha(Z_{p^k}) \subseteq Z_{p^k} \cap Z_{p^\infty}(nZ)$. We claim that the submodule $Z_{p^{2k}}$ of Z_{p^∞} is not fully strongly ($Z_{p^{2k}}$)-stable, for each k > 1, let f : $Z_{p^k} \to Z_{p^{2k}}$ be the inclusion mapping, so f (Z_{p^k}) = Z_{p^k}, but $Z_{p^k} \cap (Z_{p^{2k}} \cdot p^{2k}Z) = 0$. This shows that $Z_{p^{2k}}Z$ is not fully strongly ($Z_{p^{2k}}$ Z)-stable.

Let M be an R-module and A a non-zero right ideal of R. We say that a submodule N of M is A-pure, if NA = N ∩ MA.

In the following we show that certain submodules inherit the property of full strong stability relative to a non-zero ideal.

Proposition 2.2: Let M be a fully strongly A-stable module. Then every A-pure submodule of M is fully strongly A-stable. In particular, every pure (and hence direct summand) submodule of fully strongly A-stable module is fully strongly A-stable.

Proof: Let N be A-pure submodule of M. For each submodule K of N and R-homomorphism f : K → N, put $g = i_N \circ f : K \to M$ where i_N is the inclusion mapping of N into M, then f(K) = g(K) ⊆ K ∩ MA ⊆ K∩N ∩MA = K ∩NA. Thus N is fully strongly A-stable.

The proof of following proposition is immediate

Proposition 2.3: Let M be an R-module and non-zero right ideal A of R. Then

1- M is fully strongly A-stable if and only if each cyclic submodule of M is strongly A-stable.

2- M is fully strongly A-stable for each non-zero right ideal of R. if and only if it is fully strongly (aR)-module for each non-zero element a in R.

Next, we discuss the direct sums of fully strongly A-stable modules. The Z-module Q / Z is not fully strongly A-stable for each ideal A of Z, while $Q / Z \cong \oplus_p Z_{p^\infty}$ where the sum has been taken over all primes. This shows that fully strongly A-stable modules are not closed under direct sum. In the following we consider conditions which guarantee full strong A-stability of finite direct sum.

Proposition 2.4 : Let $M = \oplus_{i=1}^{n} M_i$ where M_i is an R-module for each i with $r_R(M_i) + \cap_{j \neq i} r_R(M_j) = R$ and A a non-zero right ideal of R. Then M is fully strongly A-stable if and only if each M_i is fully strongly A-stable.

Proof: we shall prove the case $M = M_1 \oplus M_2$ and the proposition then follows by induction on n. Let K be a submodule of M. The condition $r(M_1) + r(M_2) = R$ implies that there are submodules N_1 of M_1 and N_2 of M_2 such that $K = N_1 \oplus N_2$ [1]. Let $\theta: K \to M$ be an R-homomorphism. put $\theta_i = \pi_i \theta\, j_i$ $(i = 1,2)$ where $j_i : N_i \to M$ is the injection mapping and $\pi_i : M \to M_i$ is the natural projection. Hence $\theta = \theta_1 + \theta_2$ and $\theta(K) = \theta_1(N_1) \oplus \theta_2(N_2) \subseteq (N_1 \cap M_1A) \oplus (N_2 \cap M_2A) \subseteq (M_1 \oplus M_2)A = K \cap MA$. This implies that M is fully strongly A-stable. The converse follows from proposition (2.2).

Next we note characterizations of fully strongly A-stable modules.

Theorem 2.5: Let M be an R-module and non-zero right ideal A of R. Then the following conditions are equivalent:

1- M is fully strongly A-stable,
2- Each cyclic submodule of M is strongly A-stable,
3- $\ell_M(r_R(x)) = xR \cap MA$ for each x in M,
4- $r_R(x) \subseteq r_R(y)$ implies $y \in xR \cap MA$ for each x in M and y in MA,
5- $\ell_M(aR + r_R(x)) = \ell_M(aR) \cap xR \cap MA$ for each x in M and a in R.
6- For each R-homomorphism $\alpha: xR \to M$, there is r in R such that $\alpha(x) = xr \in MA$.

Proof: (1) $\Leftrightarrow$ (2): Follows from Proposition (2.3)

(4) $\Rightarrow$ (3): It always $xR \cap MA \subseteq xR \subseteq \ell_M(r_R(x))$. If $y \in \ell_M(r_R(x))$, then $r_R(x) \subseteq r_R(y)$, so $y \in xR \cap MA$, by (4), and hence $\ell_M(r_R(x)) \subseteq xR \cap MA$.

(3) $\Rightarrow$ (4): Let $x \in M$ and $y \in MA$. If $r_R(x) \subseteq r_R(y)$, then $\ell_M(r_R(y)) \subseteq \ell_M(r_R(x))$ so by (3), $yR \cap MA \subseteq xR \cap MA$, but $y \in yR \cap MA$, then $y \in xR \cap MA$.

(3) $\Rightarrow$ (5): $\ell_M(aR + r_R(x)) = \ell_M(aR) \cap \ell_M(r_R(x)) = \ell_M(aR) \cap xR \cap MA$

(5) $\Rightarrow$ (6): Let $\alpha: xR \to M$ be an R-homomorphism. Then $\alpha(x)R \subseteq \ell_M(r_R\alpha(x)) \subseteq \ell_M(r_R(x)) = xR \cap Ma$, by take $a = 0$ in (5), so $\alpha(x) = xr \in MA$ for some r in R.

(6) $\Rightarrow$ (2): Let $\alpha: xR \to M$ be an R-homomorphism. Then by (6), there is r in R such that $\alpha(x) = xr \in MA$. For each $u = xt \in xR$, where t in R,

so $\alpha(u) = \alpha(xt) = \alpha(x)t = (xr)t \in xR \cap MA$ and hence $\alpha(xR) \subseteq xR \cap MA$. This shows that xR is strongly A-stable submodule in M.
As we have mentioned that an R-module M is fully stable if and only if it is fully strongly R-stable, so by take A = R and M = R in Theorem (2.5) we the following corollaries respectively.

Corollary2.6: ([1], theorem (3.6)) Let M be an R-module. Then the following conditions are equivalent:

1- M is fully stable,
2- Each cyclic submodule of M is stable,
3- $\ell_M(r_R(x)) = xR$ for each x in M,
4- $r_R(x) \subseteq r_R(y)$ implies $y \in xR$ for each x, y in M ,
5- Each R-homomorphism $\alpha: xR \to M$ is a right multiplication by an element of R.

Corollary 2.7: The following are equivalent for a ring R and a non-zero right (respt. left) ideal A of R.

1- R is right (respt. left) fully strongly A-stable,
2- Each right (respt. left) principal ideal of R is strongly A-stable,
3- $\ell_R(r_R(x)) = xR \cap RA$ (respt. $r_R(\ell_R(x)) = Rx \cap AR$) for each x in R,
4- $r_R(x) \subseteq r_R(y)$ implies $y \in xR \cap RA$ (respt. $\ell_R(x) \subseteq \ell_R(y)$) implies $y \in Rx \cap AR$)for each x in R and y in RA,
5- $\ell_R(aR + r_R(x)) = \ell_R(aR) \cap xR \cap MA$ (respt. $r_R(Ra + \ell_R(x)) = r_R(Ra) \cap Rx \cap AR$) for each x , a in R.
6- For each R-homomorphism $\alpha : xR \to M$ (respt. $\alpha : Rx \to M$), there is r in R such that $\alpha(x) = xr \in RA$ (respt. $\alpha(x) = rx \in AR$).

We direct our attention for conditions (3) and (6) of Theorem (2.5) for each submodule. We have proved that an R-module M is fully strongly A-stable if and only if for each cyclic submodule xR and R-homomorphism $\alpha: xR \to M$, there is an element r in R such that $\alpha(x) = xr \in MA$ if and only if $\ell_M(r_R(x)) = xR \cap MA$ for each x in M.

Proposition 2.8 : Let M be an R-module and A a non-zero right ideal of R with $r_R(N \cap K) = r_R(N) + r_R(K)$ for each finitely generated submodules N and K of M. Then the following statements are equivalent:

1- M is fully strongly A-stable,
2- For each R-homomorphism $\alpha: x_1R + x_2R + \dots + x_nR \to M$, there is t in R such that $\alpha(\sum_{i=1}^{n} x_i r_i) = (\sum_{i=1}^{n} x_i r_i)t \in MA$ where $r_1, r_2, \dots, r_n$ in R.

Proof : (2) ⇒ (1) : follows from Theorem (2.5**)**

(1) ⇒ (2): Let $N = x_1R + x_2R + \dots + x_nR$ be a finitely generated submodule of M and $\alpha: N \to M$ be an R-homomorphism. We use

induction on n. For n = 1, this is just theorem (2.5). Suppose that the statement holds for m < n, there exist two element r, s in R such that $\alpha(\sum_{i=1}^{n-1} x_i r_i) = (\sum_{i=1}^{n-1} x_i r_i)r \in MA$ and $\alpha(x_n r_n) = x_n r_n s \in MA$. Now, let $y = \sum_{i=1}^{n-1} x_i r_i = x_n r_n$, then $\alpha(y) = yr = ys$, so $r - s \in r_R(y)$, but $y \in \sum_{i=1}^{n-1} x_i R \cap x_n R$, there exist $u \in r_R(\sum_{i=1}^{n-1} x_i R)$ and $v \in r_R(x_n R)$ such that $r-s = u + v$. Put $t = r - u = s + v$. let $z = \sum_{i=1}^{n-1} x_i r_i$. Then

$\alpha(z) = \alpha(\sum_{i=1}^{n-1} x_i r_i) + \alpha(x_n r_n) = (\sum_{i=1}^{n-1} x_i r_i)r + (x_n r_n)s = (\sum_{i=1}^{n-1} x_i r_i)r - (\sum_{i=1}^{n-1} x_i r_i)u + (x_n r_n))s + (x_n r_n)v = (\sum_{i=1}^{n-1} x_i r_i)t + (x_n r_n))t = zt \in MA$

Corollary 2.9: Let M be a noetherian R-module and A a non-zero right ideal of R with $r_R(N \cap K) = r_R(N) + r_R(K)$ for each two submodules N and K of M. Then the following statements are equivalent:

1- M is fully strongly A-stable,

2- Given a submodule N of M and R-homomorphism $\alpha : N \to M$, for each x in N, there exists an element r in R such that $\alpha(x) = xr \in MA$

Lemma 2.10: Let M be a fully strongly A-stable R-module such that for given x in M and right ideal I of R, each R-homomorphism of xI into M can be extended to one from xR into M. Then if a submodule N of M satisfies $\ell_M(r_R(N)) = N \cap MA$, then so does N+xR.

Proof: Denote $r_R(N)$ and $r_R(xR)$ by B and C respectively. Then by Theorem (2.5) and assumption, we have $\ell_M(C) = xR \cap MA$ and $\ell_M(B) = N \cap MA$. Since $r_R(N + xR) = B \cap C$, it is enough to show that $\ell_M(B \cap C) \subseteq (N + xR) \cap MA$. Let $y \in \ell_M(B \cap C)$. $\theta: xB \to M$ is well-defined by $\theta(xb) = yb$. The hypothesis implies that θ can be extended to $\alpha: xR \to M$, so $\alpha(x) \in xR \cap MA$. For each b in B, $\alpha(x)b = \alpha(xb) = yb$ implies that $\alpha(x) - y \in \ell_M(B) = N \cap MA$, so $\alpha(x) - y = n$ for some $n \in N \cap MA$ or $y = -n + \alpha(x) \in N \cap MA + Rx \cap MA \subseteq (N + xR) \cap MA$.

Proposition 2.11: Let M be an R-module and non-zero ideal A of R, such that for given x in M and right ideal I of R, each R-homomorphism of xI into M can be extended to one of xR into M. Then the following are equivalent:

1- M is fully strongly A-stable,

2- $\ell_M(r_R(N)) = N \cap MA$ for each finitely generated submodule N of M.

Proof: (1) ⇒ (2): Let $N = \sum_{i=1}^{n} x_i R$ be a finitely generated submodule of M. we use induction on n. Theorem (2.5) implies that (2) is true for n = 1. Suppose that $\ell_M(r_R(K) = K \cap MA$ for m -generated submodule K of M where m < n. Then lemma (2.10) implies (2) for (m+1)-generated submodule of M.

(2) ⇒ (1) Follows from Theorem (2.5).

Note that quasi-injective modules ([4],definition (6.70)) are satisfying the extension condition of Proposition (2.11). Then, we have the following corollary.

<u>Corollary 2.12</u>: Let M be a Noetherian quasi-injective R-module and A a non-zero right ideal of R. Then M is fully strongly A-stable if and only if $\ell_M(r_R(N)) = N \cap MA$ for each submodule N of M.

ENDOMORPHISM RING

Let M be an R-module with $S=End_R(M)$, its endomorphism ring and A a non-zero right ideal of R. Suppose R is commutative and M is fully strongly A-stable. Then for each α, β in S and x in M, there are s, t in R such that $\alpha(x) = xs \in MA$ and $\beta(x) = xt \in MA$, so $\alpha\beta = \beta\alpha$. Since MA is a fully invariant submodule of M, then $\alpha\beta(x) = \beta\alpha(x) =\beta(xs) \in \beta(MA) \subseteq MA$.

The above suggest the following:

Let M be an R-module and S be its endomorphism ring. We say that S is Strongley commutative relative to a right ideal A of R (simpley strongly A-commutative, if $\alpha\beta(x) = \beta\alpha(x) \in MA$ for each α, β in S and x in M.

<u>Proposition 3.1</u>: Let R be a commutative ring, and M a fully strongly A-stable R-module. Then S is strongly A-commutative.

The converse of the above proposition may not be true for example, it is well known that $End_Z(Q) \cong Q$ ([5], page 216) which is a commutative. Since Q(mZ) = Q for each nonzero m in Z, thus $End_Z(Q)$ is strongly (mZ)-commutative for each non-zero m in Z, while Q is not fully strongly (mZ)-stable for each ideal mZ of Z.

Now, we discuss the converse in certain class of modules.

<u>Proposition 3.2</u>: Let M be an R-module in which every cyclic submodule is direct summand. If $S = End_R(M)$ is strongly A-commutative, then M is fully strongly A-stable, where A is a non-zero right ideal of R.

Proof : Let N = xR be a cyclic submodule of M and $\alpha : N \to M$ be an R-homomorphism. Then $M = N \oplus L$ for some submodule L of M. α can be extended to $\beta \in S$, by putting $\beta(L) = 0$. For each $w = x+y \in M$ where $x \in N$ and $y \in L$, define $h: M \to M$ by $h(x+y) = x$. Put $\alpha(x) = u+v$ for some $u \in N$ and $v \in L$. Now $(h\circ\beta)(w) = h(\beta(x+y)) = h(\alpha(x)) = h(u+v) = u$, but $\beta \circ h(w) = u+v$. Strong A-commutative of S implies $u = u+v \in MA$, so $v = 0$ and hence $f(x) = u \in N \cap MA$. This shows that N is strongly A-stable, so by Theorem (2.5), M is fully strongly A-stable.

Recall that an R-module M is regular if for each m ∈ M, there is α∈ M^*=Hom_R (M,R) such that m = mα(m). In regular modules each cyclic submodule is direct summand [6, Theorem(1.6)]. Then we have.

Corollary 3.3: Let M be a regular R-module (R is commutative ring) and A a non-zero ideal of R. Then M is fully strongly A-stable if and only if S = End_R(M) is strongly A-commutative.

In ([6],theorem(5.2)), Zelmanowitz proved that for a regular R-module M, M is quasi-injective if and only if End_R(M) is self-injective ring. We shall regard this as a motivation for the following

Theorem 3.4: Let M be a regular R-module (R is commutative ring) and A a non-zero ideal of R. Then M is fully strongly A-stable, if and only if S = End_R(M) is right fully strongly Hom_R (M,MA)-stable.

First we need the following lemma.

Lemma 3.5: Let M be an R-module in which every cyclic submodule is direct summand and A a non-zero right ideal of R. If S = End_R(M) is fully strongly Hom_R (M,MA)-stable, then M is fully strongly A-stable.

Proof: Let N be a cyclic submodule of M and α : N → M an R-homomorphism. I = Hom_R(M,N) is a right ideal of S. θ: I → S is well defined by θ(f) = α∘ f , for each f ∈I. full strong K = Hom_R(M,MA) -stability of S implies that θ(I)\subseteq I ∩ KS=I ∩ K, that is for each f ∈ I, α ∘ f ∈ I ∩ K, so α ∘ f : M → N∩MA. Since N is direct summand, then α ∘ π_N: M → N ∩ MA where π_N is the natural projection of M onto N. Since π_N is onto, then α(N) = α(π_N (N)) = (α ∘ π_N)(N) so, α : N → N∩MA. Then α(N) ⊆ N∩MA, this shows that M is fully strongly A-stable, Theorem (2.5).

Corollary 3.6: Let M be a regular R-module and A a non-zero right ideal of R. If S = End_R(M) is a right fully strongly Hom_R(M,MA)-stable, then M is fully strongly A-stable.

Proof of theorem (3.4): Let K = Hom_R(M,MA) and M a regular fully strongly A-stable R-module. Then by [6, theorem (3.4)], cent(S) is a regular ring, Proposition (3.1) implies that S is strongly A-commutative, and so commutative ring. Thus S is fully stable ring [1, proposition (1.2.2)]. For each α ∈ S and S-homomorphism β : (α) →S, we have β(α S) ⊆ αS, but β(α) ∈ S and since S is strongly A-commutative, then α∘β = β ∘ α and Im(β α) ⊆ MA, so β∘ α ∈ K, that is β(α) ∈ K and hence β(α) ⊆ α S ∩ K = α S ∩ KS, this shows that S is fully strongly K-stable. The other direction follows from Corollary (3.6)

We conclude this section with some conditions that an endomorphism ring is fully strongly stable relative to an ideal.

Recall that, for two R-modules B and C, B generates C if C = $\sum Im(\varphi)$ where the sum runs over all R-homomorphism φ : B → C. Dually, C cogenerates B if $0 = \cap \ker(\varphi)$ where the intersection runs over all R-homomorphism $\varphi : B \to C$. This is equivalent to saying that for each non-zero R-homomorphism $\lambda: L \to B$, there exists an R-homomorphism $\varphi: B \to C$ such that $\varphi\lambda \neq 0$([7], theorem(3.3.3)).

<u>Theorem 3.7</u>: Let M be an R-module, $S = End_R(M)$ and W a non-zero right ideal of S. Then

1- Assume that M generates $\ker(\beta)$ for each β in S. Then S is a right fully strongly W-stable if and only if $\ker(\beta) \subseteq \ker(\delta)$ implies that $\delta \in \beta S \cap SW$.

2- Assume that M cogenerates $M/\beta(M)$ for each $\beta \in S$. Then S is a left fully strongly W-stable if and only if $\delta(M) \subseteq \beta(M)$ implies that $\delta \in S\beta \cap WS$.

Proof : 1- For each $\alpha \in r_S(\beta)$, then $Im(\alpha) \subseteq \ker(\beta) \subseteq \ker(\delta)$, so $\alpha \in r_S(\delta)$. Thus $r_S(\beta) \subseteq r_S(\delta)$ implies that $\delta \in \beta S \cap SW$, corollary 2.7. Conversely, if $\delta \in \ell_S(r_S(\beta))$ and $x \in \ker(\beta)$, then $\beta(x) = 0$ and $x = \sum_{i=1}^{n} \alpha_i(m_i)$ where $m_i \in M$ and $\alpha_i : M \to \ker(\beta)$, hence $0 = \sum_{i=1}^{n} \beta\alpha_i(m_i)$, thus $\beta\alpha_i = 0$ for each i, then $\alpha_i \in r_S(\beta)$, so $\delta\alpha_i = 0$ for each i. Then $\delta(x) = \sum_{i=1}^{n} \delta\alpha_i(m\ ) = 0$ implie that $x \in \ker(\delta)$ Thus $\ker(\beta) \subseteq \ker(\delta)$. The hypothesis implies that $\delta \in \beta S \cap SW$,so by Corollary (2.7), S is right fully strongly W-stable.

2- For each $\alpha \in \ell_S(\beta)$, then $\alpha\beta = 0$, $Im(\delta) \subseteq Im(\beta) \subseteq \ker(\alpha)$ implies that $Im(\delta) \subseteq \ker(\alpha)$ and $\alpha \in \ell_S(\delta)$. Hence $\ell_S(\beta) \subseteq \ell_S(\delta)$. Corollary (2.7) implies that $\delta \in S\beta \cap WS$. Conversely, if $\delta \in r_S(\ell_S(\beta))$ and $\delta(M) \not\subseteq \beta(M)$, then there is $m_0 \in M$ such that $\delta(m_0) \notin \beta(M)$, thus the natural epimorphism $\upsilon: M \to M / \beta(M)$ is non-zero. The cogeneration hypotheses implies there is an R-homomorphism $\sigma : M \to M$ such that $\sigma\upsilon \neq 0$, so $\sigma(\delta(m_0) + \beta(M)) \neq 0$. $f : M \to M$ is well defined by $f(m) = \sigma(m + \beta(M))$ for $m \in M$. Then $f(\delta(m_0)) = f\delta(m_0) = \sigma(\delta(m_0) + \beta(M)) \neq 0$, so $f\delta \neq 0$, while $f\beta = 0$, hence $f \notin \ell_S(\beta)$, but $f\delta \neq 0$ which is a contradiction. Thus $\delta(M) \subseteq \beta(M)$,so the hypothesis implies that $\delta \in S\beta \cap WS$, and then $r_S(\ell_S(\beta)) = S\beta \cap WS$. Corollary (2.7) implies that S is left fully strongly W-stable.

RESIDUAL CONDITION

Let R be a ring with identity and A a non-zero ideal of R.Then by theorem (2.5). R is fully strongly A-stable if and only if $\ell_R(r_R(a)) = aR \cap RA$ for each $a \in R$. If R is commutative ring, then the last condition is equivalent to the condition $[r_R(R): r_R(a)] = [aR \cap RA: R]$ for each $a \in R$ while for arbitrary R-module M, the residual condition does not equivalent to full strong A-stability of M, for example, the Z-module Q satisfies the condition $[r_R(Q) : r_R(x)] = [xZ \cap Q(mZ) : Q]$ where x in Q, but Q is not fully strongly (mZ) -stable for each non-zero $m \in Z$.

First, we prove the following

Prpopostion 4.1: Let M be an R-module and A a non-zero right ideal of R. If M is fully strongly A-stable, then $[r_R (M): r_R (x)] = [xR \cap MA : M]$ for each $x \in M$.

Proof : Let $w \in [r_R (M) : r_R (x)]$ and $m \in M$. Define $\theta : xR \to M$ by $\theta(xr) = mrw$, for $r \in R$. If $xr = 0$ and since $r_R (w) \subseteq r_R (m)$, then $mrw = 0$, thus θ is well defined. It is clear that θ is R-homomorphism. Then Theorem (2.5) implies that there is $t \in R$ such that $\theta(x) = xt \in MA$, thus $mw \in xR \cap MA$. Therefor $w \in [xR \cap MA : M]$ and hence $[r_R (M) : r_R (x)] \subseteq [xR \cap MA : M]$. The other inclusion is always true.

We need the following lemma which appears in ([1], lemma(3.2.4)), before considering the converse of Proposition (4.1)

Lemma 4.2: Let M be a multiplication R-module (R is a commutative ring) and N a submodule of M. For each R-homomorphism $\theta: N \to M$ we have

1- $[\theta(N) : M] \subseteq [r_R (M) : r_R (N)]$

2- $\theta(N) \subseteq M[r_R (M) : r_R (N)]$

Theorem 4.3: Let M be a multiplication R-module, (R is a commutative ring) and A a non-zero ideal of R. If $[r_R (M): r_R (x)] \subseteq [xR \cap MA : M]$ for each $x \in M$, then M is fully strongly A-stable.

Proof: Let xR be a cyclic submodule of M. For each $m \in M$ and $e \in [r_R(M) : r_R(x)]$, $\alpha_{(m,e)} : xR \to M$ is well defined by $\alpha_{(m,e)} (xr) = mre$, for $r \in R$. By the choice of e, m and the condition $[r_R(M) : r_R (x)] \subseteq [xR \cap MA : M]$ we have $\alpha_{(m,e)} (xR) \subseteq xR \cap MA$. Now for each R-homomorphism $\alpha : xR \to M$, by Lemma (4.3) we get $\alpha(xR) \subseteq M[r_R (M) : r_R (x)]$, thus $\alpha(xr) = \alpha(x)r = \sum_{i=1}^{n} m_i r e_i$ for some $m_i \in M$ and $e_i \in [r_R(M):r_R(x)]$ so $\alpha(xr) = \sum_{i=1}^{n} \alpha_{(m_i,e_i)}(xr)$. Therefore $\alpha = \sum_{i=1}^{n} \alpha_{(m_i,e_i)}$ and hence hence $\alpha(xR) \subseteq xR \cap MA$. That is M is fully strongly A-stable.

Then we have the following corollary which gives a characterization of fully strongly A-stable modules in terms of residual condition.

Corollary 4.4: Let M be a multiplication R-module and A a non-zero ideal of R. Then M is fully strongly A-stable if and only if $[r_R(M):r_R(x)] = [xR \cap MA : M]$ for each $x \in M$.

As every cyclic module over a commutative ring R is multiplication, in the following we have the motivation that mentioned at beginning of this section.

Corollary 4.5: Let A be a non-zero ideal of a ring R. Then R is fully strongly A-stable if and only if $[r_R(R): r_R(x)] = [xR \cap A: R]$ for each $x \in R$.

There is no comparison between multiplication modules and the residual condition, for example Z is a multiplication Z-module which does not satisfy the residual condition while Z_{p^∞} is a fully strongly A-stable Z-module for each non-zero ideal A of Z and hence by Proposition (4.1) satisfies the residual condition, but Z_{p^∞} is not multiplication.

Definition 4.6: Let M be an R-module and A a non-zero ideal of R. A submodule N of M is called A-idempotent in M if $N = N[N \cap MA : N]$

Next, we consider modules in which each submodule is A-idempotent this is equivalent to saying that each cyclic submodule is A-idempotent.

Proposition 4.7: Let M be an R-module and A a non-zero ideal of R. If each submodule of M is A-idempotent, then M is fully strongly A-stable.

Proof: Let N be a submodule of M and $\alpha: N \to M$ an R-homomorphism. For each $n \in N = N[N \cap MA:M]$, then $n = \sum_{i=1}^{n} n_i r_i$ for some $n_i \in N$ and $r_i \in [N \cap MA : M]$. Thus $\alpha(n_i) r_i \in Mr_i \subseteq N \cap MA$ for each i and so $\alpha(N) \subseteq N \cap MA$.

The Z-module Z_8 is fully strongly (3Z)-stable, Theorem (2.5). Consider the submodule $N = \{\bar{0}, \bar{4}\}$ of Z_8. $N[N \cap Z_8(3Z) : Z_8] = N[N : Z_8] = N(4Z) = 0 \neq N$. This shows Z_8 has a submodule which is not (3Z) -idempotent.

For the converse of Proposition (4.7), recall that an R-module M is prime if $r_R(M) = r_R(N)$ for each non-zero submodule N of M.

Theorem 4.8: Let M be a prime R-module and A a non-zero ideal of R. Then the following statements are equivalent

1- M is fully strongly A-stable

2- $[r_R(M) : r_R(x)] = [xR \cap MA:M]$ for each $x \in M$

3- Every submodule of M is A-idempotent.

Proof: (1) ⇒ (2): Follows from Proposition (4.1)

(2) ⇒ (3): Let N be a submodule of M, and a non-zero element n in N (no loss of generality if we assume that N is non-zero). Then by (2), $R = [r_R(M) : r_R(n)] = [nR \cap MA : M] \subseteq [N \cap MA:M] \subseteq R$, so $R = [N \cap MA : M]$ and hence $N = N[N \cap MA:M]$

(3) ⇒ (1): Follows from Proposition (4.7)

REFRENCESS

1. Abbas, M. S., On fully stable modules, Ph.D. Thesis, Univ. of Baghdad, 1990.
2. Abbas M. S. and A. M. Sharky, Fully stable modules relative to an ideal, Al-Mustansiriya J. Sc. Vol. (21).No. 6, 2010, 392-400.
3. Barnard, A. Multiplication modules, Algebra, 71, 174-178, (1981).
4. Lam, T. y., Lectures on Modules and Rings, Springer-Verlag, Berlim, New York, 1998
5. Fuchs, L. Abelian Group, Academic press, 1960.
6. Zelmanowitz, J., Regular modules, Trans. Amer. Math. Soc., 163 (1972), 341-355.
7. Kasch, F. Modules and Rings, Academic Press, Lordon, New York, 1982.