Alternating Direction Implicit Formulation of the Cosine-based DQM for Solving Navier-Stokes Equation

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Abstract

In this article, we have discussed a new application of alternating direction implicit formulation of the differential quadrature method (ADI-DQM) (Al-Saif A.S.J. et al. 2011,2012) [1, 2] on Navier-Stokes equations . the weighting coefficient computing by Cosine expansion based. Numerical results of one example, show that the present method has been high accuracy ,good convergence comparing with using weighting coefficients is Lagrange interpolation polynomial.

Key words :Cosine -based differential quadrature method (CDQM), Navier-Stokes equations, ADI, accuracy.

1. Introduction

Consider the two-dimensional Navier-Stokes equations:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$(x,y)\in\Omega\ ,\quad t>0 \qquad (1.1a)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$(x,y)\in\Omega\ ,\quad t>0 \qquad (1.1b)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1.1c)$$

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=-(\frac{\partial u}{\partial x})^2-(\frac{\partial v}{\partial y})^2-2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad (1.1d)$$

the computational domain is taken as $\Omega=\{(x,y): a\le x,y\le b\}$

with initial condition

$$u(x,y,0)=\emptyset_1(x,y)\ , v(x,y,0)=\emptyset_2(x,y) \qquad (1.1e)$$

and the boundary conditions

$$\left.\begin{matrix} u(x,y,t)=f(x,y,t)\\ v(x,y,t)=g\,(x,y,t)\end{matrix}\right\}\ (x,y)\in\partial\,\Omega$$

$$t>0 \qquad (1.1d)$$

where Re is the Reynolds number , u and v are velocity components and $\emptyset_1,\emptyset_2,f$ and g are the known functions . For a positive integer n, let $h=(b-a)\,/\,n$ denote the step size of spatial space and Δt is the step size with respect to time.

The stream function $\psi(x,y,t)$ is defined for two- dimensional flows: the partial derivatives of the stream function are linked with the velocity components through the relation:

$$\frac{\partial\psi}{\partial x}=-v\ ,\qquad \frac{\partial\psi}{\partial y}=-u \qquad (1.2)$$

The equation (1.1a) is differentiated with respect to y and the second equation (1.1b) is differentiated with respect to . Then, equation (1.1a) is subtracted from the equation (1.1b) one, so that the pressure is eliminated.

Substituting the definition of vorticity $\omega = v_x - u_y$ yields the vorticity transport equation as:

$$\frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y} - \frac{1}{Re}\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right) = 0 \qquad (1.3)$$

Substituting (1.2) in (1.3) we get :

$$\omega_t + \psi_y\omega_x - \psi_x\omega_y = \frac{1}{Re}(\omega_{xx} + \omega_{yy}) \qquad (1.4)$$

The Navier- Stokes equations are the fundamental nonlinear partial differential equations in almost every real situation that describe the motion of fluid, i.e. liquids and gases. These equations, named after Claude-Louis Navier (1822) and George Gabriel Stokes (1848) and these equations are one of the most useful sets of equations because they can be used to describe many different engineering problems. The solution of these equations describes the velocity of the fluid at a given point in space and time, and is called a velocity field. They may be used to model weather, ocean currents, water flow in a pipe, flow around an airfoil, and motion of stars inside a galaxy, the design of aircraft and cars, the study of blood flow, the design of power stations, etc. Many researchers seek the solution of the Navier- Stokes equations such as [3, 4] .The development of new techniques from the standpoint of computational efficiency and numerical accuracy is of primal interest. Since it has been developed, several researchers have applied successfully the differential quadrature method to solve a variety of problems in different fields of science and engineering. In this work weighting coefficients computing by Cosine expansion based comparing with Lagrange interpolation polynomial.

2. Differential Quadrature Method

The differential quadrature is a numerical technique used to solve the initial and boundary value problems. This method was proposed by Bellman in the early 70s . [5]. The essence of the method is that the partial (ordinary) derivatives of a function with respect to a variable in governing equation are approximated by a weighted linear sum of function values at all discrete points in that direction, then the equation can be transformed into a set of ordinary differential equations or algebraic equations.

The first –order and the second –order derivatives of a function $\omega(x, y)$ at a point $x = x_i$ along any line $y = y_j$ parallel to the x–axis, can be approximated by DQM may be written as :

$$\left.\frac{\partial\omega}{\partial x}\right|_{x=x_i} = \sum_{k=1}^{N} A_{ik}^{(1)}\omega(x_k, y) \qquad i = 1,2, \dots, N \qquad (2.1)$$

$$\left.\frac{\partial^2\omega}{\partial x^2}\right|_{x=x_i} = \sum_{k=1}^{N} A_{ik}^{(2)}\omega(x_k, y) \qquad i = 1,2, \dots, N \qquad (2.2)$$

The first –order and the second –order derivatives of a function $\omega(x , y)$ at a point $y = y_j$ along any line $x = x_i$ parallel to the y–axis, can be approximated by DQM may be written as :

$$\left.\frac{\partial\omega}{\partial y}\right|_{y=y_j} = \sum_{l=1}^{M} B_{jl}^{(1)}\omega(x, y_l) \qquad j = 1,2, \dots, M \qquad (2.3)$$

$$\left.\frac{\partial^2\omega}{\partial y^2}\right|_{y=y_j} = \sum_{l=1}^{M} B_{jl}^{(2)}\omega(x, y_l) \qquad j = 1,2, \dots, M \qquad (2.4)$$

where $A_{ik}^{(1)}\ and\ B_{jl}^{(1)}$ are the respective weighting coefficients of the first order derivatives with respect to x and y respectively , $A_{ik}^{(2)}\ and\ B_{jl}^{(2)}$ are the respective weighting coefficients of the second order derivatives with respect to x and y respectively, such that [6].

$$A_{ik}^{(1)} = \frac{-\alpha p(x_i) sinx_i}{(\cos x_i - \cos x_k)p(x_k)}$$
$$for \quad i \neq k \tag{2.5}$$

$$A_{ii}^{(1)} = -\sum_{k=1, i\neq k}^{N} A_{ik}^{(1)} \tag{2.6}$$

$$A_{ik}^{(2)} = A_{ik}^{(1)}\left(2A_{ii}^{(1)} + \frac{2 \propto sinx_i}{\cos x_i - \cos x_k} + \propto cotx_i\right)$$

$$for \quad i \neq k \tag{2.7}$$

$$A_{ii}^{(2)} = -\sum_{k=1, i\neq k}^{N} A_{ik}^{(2)} \tag{2.8}$$

Where $\propto = \frac{\pi}{b-a}$ and $p(x_i) = \prod_{k=1, i\neq k}^{N}(\cos x_i - \cos x_k)$, In the same procedure can by written $B_{jl}^{(1)} and\ B_{jl}^{(2)}$

By differential quadrature method, we approximate the partial derivatives of the equation(1.4). Using equations (2.1) , (2.2) ,(2.3) and (2.4) in equation (1.4), we obtain the system of ordinary differential equation as:

$$\left.\frac{\partial \omega}{\partial t}\right|_{ij}^{n} + \sum_{k=1}^{N} \psi_{y_{ij}} A_{ik}^{(1)} \omega_{kj} - \sum_{l=1}^{M} \psi_{x_{ij}} B_{jl}^{(1)}\, \omega_{il}$$
$$= \frac{1}{Re}\left(\sum_{k=1}^{N} A_{ik}^{(2)} \omega_{kj} + \sum_{l=1}^{M} B_{jl}^{(2)} \omega_{il}\right) \tag{2.9}$$

Approximation the first-order derivatives with respect to the temporal variable in the equation (2.9) by using the forward differences and arrangement the terms of equation (2.9) ,we obtain the system of algebraic equation as:

$$\frac{\omega_{ij}^{n+1} - \omega_{ij}^{n}}{\Delta t} + \sum_{k=1}^{N}\left(\psi_{y_{ij}}^{\ n} A_{ik}^{(1)} - \frac{1}{Re} A_{ik}^{(2)}\right)\omega_{kj}^{n} -$$

$$\sum_{l=1}^{M}(\psi_{x_{ij}}^{\ n} B_{jl}^{(1)} + \frac{1}{Re} B_{jl}^{(2)})\omega_{il}^{n} = 0 \tag{2.10}$$

3. Alternating Direction Technique of the DQM

The alternating direction implicit technique was introduced in the mid-50s by Peaceman and Rachford [7] for solving equations, which result from finite difference discretization of partial differential equations (PDEs). From iterative method's perspective, ADI method can be considered as special relaxation method, where a big system is simplified into a number of smaller systems such that each of them can be solved efficiently and the solution of the whole system is got from the solutions of the sub-systems in an iterative way. Using alternating direction implicit method into equation (2.10), we get the systems of algebraic equations in the form:

$$\frac{\omega_{ij}^{n+\frac{1}{2}} - \omega_{ij}^{n}}{\frac{\Delta t}{2}} + \sum_{k=1}^{N}\left(\psi_{y_{ij}}^{\ n} A_{ik}^{(1)} - \frac{1}{Re} A_{ik}^{(2)}\right)\omega_{kj}^{n+\frac{1}{2}}$$
$$-\sum_{l=1}^{M}(\psi_{x_{ij}}^{\ n} B_{jl}^{(1)} + \frac{1}{Re} B_{jl}^{(2)})\omega_{il}^{n} = 0 \tag{3.1}$$

$$\frac{\omega_{ij}^{n+1} - \omega_{ij}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}} + \sum_{k=1}^{N}\left(\psi_{y_{ij}}^{\ n+\frac{1}{2}} A_{ik}^{(1)} - \frac{1}{Re} A_{ik}^{(2)}\right)\omega_{kj}^{n+\frac{1}{2}}$$

$$-\sum_{l=1}^{M}(\psi_{x_{ij}}^{n+\frac{1}{2}}B_{jl}^{(1)}+\frac{1}{Re}B_{jl}^{(2)})\omega_{il}^{n+1}=0 \qquad (3.2)$$

Formula (3.1) is used to compute function values at all interval mesh points along rows and known as horizontal traverse or x −sweep. While, Formula (3.2) is used to compute function values at all interval mesh points along columns and known as vertical traverse or y −sweep.

4. Numerical Experiments and Discussion

In this section, we apply ADI-CDQM and ADI-DQM on one test problem which are also considered by other researchers.

Problem [(8)]

The exact solutions to the equation (1.4) can be written as:

$$\omega(x,y,t)=\frac{2B_0}{Rec}+\frac{2}{Rec}e^{\frac{Rec}{2}(x+y+ct+\xi_0)} \qquad (4.1)$$

$$\psi(x,y,t)=\frac{-2}{(Re)^2c^2}\left[\alpha+\frac{2B_0}{Rec}+\frac{2}{Rec}e^{\frac{Rec}{2}(x+y+ct+\xi_0)}\right] \qquad (4.2)$$

The boundary condition can be obtained easily from (4.1) by using $x,y=0,1$. In this problem, we found numerical results for ω and using equally spaced grid points. In Tables (1,…,4) we shows the errors obtained in solving problem with the ADI-DQM by using Weighting coefficients are Lagrange interpolated polynomials and Cosine expansion based at t=1, t=0.1, $\Delta t=0.001$, $Re=10,100$ and $x,y\in[0,1]$ for different values of h.. In Figs. (1) and (2) we shows the exact and approximate solutions for $t=0.1$, $\Delta t=0.001$ and $Re=10,100$ respectively. The numerical results given in tables (1,…,4) confirm that ADI-DQM by using weighting coefficients Cosine expansion based has been , high accuracy, good convergence compare with weighting coefficients Lagrange interpolated polynomials.

Table 1. Errors obtained by **ADI-DQM** for problem with t=0.1, $\Delta t=0.001$, $\xi_0=0$ $Re=10$ *and* $B_0=c=0.1$

	Max|error| for the ω by ADI-DQM with weighting coefficients is	
$N\times M$	***Cosine expansion based*** *(ADI-CDQM)*	***Lagrange interpolated polynomials*** *(ADI-DQM)*
5×5	1.757778E-08	3.564035E-08
10×10	1.265919E-08	1.353973E-07
15×15	1.163422E-08	2.450875E-07

Table 2. Errors obtained by **ADI-DQM** for problem with t=0.1, $\Delta t=0.001$, $\xi_0=0$ $Re=100$ *and* $B_0=c=0.1$

	Max|error| for the ω by ADI-DQM with weighting coefficients is	
$N\times M$	***Cosine expansion based*** *(ADI-CDQM)*	***Lagrange interpolated polynomials*** *(ADI-DQM)*
5×5	1.565777E-06	1.485690E-06
10×10	4.703395E-08	1.212858E-07
15×15	4.387702E-08	1.173604E-07

Table 3. Errors obtained by **ADI-DQM** for problem with t=1, $\Delta t = 0.001$, $\xi_0 = 0$ $Re = 10$ *and* $B_0 = c = 0.1$

	Max|error| for the ω by ADI-DQM with weighting coefficients is	
$N \times M$	***Cosine expansion based*** *(ADI-CDQM)*	***Lagrange interpolated polynomials*** *(ADI-DQM)*
5×5	1.743789E-06	1.895389E-06
10×10	1.314255E-06	1.965212E-06
15×15	1.213379E-06	2.216335E-06

Table 4. Errors obtained by **ADI-DQM** for problem with t=1, $\Delta t = 0.001$, $\xi_0 = 0$ $Re = 100$ *and* $B_0 = c = 0.1$

	Max|error| for the ω by ADI-DQM with weighting coefficients is	
$N \times M$	***Cosine expansion based*** *(ADI-CDQM)*	***Lagrange interpolated polynomials*** *(ADI-DQM)*
5×5	2.498984E-04	2.499745E-04
10×10	1.553376E-05	1.582487E-05
15×15	7.024219E-06	9.673604E-06

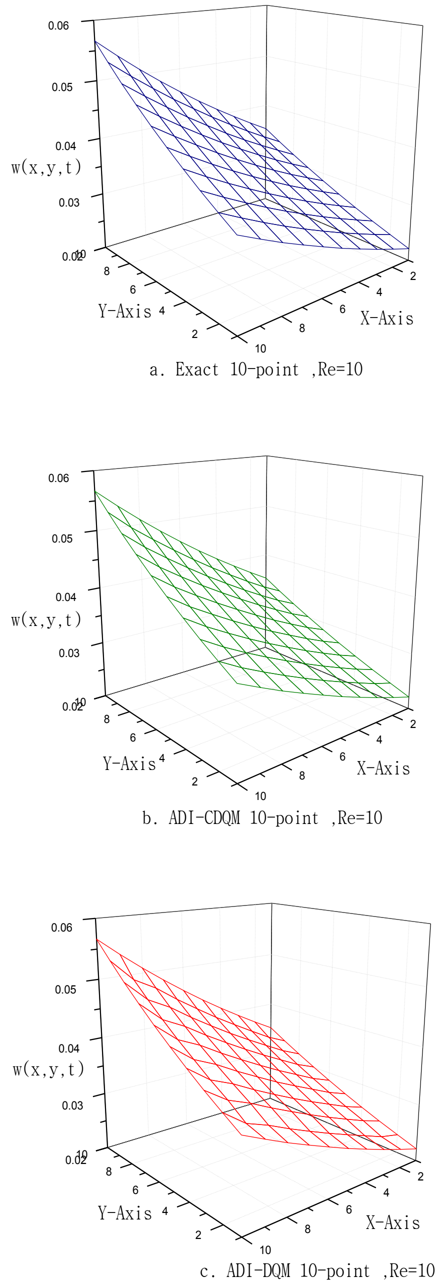
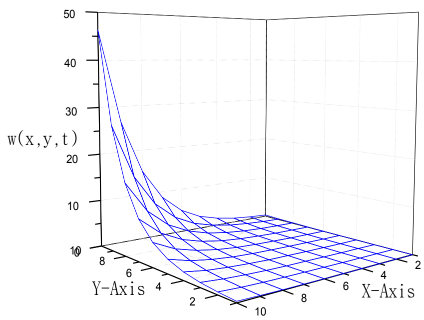
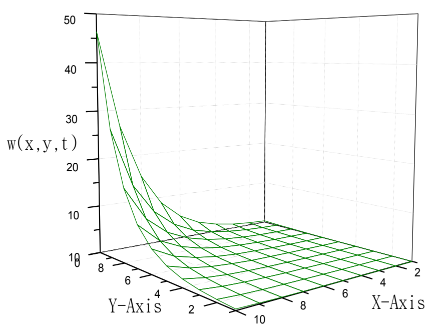


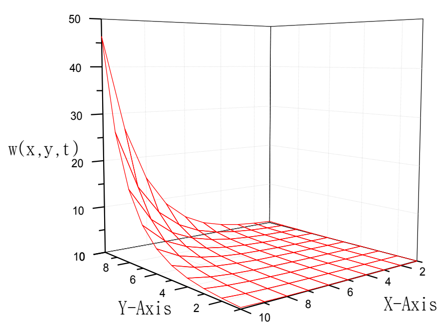
Fig. 1 Exact and approximate solution of the problem with, t=0.1 $Re = 10$ and $\Delta t = 0.001$



a. Exact 10-point ,Re=100



b- ADI-CDQM 10-point ,Re=100



c. ADI-DQM 10-point ,Re=100

Fig. 2 Exact and approximate solution of the problem with, t=0.1 $Re = 100$ and $\Delta t = 0.001$

7. Conclusions

In this work, we employed the ADI-DQM by using weighting coefficients are Lagrange interpolated polynomials and Cosine expansion based. The methods applied for the solution of the Navier-Stokes equations in two-dimension. The numerical results show that the Cosine expansion based has the higher accuracy and convergence comparing with Lagrange interpolated polynomials. The accuracy of the method depends on the number and type of grid points chosen for DQM.

The stability of the DQM applied depends on the eigenvalues of differential quadrature discretization matrices. These eigenvalues in turn vary much depend on the distribution of grid points.

The results, show that is of high accuracy can be obtained if the number of grid points is large, while the stability requires the inverse.

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صيغة الاتجاه الضمني المتناوب لطريقة التفاضل التربيعي لحل معادلة نافير– ستوكس

محمد جواد كاظم الكناني

المديرية العامة لتربية ذي قار

الخلاصة

في هذا البحث ناقشة تطبيق جديد على صيغة الاتجاه الضمني المتناوب لطريقة التفاضل التربيعي على معادلة نافير– ستوكس ثنائية البعد وذلل من خلال حساب معاملات الوزن باستخدام متعددة حدود الجيب تمام . النتائج العددية بينت ان الطريقة الحالية تمتلك دقة عالة وتقارب جيد مقارنة مع استخدام متعددة حدود لاكرانج .