تعميم تمركزات الجوردان على الحلقات من نمط - Γ

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الخلاصة:
في هذا البحث عممنا نتيجة Md.Fazlud, A.C.Paul , B. Zalar على تعميم تمركز حلقة من النمط - Γ وكذلك برهنا ان كل تعميم تمركزات= الجوردان في حلقة اولية تامة من النمط (2) في حلقة من النمط - Γ طليقة الالتواء هي تعميم تمركزات في نفس الحلقة من النمط - Γ.

الكلمات المفتاحية: تعميم تمركز ، تعميم تمركز جوردان، حلقة أولية تامة من النمط - Γ.

Jordan Generalized Centralizer on Γ-Rings

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Abstract:
In this paper we generalize the results of Md.Fazlud Hoque, and A.C.Paul and B.Zalar on generalized centralizer of completely prime Γ-ring. We prove that every Jordan generalized centralizer of a 2-torsion free completely prime Γ-ring M is a generalized centralizer.

Keywords: Generalized centralizer, Jordan generalized centralizer, completely prime Γ-ring.

1. Introduction:
Let M and Γ be two additive abelian groups. Then M is called a Γ-ring if for any x, y, z ∈ M and α, β ∈ Γ the following conditions are satisfied
(1) $x\alpha y \in M$,

(2) $(x + y)\alpha z = x\alpha z + y\alpha z$,
$x(\alpha + \beta)z = x\alpha z + x\beta z$,
$x\alpha(y + z) = x\alpha y + x\alpha z$ and
(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$.

The notion of a Γ-ring was introduced by Nobusawa [6] and generalized by Barnes [1] as defined above. Many properties of Γ-ring were obtained by Barnes [1], Kyuno [5], Nobusawa [6] and others.

S.Chakraborty and A.Chandra Paul [3], defined 2-torsion free as follows,

Let M be a Γ-ring. Then M is said to be 2-torsion free if $2a = 0$ implies $a = 0$ for all $a \in M$. Besides, M is called a prime gamma ring if, for all $a, b \in M$, $a\Gamma M\Gamma b = (0)$ implies either $a = 0$ or $b = 0$. And, M is called semiprime if $a\Gamma M\Gamma a = (0)$ with $a \in M$ implies $a = 0$.

Note that every prime Γ-ring is obviously semiprime. And M is called a completely prime if $a\Gamma b = (0)$ implies $a = 0$ or $b = 0$ where $a, b \in M$. It is clear that $a\Gamma b\Gamma a\Gamma b \subset a\Gamma M\Gamma b$, therefore every completely prime Γ-ring is prime.

Let M be a Γ-ring and $d: M \longrightarrow M$ an additive map. Then d is called a derivation if

$$d(x\alpha y) = d(x)\alpha y + x\alpha d(y), \text{ for all } x, y \in M \text{ and } \alpha \in \Gamma.$$

and d is called a Jordan derivation if:

$$d(x\alpha x) = d(x)\alpha x + x\alpha d(x), \text{ for all } x \in M \text{ and } \alpha \in \Gamma.$$

If M is a Γ-ring, then $[x,y]_\alpha = x\alpha y - y\alpha x$ is known as the commutator of x and y with respect to α, where $x, y \in M$ and $\alpha \in \Gamma$.

Y.Ceven and M.Ali Özturk [3], defined generalized derivation as follows, an additive mapping $F: M \longrightarrow M$ is called a generalized derivation if there exists a derivation $d: M \longrightarrow M$ such that $F(x\alpha y) = F(x)\alpha y + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. Finally F is called a Jordan generalized derivation if there exists a derivation $d: M \longrightarrow M$ such that $F(x\alpha x) = F(x)\alpha x + x\alpha d(x)$ for all $x \in M$ and $\alpha \in \Gamma$.

B.Zalar [10], defined left centralizer as follows, an additive mapping $T: R \longrightarrow R$ is called left (resp. right) centralizer if $T(x,y) = T(x)y$ (resp. $T(xy) = xT(y)$, for all $x, y \in R$. If T is a left and right centralizer, then T is a centralizer, an additive mapping $T: R \longrightarrow R$ is called Jordan left (right) centralizer in case $T(x^2) = T(x)x$ (resp. $T(x) = xT(x)$), for all $x \in R$. Obviously every left (right) centralizer a Jordan left (right) centralizer. The converse is not true in general. Md.Fazlul Hoque and A.C.Paul [4] proved that every centralizer of semiprime Γ-ring is a Jordan centralizer, but the converse is true by condition.

Joso Vakman [7,8,9] developed some remark able results using centralizers on prime and semiprime rings.

Md.Fazlul Hoque and A.C.Paul [4], prove that every Jordan left centralizer (Jordan centralizer) of a 2-torsion free semiprime Γ-ring satisfiying ($x\alpha y\beta z = x\beta y\alpha z$, for all x, y, z ∈ M and α, β ∈ Γ) is a left centralizer (centralizer).

In this paper, we introduce and study some results concerning the generalized centralizer and Jordan generalized centralizer on Γ-ring.

2. Generalized Centralizer

Now we will introduce the definition of generalized centralizer and Jordan generalized centralizer on Γ-ring and other concepts which will be used in our work.

<u>2.1 Definition:</u>

Let M be a Γ-ring and $f : M \longrightarrow M$ be an additive mapping, then f is said to be **a generalized centralizer of M** if there exists a centralizer T of M such that

$f(x\alpha y + y\beta x) = f(x)\alpha y + y\beta T(x)$, for all x, y ∈ M, α, β ∈ Γ.

We give below an example of generalized centralizer on Γ-ring.

<u>2.2 Example:</u>

Let $M = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in \mathbb{Z}, \text{ where } \mathbb{Z} \text{ is a integer number} \right\}$ be a Γ-ring of 2×2 matrices $\Gamma = \left\{ \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} : n \in I \right\}$ with respect to the usual operation of addition and multiplication, let $f : M \longrightarrow M$ be a generalized centralizer defined as $f\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ for all x ∈ M, if there exists a centralizer T: $M \longrightarrow M$ be additive mapping defined as $T\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ we use the usual addition and multiplication on matrices of $M\times\Gamma\times M$. Then f is a generalized centralizer.

2.3 Definition:

Let M be a Γ-ring and $f:M \longrightarrow M$ be additive mapping of M. Then f is called **Jordan generalized centralizer of M** if satisfies:

$f(x\alpha x + x\alpha x) = f(x)\alpha x + x\alpha T(x)$, for all $x \in M$, $\alpha \in \Gamma$.

In example (2.2) also f is Jordan generalized centralizer

It is clear that every generalized centralizer of M is a Jordan generalized centralizer, but the converse is not true in general.

In the following Lemma we give the properties of generalized centralizer.

2.4 Lemma:

Let M be a Γ-ring and f be a Jordan generalized centralizer of M. Then for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, the following statements hold

(i) $f(x\alpha y + y\alpha x) = f(x)\alpha y + y\alpha T(x)$.

(ii) $f(x\alpha y\beta x + x\beta y\alpha x) = f(x)\alpha y\beta x + x\alpha y\beta T(x)$.

(iii) $f(x\alpha y\alpha z + z\alpha y\alpha x) = f(x)\alpha y\alpha z + z\alpha y\alpha T(x)$.

In particular $x\alpha y\beta x = x\beta y\alpha x$

(iv) $f(x\alpha y\beta z + z\alpha y\beta x) = f(x)\alpha y\beta z + z\alpha y\beta T(x)$.

Proof:

(i) $f((x + y)\alpha(x + y)) = f(x\alpha x + x\alpha y + y\alpha x + y\alpha y)$

$= f(x\alpha x + y\alpha y) + f(x\alpha y + y\alpha x)$

$= f(x)\alpha x + y\alpha T(y) + f(x)\alpha y + y\alpha T(x)$ …(1)

On the other hand

$f((x + y)\alpha(x + y)) = f(x\alpha x + x\alpha y + y\alpha x + y\alpha y)$

$= f(x\alpha x + y\alpha y) + f(x\alpha y + y\alpha x)$

$= f(x)\alpha x + y\alpha T(y) + f(x\alpha y + y\alpha x)$ …(2)

By comparing (1) and (2) we get:

$f(x\alpha y + y\alpha x) = f(x)\alpha y + y\alpha T(x)$.

(ii) Replacing $x\beta y + y\beta x$ for y in (i)

$= f(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x)$

$= f(x)\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha T(x)$

$= f(x)\alpha x\beta y + f(x)\alpha(y\beta x) + x\beta y\alpha T(x) + y\beta x\alpha T(x)$ …(3)

On the other hand

$= f(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x)$

$= f(x\alpha x\beta y + x\alpha y\beta x + x\beta y\alpha x + y\beta x\alpha x)$

$= f(x\alpha(x\beta y) + y\beta(x\alpha x)) + f(x\alpha y\beta x + x\beta y\alpha x)$

$= f(x)\alpha x\beta y + y\beta T(x\alpha x) + f(x\alpha y\beta x + x\beta y\alpha x)$

$= f(x)\alpha x\beta y + y\beta x\alpha T(x) + f(x\alpha y\beta x + x\beta y\alpha x)$ …(4)

By comparing (3) and (4) we get

$f(x\alpha y\beta x + x\beta y\alpha x) = f(x)\alpha y\beta x + x\alpha y\beta T(x).$

(iii) Replacing $x + z$ for x and α for β in (ii) we get:

$f((x + z)\alpha y\alpha(x + z) + (x + z)\alpha y\alpha(x + z))$

$= f(x + z)\alpha y\alpha(x + z) + (x + z)\alpha y\alpha T(x + z)$

$= f(x)\alpha y\alpha x + f(x)\alpha y\alpha z + f(z)\alpha y\alpha x + f(z)\alpha y\alpha z + x\alpha y\alpha T(x) + x\alpha y\alpha T(z) + z\alpha y\alpha T(x) + z\alpha y\alpha T(z)$ …(5)

On the other hand

$f((x + z)\alpha y\alpha(x + z) + (x + z)\alpha y\alpha(x + z))$

$= f(x\alpha y\alpha x + x\alpha y\alpha z + z\alpha y\alpha x + z\alpha y\alpha z + x\alpha y\alpha x + x\alpha y\alpha z + z\alpha y\alpha x + z\alpha y\alpha z)$

$= f(x\alpha y\alpha x + x\alpha y\alpha x) + f(z\alpha y\alpha x + z\alpha y\alpha x) + f(z\alpha y\alpha x + x\alpha y\alpha z) + f(x\alpha y\alpha z + z\alpha y\alpha x)$

By lemma (2.4) (ii) we get:

$= f(x)\alpha y\alpha x + x\alpha y\alpha T(x) + f(z)\alpha y\alpha x + z\alpha y\alpha T(x) + f(z)\alpha y\alpha x + x\alpha y\alpha T(z) + f(x\alpha y\alpha z + z\alpha y\alpha x)$ …(6)

By comparing (5) and (6) we get:

$f(x\alpha y\alpha z + z\alpha y\alpha x) = f(x)\alpha y\alpha z + z\alpha y\alpha T(x)$

(iv) Replacing $x + z$ for x in (ii) we get:

$= f((x + z)\alpha y\beta(x + z) + (x + z)\beta y\alpha(x + z))$

$= f(x + z)\alpha y\beta(x + z) + (x + z)\beta y\alpha T(x + z)$

$= f(x)\alpha y\beta x + f(x)\alpha y\beta z + f(z)\alpha y\beta x + f(z)\alpha y\beta T(z) + x\beta y\alpha T(x) +$

$x\beta y\alpha T(z) + z\beta y\alpha T(x) + z\beta y\alpha T(z)$ …(7)

On the other hand

$= f((x + z)\alpha y\beta(x + z) + (x + z)\beta y\alpha(x + z))$

$= f(x\alpha y\beta x + x\alpha y\beta z + z\alpha y\beta x + z\alpha y\beta z + x\beta y\alpha x + x\beta y\alpha z + z\beta y\alpha x + z\beta y\alpha z$

$= f(x\alpha y\beta x + x\beta y\alpha x) + f(z\beta y\alpha x + x\beta y\alpha z) + f(z\alpha y\beta z + z\beta y\alpha z) + f(x\alpha y\beta z + z\alpha y\beta x)$

$= f(x)\alpha y\beta x + x\beta y\alpha T(x) + f(z)\beta y\alpha x + x\beta y\alpha T(z) + f(z)\alpha y\beta z + z\beta y\alpha T(z) + f(x\alpha y\beta z + z\alpha y\beta x)$ …(8)

By comparing (7) and (8) we get

$f(x\alpha y\beta z + z\alpha y\beta x) = f(x)\alpha y\beta z + z\alpha y\beta T(x)$.

2.5 Definition:

Let M be a Γ-ring and f be a generalized centralizer of M then for every x, y∈ M and α ∈ Γ we define $\delta(x,y)_\alpha : M\times\Gamma\times M \longrightarrow M$ by $\delta(x,y)_\alpha = f(x\alpha y) - x\alpha T(y)$, for all x, y∈ M and α ∈ Γ.

Now we give in the following lemma the properties of $\delta(x,y)_\alpha$.

2.6 Lemma:

Let M be a Γ-ring and f be generalized centralizer then for all x, y, z ∈ M and α, β ∈ Γ.

(i) $\delta(x + z,y)_\alpha = \delta(x,y)_\alpha + \delta(z,y)_\alpha$

(ii) $\delta(x,y + z)_\alpha = \delta(x,y)_\alpha + \delta(x,z)_\alpha$

(iii) $\delta(x,y)_{(\alpha+\beta)} = \delta(x,y)_\alpha + \delta(x,y)_\beta$

Proof:

(i) $\delta(x + z,y)_\alpha = f((x + z)\alpha y) - (x + z)\alpha T(y)$

$= f(x\alpha y + z\alpha y) - x\alpha T(y) - z\alpha T(y)$

$= f(x\alpha y) - x\alpha T(y) + f(z\alpha y) - z\alpha T(y)$
$= \delta(x,y)_\alpha + \delta(z,y)_\alpha$

(ii) $\delta(x,y+z)_\alpha = f(x\alpha(y+z)) - x\alpha T(y+z)$
$= f(x\alpha y + x\alpha z) - x\alpha T(y) - x\alpha T(z)$
$= f(x\alpha y) - x\alpha T(y) + x\alpha z - x\alpha T(z)$
$= \delta(x,y)_\alpha + \delta(x,z)_\alpha$

(iii) $\delta(x,y)_{(\alpha+\beta)} = f(x(\alpha+\beta)y) - x(\alpha+\beta)T(y)$
$= f(x\alpha y) + f(x\beta y) - x\alpha T(y) - x\beta T(y)$
$= f(x\alpha y) - x\alpha T(y) + f(x\beta y) - x\beta T(y)$
$= \delta(x,y)_\alpha + \delta(x,y)_\beta$

2.7 Remark:

Note that f is generalized centralizer of Γ-ring M into Γ-ring M if and only if $\delta(x,y)_\alpha = 0$ for all $x, y \in M$, $\alpha \in \Gamma$.

3. Main Results

In this section we introduce our main results.

3.1 Theorem:

Let M be a 2-torsion free completely prime Γ-ring such that $0 \neq x \in Z(M)$ and f be a Jordan generalized centralizer of M such that $x\alpha y\beta x = x\beta y\alpha x$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Then $\delta(x,y)_\alpha = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Proof:

Since f is a Jordan generalized centralizer of M then for all $0 \neq x \in M$, $y \in M$ and $\alpha, \beta \in \Gamma$. By Lemma (2.4) (ii)

$f(x\alpha y\beta x + x\beta y\alpha x) = f((x\alpha y)\beta x + x\beta(y\alpha x))$
$= f(x\alpha y)\beta x + x\beta T(y\alpha x)$
$= f(x\alpha y)\beta x + x\beta T(y)\alpha x$ ...(9)

On the other hand since $x\alpha y\beta x = x\beta y\alpha x$

$f(x\beta y\alpha x + x\alpha y\beta x) = f((x\beta y)\alpha x + x\alpha(y\beta x))$
$= f(x\beta y)\alpha x + x\alpha T(y\beta x)$
$= f(x\beta y)\alpha x + x\alpha T(y)\beta x$ ...(10)

By comparing (9) and (10) we get

$f(x\alpha y)\beta x - x\alpha T(y)\beta x - f(x\beta y)\alpha x + x\beta T(y)\alpha x = 0$

$f(x\alpha y)\beta x - x\alpha T(y)\beta x = f(x\beta y)\alpha x - x\beta T(y)\alpha x = 0$

$\delta(x,y)_\alpha\beta x = \delta(x,y)_\beta\alpha x = 0$

$\delta(x,y)_\alpha\beta x = 0$

Since M is completely prime Γ-ring and $x \neq 0$ then

$\delta(x,y)_\alpha = 0$.

3.2 Corollary:

Every Jordan generalized centralizer of 2-torsion free completely prime Γ-ring M is generalized centralizer of M

Proof:

By Theorem (3.1) we get $\delta(x,y)_\alpha = 0$ and by Remark (2.7) we get every Jordan generalized centralizer is a generalized centralizer.

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