Using Boubaker Polynomials Method for Solving Mixed Linear Volterra-Fredholm Integral Equations

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Abstract

In this paper, the Boubaker Polynomials is used to find an approximate solution for mixed linear Volterra-Fredholm integral equations of the second kind. Three examples are considered and the simulation results were discussed numerically and displayed graphically. By increasing the n order of Boubaker Polynomials, we can improve the accuracy of results.

Keywords: Boubaker Polynomials Method, Mixed linear Volterra-Fredholm Integral Equations.

1. Introduction

Broadly, the approximation of functions play an important role in real and applied field. In mathematics, they have been used to solve non-linear equations, Ordinary differential equations and Partial differential equations. These functions, reduce the original problems to those of solving a system of linear algebraic equations. Such as Bernstein Polynomials [1] and Boubaker polynomials [2]. Recently, several researchers have been successfully applying Bernstein polynomials to various linear and nonlinear integral equations. Intisar [3] used Bernstein Polynomials method to solve high-order nonlinear mixed Volterra-Fredholm integro equations. On the other hand, Boubaker polynomials, which have useful properties such as the capability to represent functions at different levels of resolution. There are many researchers have used Boubaker polynomials in their applications in different problems in physics and applied science such as Yücel and Boubaker [4] used the Boubaker Polynomials expansion scheme to compare and confirm solution to differently established applied physics nonlinear problems in the field of fluids motion and waves dynamics. Dada *et al.* [5] presented a general analytical solutions to the Neutron Boltzmann Transport equation using Boubaker Polynomial expansion scheme. Yalçınbaş and Tuğçe [6] are introduced collocation method, which is based on Boubaker polynomials, for the approximate solutions of mixed linear integro-differential-difference equations.

In this work, we will discuss the solution of the mixed linear Volterra-Fredholm differential equation of the second kind which can be expressed in general form as follows [7] [8]:

$$u(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)y(t)dt + \lambda_2 \int_a^b k_2(x,t)y(t)dt \qquad \ldots (1)$$

where, $k_1(x,t)$, $k_2(x,t)$ and $f(x)$ are known functions, a,b are constant values, and $u(x)$ is the unknown function to be determined.

In the present study, we introduce the approximation method to solve the linear Volterra-Fredholm differential equations of the second kind by using Boubaker polynomials method. The paper is organized as follows: The Boubaker polynomials is described in the second section. While the Boubaker polynomials approximation method and the numerical finding to illustrate the accuracy and applicability of the propose method are presented in the third and fourth section.

2. Boubaker Polynomials

The definition of Boubaker polynomials was appeared firstly in a physical study in an attempt to get an analytical solution to heat equation [9]. According to the first definition, the Boubaker polynomials have the following expression [10]:

$$B_n(x) = \sum_{k=0}^{\xi(n)} \left[\frac{(n-4p)}{(n-p)} C_{n-p}^{p} C_{n-p}^{p} \right] \cdot (-1)^p \cdot x^{n-2p} \qquad \ldots (2)$$

where,

$\xi(n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n + ((-1)^n - 1)}{4}$, where the symbol $\lfloor \ \rfloor$ designates the floor function.

The Boubaker polynomials, which are a polynomial sequence with integer coefficients, have the explicit monic expression as follow [2]:

$$\begin{aligned}
B_0(x) &= 1 \\
B_1(x) &= x \\
B_2(x) &= x^2 + 2 \\
B_3(x) &= x^3 + x \\
B_4(x) &= x^4 - 2 \\
B_5(x) &= x^5 - x^3 - 3x \\
&\vdots \\
B_m(x) &= x \cdot B_{m-1}(x) - B_{m-2}(x), \qquad \text{for } m > 2.
\end{aligned} \qquad \ldots (3)$$

3. Boubaker polynomials Approximation Method for Mixed Linear Volterra-Fredholm Integral Equations

In this section, we will consider the Boubaker polynomials approximation solution. Any function $u(x) \in L^2([a,b])$ can be expanded into a Boubaker polynomials series of finite terms [2], [9], [10]:

$$u(x) = b_0 B_0(x) + b_1 B_1(x) + b_2 B_2(x) + \cdots + b_n B_n(x) \;,\; -\infty \le a \le x \le b \le \infty \qquad \ldots (4)$$

where $B_0(x), B_1(x), B_2(x), \ldots, B_n(x)$ are Boubaker polynomials terms which was defined in equation (3), $b_0, b_1, b_2, \ldots, b_n$ are unknown Boubaker polynomials coefficients, then equation (4) can be decomposed as

$$u(x) = \sum_{i=0}^{n} b_i B_i(x) \qquad \ldots (5)$$

By utilizing equation (5) and substituting into equation (1), we obtain

$$\sum_{i=0}^{n} b_i B_i(x) = f(x) + \lambda_1 \int_a^x k_1(x,t) \sum_{i=0}^{n} b_i B_i(t) dt + \lambda_2 \int_a^b k_2(x,t) \sum_{i=0}^{n} b_i B_i(t) dt \qquad \ldots (6)$$

The equation (6) can be rewritten in a simplified form as

$$\begin{aligned} b_0 B_0(x) + b_1 B_1(x) + b_2 B_2(x) + \ldots + b_n B_n(x) &= f(x) \\ &+ \int_a^x k_1(x,t)\left[b_0 B_0(t) + b_1 B_1(t) + b_2 B_2(t) + \ldots + b_n B_n(t)\right] dt \\ &+ \int_a^b k_2(x,t)\left[b_0 B_0(t) + b_1 B_1(t) + b_2 B_2(t) + \ldots + b_n B_n(t)\right] dt \end{aligned} \qquad \ldots (7)$$

Sequentially, substitute the recurrence relation of Boubaker polynomials terms as in equation (3) into equation (7), we obtain:

$$\begin{aligned} b_0 + b_1 x + b_2 (x^2 + 2) + \ldots + b_n (x B_{n-1}(x) - B_{n-1}(x)) &= f(x) \\ &+ \int_a^x k_1(x,t)\left[b_0 + b_1 t + b_2 (t^2 + 2) + \ldots + b_n (t B_{n-1}(t) - B_{n-1}(t))\right] dt \\ &+ \int_a^b k_2(x,t)\left[b_0 + b_1 t + b_2 (t^2 + 2) + \ldots + b_n (t B_{n-1}(t) - B_{n-1}(t))\right] dt \end{aligned} \qquad \ldots (8)$$

Next, by integrating the terms in the right hand side of equation (8), then this equation is simplified and represented as a linear equation include x as a variable. Then substitute the collocation points x_i in the interval $[a,b]$ which can be calculated as follows

$$h = \frac{b-a}{n} \quad ,$$

$$x_i = a + i\, h \;, \quad i = 0, 1, 2, \ldots, n \qquad \ldots (9)$$

Finally, a system of linear equations involve $(n+1)$ unknown coefficients b_i, $i = 0, 1, 2, \ldots, n$. are established. The unknown coefficients which can be determined by solving the linear system equations by using the MATLAB solver [11].

4. Numerical Examples and Discussions

In this section, three numerical examples are given to demonstrate the accuracy and effectiveness of the proposed method. Two of these examples have exact solutions while the third one without exact solution. In addition, the numerical results to the examples are discussed numerically and displayed graphically. All the computations were carried out using MATLAB®, version 13.

Example 1

Consider the linear Volterra-Fredholm integral equation:

$$u(x)=-\frac{2}{5}x^7-\frac{5}{4}x^4+x^3-\frac{59}{20}x+1+\int_0^1 x(y+1)u(y)dy-\int_0^x (2x^2y)u(y)dy, \quad , \quad 0\le x\le 1 \ldots (10)$$

with initial condition $u(0)=1$ and the exact solution which is $u(x)=1+x^3$.

The numerical results of example 1 by using Boubaker polynomials method as explained in section 3 are shown in Table1 and Figure (1). For instance Table 1 has simulation results obtained from three orders of Boubaker polynomials as well as the exact solution. The results show that, when the order of Boubaker polynomials n increase the errors will be decreased as in Table 1. Figure (1) shows the comparison between the exact solution and the approximate solutions for various n orders.

Table. (1): Simulation results for Example 1 using Boubaker Polynomials Method for different *n=1, 2, 3* and $Error=\left|u_{Exact}-u_{approx,n}\right|$

x	*Exact solution*	Boubaker Polynomials Method *Approximate solution*					
		n=1	*Error*	*n=2*	*Error*	*n=3*	*Error* $\left\|u_{Exact}-u_{aprox,n=3}\right\|$
0	1.0000	1.00000	0	1.0000	0.00000	1.0000	0
0.1	1.0010	1.01000	0.009	0.9762491262	0.024751	1.00100158	0.0000
0.2	1.0080	1.02000	0.012	0.9738123502	0.034188	1.00800218	0.0000
0.3	1.0270	1.03000	0.003	0.9926896719	0.03431	1.02700219	0.0000
0.4	1.0640	1.04000	0.024	1.0328810914	0.031119	1.06400198	0.0000
0.5	1.1250	1.05000	0.075	1.0943866086	0.030613	1.12500195	0.0000
0.6	1.2160	1.06000	0.156	1.1772062236	0.038794	1.216002472	0.0000
0.7	1.3430	1.07000	0.273	1.2813399364	0.06166	1.343003929	0.0000
0.8	1.5120	1.08000	0.432	1.4067877469	0.105212	1.512006704	0.0000
0.9	1.7290	1.09000	0.639	1.5535496552	0.17545	1.729011179	0.0000
1	2.0000	1.10000	0.9	1.7216256612	0.278374	2.000017737	0.0000

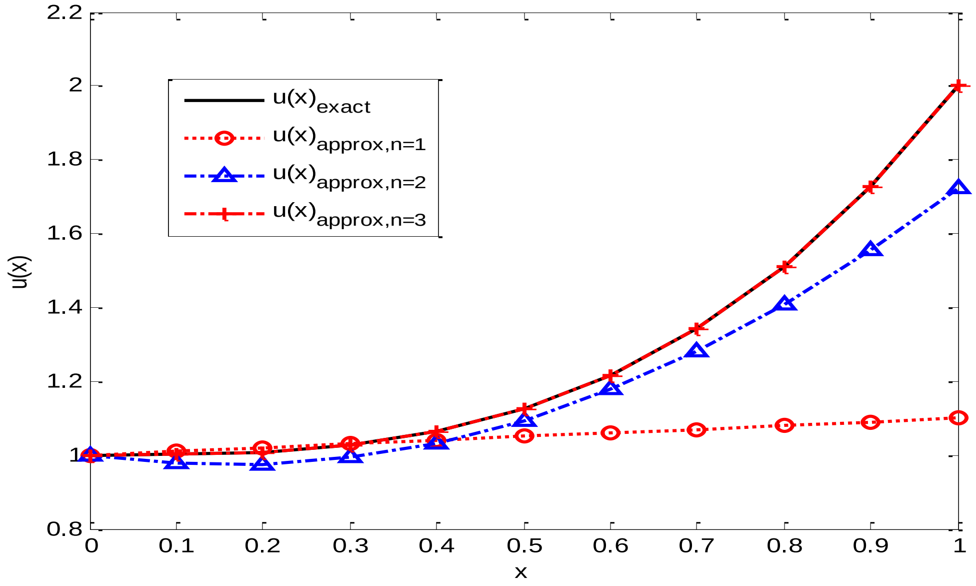

Figure (1): Comparison of the approximation solutions using Boubaker Polynomials Method and exact solution for the example 1

Example 2

Consider the linear Volterra-Fredholm integral equation as follows:

$$u(x) = -\frac{26}{3} - \frac{2}{3}x + 12x^2 - \frac{1}{3}x^3 - x^4 + \int_0^1 (x+t+1)u(t)dt + \int_0^x (x-t)u(t)dt, \quad , \quad 0 \le x \le 1 \qquad \ldots (11)$$

with initial condition $u(0) = 0$ and the exact solution which is $u(x) = 2x + 12x^2$. where the absolute errors in Table (2) are the values of $|u_{Exact} - u_{approx,n}|$ at chosen points in the interval $[0,1]$.

Table. (2): Simulation results for Example 2 by using Boubaker Polynomials Method for different *n=1, 2*

x	*Exact solution*	Boubaker Polynomials Method			
		n=1 Approximate solution	*Error* $\lvert u_{Exact} - u_{approx,n=1}\rvert$	*n=2 Approximate solution*	*Error* $\lvert u_{Exact} - u_{approx,n=2}\rvert$
0	0.0000	-2.4706	2.4706	-0.0000	0.0000
0.1	0.3200	-1.2824	1.6024	0.3200	0.0000
0.2	0.8800	-0.0941	0.9741	0.8800	0.0000
0.3	1.6800	1.0941	0.5859	1.6800	0.0000
0.4	2.7200	2.2824	0.4376	2.7200	0.0000
0.5	4.0000	3.4706	0.5294	4.0000	0.0000
0.6	5.5200	4.6588	0.8612	5.5200	0.0000
0.7	7.2800	5.8471	1.4329	7.2800	0.0000
0.8	9.2800	7.0353	2.2447	9.2800	0.0000
0.9	11.5200	8.2236	3.2964	11.5200	0.0000
1	14.0000	9.4118	4.5882	14.0000	0.0000

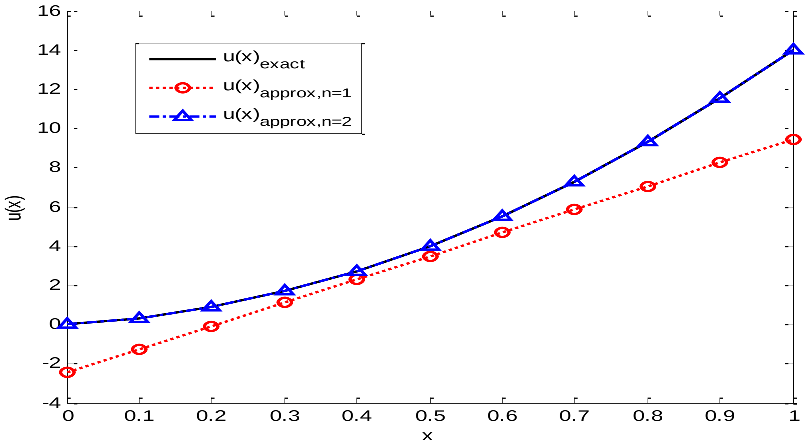

Figure (2): Comparison of the approximation solutions using Boubaker Polynomials Method and exact solution for the example 2

Example 3

Consider the linear Volterra-Fredholm integral equation as follows [8]:

$$u(x) = 2\cos x - x\cos 2 - 2x\sin 2 + x - 1 + \int_0^2 xtu(t)dt + \int_0^x (x-t)u(t)dt, \quad , \quad 0 \le x \le 2 \quad \dots (12)$$

The Figure (3) shows the approximation solutions for different order of n, where the absolute errors in Table (3) are the values of $\left| u_{approx,n+1} - u_{approx,n} \right|$ for order n=1,2,3,4,5 at chosen points on interval [0,2] .

Table. (3): Approximation results for Example 3 using Boubaker Polynomials Method with $\text{Error} = \left| u_{approx,n+1} - u_{approx,n} \right|$ **for n=1, 2, 3, 4, 5**

x	Boubaker Polynomials Method								
	n=1	*n=2*	*Error*	*n=3*	*Error*	*n=4*	*Error*	*n=5*	*Error*
0.1	0.949369	0.975751	0.026382	1.017429	0.041678	1.003185	0.014244	1.000705	0.002481
0.2	0.898738	0.946443	0.047705	1.021733	0.075291	0.995983	0.02575	0.991634	0.004349
0.3	0.848107	0.912074	0.063968	1.014038	0.101963	0.978814	0.035224	0.972942	0.005872
0.4	0.797476	0.872647	0.075171	0.995468	0.122821	0.952158	0.04331	0.944902	0.007256
0.5	0.746845	0.828159	0.081314	0.967148	0.138989	0.916552	0.050597	0.907897	0.008655
0.6	0.696214	0.778612	0.082398	0.930204	0.151593	0.872591	0.057613	0.862418	0.010173
0.7	0.645583	0.724005	0.078422	0.885761	0.161756	0.82093	0.064831	0.809052	0.011878
0.8	0.594951	0.664338	0.069387	0.834944	0.170605	0.76228	0.072663	0.748475	0.013806
0.9	0.54432	0.599612	0.055292	0.778877	0.179265	0.697412	0.081465	0.681445	0.015967
1	0.493689	0.529826	0.036137	0.718686	0.18886	0.627153	0.091532	0.608796	0.018358
1.1	0.443058	0.45498	0.011922	0.655496	0.200515	0.552391	0.103105	0.531428	0.020963
1.2	0.392427	0.375075	0.017352	0.590432	0.215357	0.474069	0.116363	0.450302	0.023766
1.3	0.341796	0.29011	0.051686	0.524619	0.234509	0.39319	0.131429	0.366431	0.026759
1.4	0.291165	0.200085	0.09108	0.459182	0.259097	0.310815	0.148367	0.280872	0.029943
1.5	0.240534	0.105001	0.135533	0.395247	0.290246	0.228064	0.167183	0.19472	0.033343
1.6	0.189903	0.004857	0.185046	0.333938	0.329081	0.146113	0.187825	0.1091	0.037012
1.7	0.139272	-0.10035	0.239619	0.27638	0.376727	0.066197	0.210183	0.02516	0.041037
1.8	0.088641	-0.21061	0.299251	0.223699	0.434309	-0.01039	0.234088	-0.05594	0.04555
1.9	0.03801	-0.32593	0.363943	0.177019	0.502952	-0.08229	0.259313	-0.13303	0.050731
2	-0.01262	-0.44632	0.433695	0.137466	0.583782	-0.14811	0.285575	-0.20493	0.056822

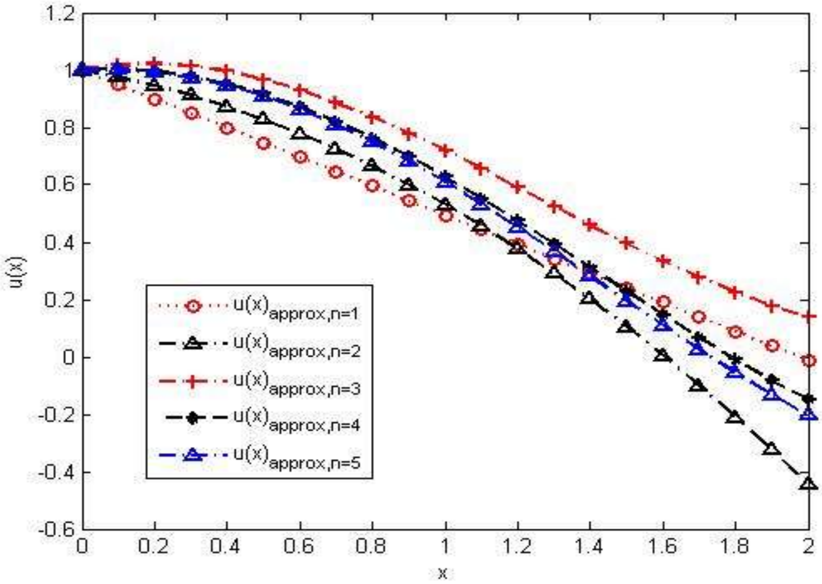

Figure (3): Approximation solutions using Boubaker Polynomials for the example 3

5. Conclusion

Most of integral equations are difficult to get analytical solution, therefor in many cases are required to obtain the approximate solutions, for this purpose, the Boubaker polynomials method for the solution of linear Volterra-Fredholm differential equations is successfully implemented. The method is based on Boubaker polynomials which reduces a linear Volterra_Fredholm differential equations to a set of linear algebraic equations that can be easily solved by using MATLAB program. From all the figures, it is clear that, the obtained results indicate that the convergence rate to the exact or approximate solution is very fast, and high accuracy on the examples can achieve by lower order of n.

6. References

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استعمال طريقة متعددة حدود بوبكر لحل معادلات فولتيرا- فريدهولم المختلطة التكاملية الخطية

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الخلاصة:

في هذا البحث استعملت طريقة متعددة حدود بوبكر لإيجاد الحل التقريبي لمعادلة فولتيرا- فريدهولوم المختلطة التكاملية الخطية من النوع الثاني. ثلاثة امثلة أُخذت بنظر الاعتبار وكانت الحلول قد نُوقشت عددياً وبُينت بيانياً. وبزيادة رتبة (n) لمتعددة حدود بوبكر ، فإننا نستطيع من تحسين دقة النتائج.