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The M-Polynomial and Nirmala index of Certain Composite Graphs

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ABSTRACT

The M-Polynomial and Nirmala index are considered as two of the most recent found and important subjects in chemical graph theory. In this paper we drive and prove the computing formula of Nirmala index from the M-Polynomial, then compute the M-Polynomial for some certain composite graphs, and the Nirmala index via the computed M-Polynomial. The composite graphs are new defined graphs $K_n(P_t)K_m$, $C_n(e)K_n$, and others obtained from simple graphs by certain graph operations such as join, corona, and cluster of any graph with some special graphs such as complete, path, …etc.

1 Introduction

Let $G = (V,E)$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$, where the order and size of G are $|V(G)| = n_G$ and $|E(G)| = m_G$ respectively[1]. The degree of a vertex u is the number of all edges incidence to u in G, which is denoted by $d_G(u)$ [1]. By pendent vertex we mean a vertex of degree one, and by *i-vertex* we mean the vertex v has degree i, and an edge joining an *i-vertex* to a *j-vertex* is denoted by (i,j)*-edge* [1, 2]. A u-v walk W_n in a connected graph G, is a sequence of vertices $(u = u_1,u_2,...,u_{n-1},u_n = v)$ in G, such that consecutive vertices in W_n are adjacent in G. A path is just a walk in which no vertex is repeated, and a path with n vertices is denoted by P_n. A closed path is called cycle, and denoted by C_n. A graph in which every two vertices are adjacent is called complete graph and denoted by K_n. A star graph S_n is a graph that has n+1 vertices, one of them has degree of n which is called the center vertex and the other n vertices have degree of one which are called pendent vertices [1, 3, 4].
Let G and H be two graphs then the vertex gluing of G and H is a new graph that constructed from G and H by identifying a vertex between them [3], the vertex gluing of G and H is denoted by $G(\text{o})H$, which is a new graph of order $n_G + n_H$ -1 and size $m_G+ m_H$ (see **Figure 1**).

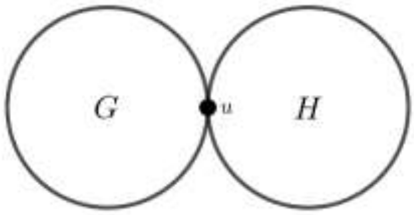

Figure 1: G(o)H

A graph in which a vertex is labeled in a special way so as to distinguish from other vertices is called a rooted graph, and the special vertex is called the root of it [5].The cluster of two graphs G and H is denoted by $G\{H\}$, which can be obtained by taking a copy of G and n_G copies of the rooted graph H such that we identify the root of the i^{th} copy of H with the i^{th} vertex of G for each $i \in \{1,2,3,..., n_G\}$[6]. For instance, the cluster of the path P_5 and the cycle C_3 is shown in the **Figure 2**.

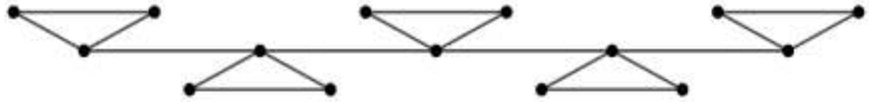

Figure 2: $P_5\{C_3\}$

The join (sum) of two graphs G and H is a new graph that denoted by $G + H$, with the vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv \,;\, u \in V(G)$ and $v \in V(H)\}$ [4]. The corona product of G and H is obtained by taking a copy of G and n_G copies of H and join the i^{th} vertex of G with each vertex of the i^{th} copy of H for each $i \in \{1,2,3,\ldots,n_G\}$ and denoted by $G \odot H$ [6]. For instance, the join and corona product of the complete graph K_3 and the path P_2 are shown in the **Figure 3** respectively [7].

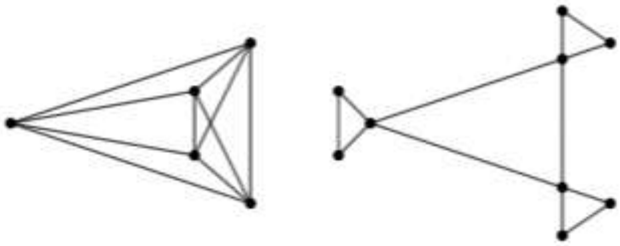

Figure 3: K_3+P_2 and $K_3 \odot P_2$

A graph polynomial is a graph invariant whose values are polynomials. An important degree-based polynomial is the M-Polynomial which is defined by Deutsch and Klavžar in 2014 [8]. For a graph G, the M-Polynomial is defined by:

$M(G,x,y) = \sum_{i \le j} m_{ij}(G) x^i y^j$ .... (1)

where $i,j \ge 1$ and m_{ij} is the number of (i,j)-edges of G, such that $i = d_G(u)$, and $j = d_G(v)$ for some vertices $u,v \in G$.

We can see that the M-Polynomial for a graph G also can be represent as:

$M(G,x,y) = \sum_{e=uv \in E(G)} x^{d_G(u)} y^{d_G(v)}$ .... (2)

Many studies have done about the M-Polynomial such as computation of M-polynomial book graph and starphene graph in [9,10]. Also Basavanagoud, and et al obtained the M-polynomial of some graph operations and cycle related graphs in [11].

A graph invariant is a number related to a graph which is structural invariant, fixed under graph automorphisms. In chemistry these invariants are known as the topological indices [2]. As a chemical descriptor, the topological index has an integer attached to the graph which features the graph, and there is no change under graph automorphism [7]. A degree based topological index of the graph G is a graph invariant of the form:

$I(G) = \sum_{e=uv \in E(G)} f(d_G(u), d_G(v))$ .... (3)

where f is a function appropriately selected for possible chemical applications [8]. Unlike the other graph polynomials through this polynomial, we can easily compute more than one degree based topological indices such as Atom bond connectivity index, Geometric connectivity index and some other indices by a certain derivative or integral or sometimes both. Some formula for computing those indices from the M-Polynomial are found in [8-14] as we illustrate some of these formulas in the following Table.

Table 1: Formulas of computing some degree based topological indices from $M(G,x,y)$

Topological indices	$f(d_G(u), d_G(v))$	Derivation from $M(G,x,y)$
Atom Bond Connectivity index	$\sum_{uv \in E(G)} \sqrt{\frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}}$	$D_x^{1/2} Q_{(-2)} J S_x^{1/2} S_y^{1/2} [M(G,x,y)]_{x=1}$ [10,14]
Geometric Arithmetic index	$\sum_{uv \in E(G)} \frac{2\sqrt{d_G(u)d_G(v)}}{d_G(u)+d_G(v)}$	$2 S_x J D_x^{1/2} D_y^{1/2} [M(G,x,y)]_{x=1}$ [10,13,14]
First Zagreb index	$\sum_{uv \in E(G)} d_G(u)+d_G(v)$	$(D_x + D_y)[M(G,x,y)]_{x=y=1}$ [8,11,12]
Second Zagreb index	$\sum_{uv \in E(G)} d_G(u)d_G(v)$	$(D_x D_y)[M(G,x,y)]_{x=y=1}$ [8,11,12]
Randic index	$\sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}}$	$\left(S_x^{1/2} S_y^{1/2}\right)[M(G,x,y)]_{x=y=1}$ [8]

where used operators are defined as[8-10,12-14]:

$$D_x = x\frac{\partial(M(G,x,y))}{\partial x}, D_y = y\frac{\partial(M(G,x,y))}{\partial y}, S_x = \int_0^x \frac{M(G,t,y)}{t}dt, S_y = \int_0^y \frac{M(G,x,t)}{t}dt,$$

$$D_x^{1/2}(M(G,x,y)) = \sqrt{x\frac{\partial(M(G,x,y))}{\partial x}}\sqrt{M(G,x,y)}, D_y^{1/2}(M(G,x,y)) = \sqrt{y\frac{\partial(M(G,x,y))}{\partial y}}\sqrt{M(G,x,y)}$$

$$S_x^{1/2}(M(G,x,y)) = \sqrt{\int_0^x \frac{M(G,t,y)}{t}dt}\sqrt{M(G,x,y)}, S_y^{1/2}(M(G,x,y)) = \sqrt{\int_0^y \frac{M(G,x,t)}{t}dt}\sqrt{M(G,x,y)}$$

$$J\big(M(G,x,y)\big) = M(G,x,x),\ Q_\alpha\big(M(G,x,y)\big) = x^\alpha M(G,x,y)$$

One of the most recent defined degree based topological indices is Nirmala index defined by Kulli in 2021 [15], which is defined as follows:

$N(G) = \sum_{uv\in E(G)} \sqrt{d_G(u)+d_G(v)}$ ……………. (4)

where $d_G(u)$, and $d_G(v)$ are degrees of vertices u and v in G respectively. Recently, some mathematical properties of Nirmala index were studied in [16], also many studies have done on Nirmala index, such as the Nirmala index of Kragujevac trees in [17] by Ivan Gutman, and et al. Also more studies can be found, for instance different versions of Nirmala index in [18]. Also on multiplicative inverse Nirmala indices and Nirmala energy in [19, 20].

In the next section, the formula of computing Nirmala index from the M-Polynomial, some important results about computing the M-Polynomial and next Nirmala index through the obtained polynomial are shown for certain graphs.

2 Results and Discussion

Theorem 2.1 *For a graph G the formula of Nirmala index can be obtained from the M-Polynomial of G as follows:*

$N(G) = \left(D_x^{1/2}J\right)[M(G,x,y)]\big|_{x=1}$ ……………… (5)

where the two operators $D_x^{1/2}$, and J are defined as above, and $M(G,x,y)$ is the M-Polynomial of the graph G.

Proof: Since $M(G,x,y) = \sum_{uv\in E(G)} x^{d_G(u)}y^{d_G(v)}$, then:

$$\begin{aligned}\left(D_x^{\frac{1}{2}}J\right)[M(G,x,y)] &= \left(D_x^{\frac{1}{2}}J\right)\left[\sum_{uv\in E(G)} x^{d_G(u)}y^{d_G(v)}\right]\\ &= \sum_{uv\in E(G)}\left(D_x^{\frac{1}{2}}J\right)\left[x^{d_G(u)}y^{d_G(v)}\right] = \sum_{uv\in E(G)} D_x^{\frac{1}{2}}\left[J\left(x^{d_G(u)}y^{d_G(v)}\right)\right]\\ &= \sum_{uv\in E(G)} D_x^{\frac{1}{2}}\left[x^{d_G(u)+d_G(v)}\right]\\ &= \sum_{uv\in E(G)} \sqrt{(d_G(u)+d_G(v))x^{d_G(u)+d_G(v)}}\sqrt{x^{d_G(u)+d_G(v)}}\\ &= \sum_{uv\in E(G)} \sqrt{d_G(u)+d_G(v)} = N(G)\end{aligned}$$

, at $x = 1$, which is the result (4).

Theorem 2.2 *Let K_n and K_m be two complete graphs, then the M-Polynomial of the vertex gluing of them is:*

$$M(K_n(\text{o})K_m,x,y) = \binom{n-1}{2}(xy)^{n-1} + \binom{m-1}{2}(xy)^{m-1} + x^{n+m-2}[(n-1)y^{n-1} + (m-1)y^{m-1}].$$

Proof: The graph $K_n(\text{o})K_m$ has $n+m-1$ vertices and $\binom{n}{2}+\binom{m}{2}$ edges.

Suppose that the vertex gluing point between them is u^*. Let $e = uv \in E(K_n(\text{o})K_m)$ then

Case 1 If $e = uv \in E(K_n)$ such that $u, v \neq u^*$ then:

$d_{K_n(\text{o})K_m}(u) = d_{K_n(\text{o})K_m}(v) = n-1.$

case 2 If $e = uv \in E(K_m)$ such that $u, v \neq u^*$ then:

$d_{K_n(\text{o})K_m}(u) = d_{K_n(\text{o})K_m}(v) = m-1.$

Case 3 If $e = u^*v$ such that $v \in V(K_n)$, then $d_{K_n(\text{o})K_m}(u^*) = n+m-2$ and $d_{K_n(\text{o})K_m}(v) = n-1$

Case 4 If $e = u^*v$ such that $v \in V(K_m)$, then $d_{K_n(\text{o})K_m}(u^*) = n+m-2$ and $d_{K_n(\text{o})K_m}(v) = m-1$

From the above cases,

$$
\begin{aligned}
M(K_n(\mathrm{o})K_m, x, y) &= \sum_{e=uv\in E(K_n(\mathrm{o})K_m)} x^{d_{K_n(\mathrm{o})K_m}(u)} y^{d_{K_n(\mathrm{o})K_m}(v)} \\
&= \sum_{e=uv\in E(K_n-u^*)} x^{(n-1)}y^{(n-1)} + \sum_{e=uv\in E(K_m-u^*)} x^{(m-1)}y^{(m-1)} \\
&\quad +(n-1)x^{n+m-2}y^{n-1} + (m-1)x^{n+m-2}y^{m-1} \\
&= \left(\binom{n}{2} - (n-1)\right)(xy)^{n-1} + \left(\binom{m}{2} - (m-1)\right)(xy)^{m-1} \\
&\quad +(n-1)x^{n+m-2}y^{n-1} + (m-1)x^{n+m-2}y^{m-1} \\
&= \binom{n-1}{2}(xy)^{n-1} + \binom{m-1}{2}(xy)^{m-1} \\
&\quad +x^{n+m-2}[(n-1)y^{n-1} + (m-1)y^{m-1}].
\end{aligned}
$$

From Theorems 2.1 and 2.2, we get the following result:

Corollary 2.1 *The Nirmala index of the graph* $K_n(o)K_m$ *is given by:*

$$
\begin{aligned}
N(K_n(\mathrm{o})K_m) &= \binom{n-1}{2}\sqrt{2(n-1)} + \binom{m-1}{2}\sqrt{2(m-1)} \\
&\quad +(n-1)\sqrt{2n+m-3} + (m-1)\sqrt{n+2m-3}.
\end{aligned}
$$

Definition 2.1 *Let* K_n , K_m *be two complete graphs and* P_t *be a path. We define a new graph* $K_n(P_t)K_m$ *by vertex gluing* K_n *and* K_m *to* P_t *at it's end points (see* ***Figure 4****).*

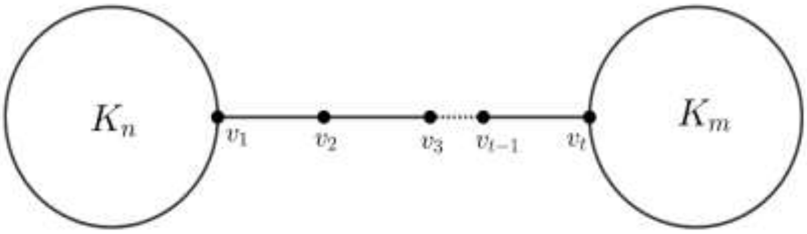

Figure 4: $K_n(P_t)K_m$

Theorem 2.3 *Let* $K_n(P_t)K_m$ *be defined as above. Then The M-Polynomial of the graph* $K_n(P_t)K_m$ *is:*

$$
\begin{aligned}
M(K_n(P_t)K_m, x, y) &= (xy)^2(y^{n-2} + y^{m-2} + t - 3) + (n-1)x^n y^{n-1} + (m-1)x^m y^{m-1} \\
&\quad + \binom{n-1}{2}(xy)^{n-1} + \binom{m-1}{2}(xy)^{m-1}.
\end{aligned}
$$

Proof: The graph $K_n(P_t)K_m$ has $n + m + t - 2$ vertices and $\binom{n}{2} + \binom{m}{2} + t - 1$ edges. For all vertex v of the graph $K_n(P_t)K_m$ there are the following possibilities of degree v ; $2, n-1, n, m-1, m$. Let $e = uv \in E(K_n(P_t)K_m)$ then based on this information we have the following illustration table (see **Table 2**).

Table 2: Edge partitions and number of edges in each partition based on degree of end vertices in each edges of the graph $K_n(P_t)K_m$

Type of edges	Number of edges
$(2,2)$	$t-3$
$(2,n)$	1
$(2,m)$	1
$(n,n-1)$	$n-1$
$(n-1,n-1)$	$\binom{n}{2} - n + 1$
$(m,m-1)$	$m-1$
$(m-1,m-1)$	$\binom{m}{2} - m + 1$
Sum of all edges	$\binom{n}{2} + \binom{m}{2} + t - 1$

Hence,

$$\begin{aligned} M(K_n(P_t)K_m,x,y) &= (t-3)(xy)^2 + x^2y^n + x^2y^m + (n-1)x^ny^{n-1} + (m-1)x^my^{m-1} \\ &+ \left[\binom{n}{2} - n + 1\right](xy)^{n-1} + \left[\binom{m}{2} - m + 1\right](xy)^{m-1} \\ &= (xy)^2(y^{n-2} + y^{m-2} + t - 3) + (n-1)x^ny^{n-1} + (m-1)x^my^{m-1} \\ &+ \binom{n-1}{2}(xy)^{n-1} + \binom{m-1}{2}(xy)^{m-1}. \end{aligned}$$

■

From Theorems 2.1 and 2.3, we get the following result:

Corollary 2.2 *The Nirmala index of the graph $K_n(P_t)K_m$ is:*

$$N(K_n(P_t)K_m) = \binom{n-1}{2}\sqrt{2(n-1)} + \binom{m-1}{2}\sqrt{2(m-1)} + (n-1)\sqrt{2n-1}$$
$$+(m-1)\sqrt{2m-1} + \sqrt{n+2} + \sqrt{n+2} + 2(t-3).$$

■

Definition 2.2 *Let K_n be a complete graph and C_n be a cycle. Suppose that we have n copies of K_n such that each copy of K_n intersects with C_n in only a unique edge and no two copies of K_n are intersected in their edges (see **Figure 5**), we denote the constructed graph by $C_n(e)K_n$.*

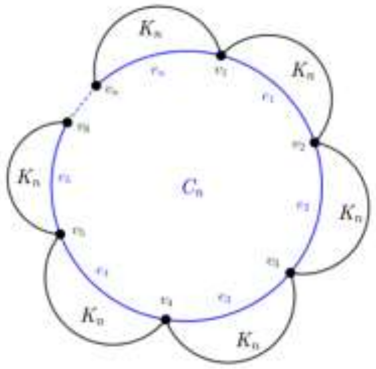

Figure 5: $C_n(e)K_n$

Theorem 2.4 *Let K_n be a complete graph and C_n be a cycle, and $C_n(e)K_n$ be defined as above, then the M-Polynomial of the $C_n(e)K_n$ is:*

$$M(C_n(e)K_n,x,y) = n(xy)^{n-1}\left[(xy)^{n-1} + 2(n-2)x^{n-1} + \binom{n-2}{2}\right].$$

Proof: We see that ta graph $C_n(e)K_n$ has $n(n-1)$ vertices and $n\binom{n}{2}$ edges. If $e = uv \in E(C_n(e)K_n)$, then there are three possible cases for e:

Case 1 If $e = uv \in E(C_n)$ then $d_{C_n(e)K_n}(u) = d_{C_n(e)K_n}(v) = 2(n-1)$,

Case 2 If $e = uv_i$ such that $u \in V(K_n)$ for some copy of K_n then $d_{C_n(e)K_n}(u) = n-1$ and $d_{C_n(e)K_n}(v_i) = 2(n-1)$, for all $i \in \{1,2,3,\dots,n\}$

Case 3 If $e = uv \in E(K_n)$ for some copy of K_n such that $u,v \neq v_i$ for all $i \in \{1,2,3,\dots,n\}$ then $d_{C_n(e)K_n}(u) = d_{C_n(e)K_n}(v) = n-1$.

Based on the above three cases we have the following table (see **Table 3**).

Table 3: Edge partitions and number of edges in each partition based on degree of end vertices in each edges of the graph $C_n(e)K_n$

Type of edges	Number of edges
$(2(n-1),2(n-1))$	n
$(2(n-1),n-1)$	$2n(n-2)$
$(n-1,n-1)$	$n\binom{n-2}{2}$
Sum of all edges	$n\binom{n}{2}$

Hence,

$$\begin{aligned} M(C_n(e)K_n,x,y) &= \sum_{e=uv\in E(C_n(e)K_n)} x^{d_{C_n(e)K_n}(u)}y^{d_{C_n(e)K_n}(v)} \\ &= n(xy)^{2(n-1)} + 2n(n-2)x^{2(n-1)}y^{n-1} + n\binom{n-2}{2}(xy)^{n-1} \\ &= n(xy)^{n-1}\left[(xy)^{n-1} + 2(n-2)x^{n-1} + \binom{n-2}{2}\right]. \end{aligned}$$

■

From Theorems 2.1 and 2.4, we get the following result:

Corollary 2.3 *The Nirmala index of the graph $C_n(e)K_n$ is:*

$$N(C_n(e)K_n) = n\sqrt{n-1}\left[2 + 2(n-2)\sqrt{3} + \sqrt{2}\binom{n-2}{2}\right].$$

Theorem 2.5 *Let G be any graph and K_n be the complete graph, then the M-Polynomial of the cluster graph of G and K_n is:*

$$M(G\{K_n\},x,y) = (xy)^{n-1}\left[M(G,x,y) + (n-1)\sum_{u\in V(G)} x^{d_G(u)} + n_G\binom{n-1}{2}\right].$$

Proof: Clearly the graph $G\{K_n\}$ has $n\,n_G$ vertices and $m_G + n_G\binom{n}{2}$ edges, where m_G *is the size of* G.

Let $e = uv \in E(G\{K_n\})$ then,

Case 1 If $e = uv \in E(G)$ then $d_{G\{K_n\}}(u) = d_G(u) + n - 1$ and $d_{G\{K_n\}}(v) = d_G(v) + n - 1$.

Case 2 If $e = uv \in E(K_n)$ such that u be one of the identified vertex and $v \in V(K_n)$, for some copy of K_n, then $d_{G\{K_n\}}(u) = d_G(u) + n - 1$ and $d_{G\{K_n\}}(v) = n - 1$.

Case 3 If $e = uv \in E(K_n)$ such that non of u and v is the identified vertex, then $d_{G\{K_n\}}(u) = d_{G\{K_n\}}(v) = n - 1$.

From the above cases

$$M(G\{K_n\},x,y) = \sum_{e=uv\in E(G\{K_n\})} x^{d_{G\{K_n\}}(u)} y^{d_{G\{K_n\}}(v)}$$
$$= \sum_{e=uv\in E(G)} x^{d_G(u)+n-1} y^{d_G(v)+n-1}$$
$$+n_G \sum_{e=uv\in E(K_n);\ u \text{ is the } i^{th} \text{ identified vertex}} x^{d_G(u)+n-1} y^{n-1}$$
$$+n_G \sum_{e=uv\in E(K_n);\ u,v \text{ are not the identified vertex}} (xy)^{n-1}$$
$$= (xy)^{n-1} M(G,x,y) + (n-1)\sum_{u\in V(G)} x^{d_G(u)+n-1} y^{n-1}$$
$$+n_G\left(\frac{n(n-1)}{2} - (n-1)\right)(xy)^{n-1}$$
$$= (xy)^{n-1}\left[M(G,x,y) + (n-1)\sum_{u\in V(G)} x^{d_G(u)} + n_G\binom{n-1}{2}\right].$$

■

From Theorems 2.1 and 2.5, we get the following result:

Corollary 2.4 *The Nirmala index of* $G\{K_n\}$ *is:*

$$N(G\{K_n\}) = \sum_{uv\in E(G)} \sqrt{d_G(u) + d_G(v) + 2(n-1)} + (n-1)\sum_{u\in V(G)} \sqrt{d_G(u) + 2(n-1)} + n_G\binom{n-1}{2}\sqrt{2(n-1)}.$$

■

Theorem 2.6 *Let G be any graph and P_n $(n \geq 3)$ be a path such that one of it's end vertices be it's root. Then the M-Polynomial of the cluster graph $G\{P_n\}$ is:*

$$M(G\{P_n\},x,y) = (xy)\left[M(G,x,y) + y\sum_{u\in V(G)} x^{d_G(u)} + (n-3)n_G xy + n_G x\right].$$

Proof: Let $e = uv \in E(G\{P_n\})$, then there are three cases:

Case 1 If $e = uv \in E(G)$. Then $d_{G\{P_n\}}(u) = d_G(u) + 1$ and $d_{G\{P_n\}}(v) = d_G(v) + 1$.

Case 2 If $e = uv \in E(P_n)$, for some copy of P_n such that u be the root vertex of P_n. Then $d_{G\{P_n\}}(u) = d_G(u) + 1$ and $d_{G\{P_n\}}(v) = 2$.

Case 3 If $e = uv \in E(P_n)$ for some copy of P_n such that u,v are not root of P_n. Then $d_{G\{P_n\}}(u) = d_{G\{P_n\}}(v) = 2$ or $d_{G\{P_n\}}(u) = 2$, $d_{G\{P_n\}}(v) = 1$.

From the above cases,

$$M(G\{P_n\},x,y) = \sum_{e=uv\in E(G\{P_n\})} x^{d_{G\{P_n\}}(u)} y^{d_{G\{P_n\}}(v)}$$
$$= \sum_{e=uv\in E(G)} x^{d_G(u)+1} y^{d_G(v)+1} + \sum_{u\in V(G)} x^{d_G(u)+1} y^2$$
$$+n_G(n-3)(xy)^2 + n_G x^2 y$$
$$= (xy)\left[M(G,x,y) + y\sum_{u\in V(G)} x^{d_G(u)} + (n-3)n_G xy + n_G x\right].$$

■

From Theorems 2.1 and 2.6, we get the following result:

Corollary 2.5 *The Nirmala index of* $G\{P_n\}$ *is:*

$$N(G\{P_n\}) = \sum_{e=uv\in E(G)} \sqrt{d_G(u) + d_G(v) + 2} + \sum_{u\in V(G)} \sqrt{d_G(u) + 3} + n_G\left[2n - 6 + \sqrt{3}\right]$$

Theorem 2.7 *Let G be any graph and C_n be a cycle graph then the M-Polynomial of the cluster graph $G\{C_n\}$ is:*

$$M(G\{C_n\}) = (xy)^2\left[M(G,x,y) + 2\sum_{u\in V(G)} x^{d_G(u)} + n_G(n-2)\right].$$

Proof: Let $e = uv \in E(G\{C_n\})$. Then

Case 1 If $e = uv \in E(G)$. Then $d_{G\{C_n\}}(u) = d_G(u) + 2$ and $d_{G\{C_n\}}(v) = d_G(v) + 2$.

Case 2 If $e = uv \in E(C_n)$, for some copy of C_n such that u be the root vertex of C_n. Then $d_{G\{C_n\}}(u) = d_G(u) + 2$ and $d_{G\{C_n\}}(v) = 2$.

Case 3 If $e = uv \in E(C_n)$ for some copy of C_n such that u,v are not root of C_n. Then $d_{G\{C_n\}}(u) = d_{G\{C_n\}}(v) = 2$.

From the above cases,

$$\begin{aligned} M(G\{C_n\},x,y) &= \sum_{e=uv\in E(G\{C_n\})} x^{d_{G\{C_n\}}(u)} y^{d_{G\{C_n\}}(v)} \\ &= \sum_{e=uv\in E(G)} x^{d_G(u)+2} y^{d_G(v)+2} \\ &+ n_G \sum_{e=uv\in E(C_n);\ u \text{ is the root vertex of } C_n} x^{d_G(u)+2} y^2 \\ &+ n_G \sum_{e=uv\in E(C_n);\ u,v \text{ are not root vertex of } C_n} (xy)^2 \\ &= (xy)^2 \sum_{e=uv\in E(G)} x^{d_G(u)} y^{d_G(v)} + 2\sum_{u\in V(G)} x^{d_G(u)+2} y^2 + (n-2)n_G(xy)^2 \\ &= (xy)^2 \left[M(G,x,y) + 2\sum_{u\in V(G)} x^{d_G(u)} + n_G(n-2) \right]. \end{aligned}$$

■

From Theorems 2.1 and 2.7, we get the following result:

Corollary 2.6 *The Nirmala index of $G\{C_n\}$ is:*

$N(G\{C_n\}) = \sum_{e=uv\in E(G)} \sqrt{d_G(u) + d_G(v) + 4} + 2\left[n_G(n-2) + \sum_{u\in V(G)} \sqrt{d_G(u) + 4}\right]$.

■

Theorem 2.8 *Let G be any graph and S_n be the star graph, such that the center vertex of S_n be it's root vertex. Then the M-Polynomial of the cluster graph $G\{S_n\}$ is*

$M(G\{Sn\},x,y) = (xy)^n M(G,x,y) + nx^n y \sum_{u\in V(G)} x^{d_G(u)}$

Proof: Let $e = uv \in E(G\{S_n\})$. Then

Case 1 If $e = uv \in E(G)$. Then $d_{G\{S_n\}}(u) = d_G(u) + n$ and $d_{G\{S_n\}}(v) = d_G(v) + n$.

Case 2 If $e = uv \in E(S_n)$, for some copy of S_n such that u be the root vertex of S_n. Then $d_{G\{S_n\}}(u) = d_G(u) + n$ and $d_{G\{S_n\}}(v) = 1$.

From the above cases,

$$\begin{aligned} M(G\{S_n\},x,y) &= \sum_{e=uv\in E(G\{S_n\})} x^{d_{G\{S_n\}}(u)} y^{d_{G\{S_n\}}(v)} \\ &= \sum_{e=uv\in E(G)} x^{d_G(u)+n} y^{d_G(v)+n} + \sum_{u\in V(G)} x^{d_G(u)+n}(n\,y) \\ &= (xy)^n M(G,x,y) + nx^n y \sum_{u\in V(G)} x^{d_G(u)}. \end{aligned}$$

■

From Theorems 2.1 and 2.8, we get the following result:

Corollary 2.7 *The Nirmala index of $G\{S_n\}$ is:*

$N(G\{S_n\}) = \sum_{e=uv\in E(G)} \sqrt{d_G(u) + d_G(v) + 2n} + n\sum_{u\in V(G)} \sqrt{d_G(u) + n + 1}$.

■

Theorem 2.9 *Let G_1 and G_2 be two graphs with vertex sets $V(G_1)$, $V(G_2)$, edge sets $E(G_1)$, $E(G_2)$, and orders n_1, n_2 respectively. Then the M-Polynomial of the join of G_1 and G_2 is*

$$\begin{aligned} M(G_1+G_2,x,y) &= (xy)^{n_2} M(G_1,x,y) + (xy)^{n_1} M(G_2,x,y) \\ &+ x^{n_2} y^{n_1} \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} x^{d_{G_1}(u)} y^{d_{G_2}(v)} \end{aligned}$$

Proof: Let $G = G_1 + G_2$, and $e = uv \in E(G)$. Then,

Case 1 If $e = uv \in E(G_1)$ then $d_G(u) = d_{G_1}(u) + n_2$ and $d_G(v) = d_{G_1}(v) + n_2$,

Case 2 If $e = uv \in E(G_2)$ then $d_G(u) = d_{G_2}(u) + n_1$ and $d_G(v) = d_{G_2}(v) + n_1$,

Case 3 If $e = uv$ such that $u \in V(G_1)$ and $v \in V(G_2)$ then $d_G(u) = d_{G_1}(u) + n_2$ and $d_G(v) = d_{G_2}(v) + n_1$.

From the above three cases,

$$M(G,x,y)=\sum_{e=uv\in E(G)} x^{d_G(u)}y^{d_G(v)}$$
$$=\sum_{e=uv\in E(G_1)} x^{d_G(u)}y^{d_G(v)}+\sum_{e=uv\in E(G_2)} x^{d_G(u)}y^{d_G(v)}+\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} x^{d_G(u)}y^{d_G(v)}$$
$$=\sum_{e=uv\in E(G_1)} x^{d_{G_1}(u)+n_2}y^{d_{G_1}(v)+n_2}+\sum_{e=uv\in E(G_2)} x^{d_{G_2}(u)+n_1}y^{d_{G_2}(v)+n_1}$$
$$+\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} x^{d_{G_1}(u)+n_2}y^{d_{G_2}(v)+n_1}$$
$$=(xy)^{n_2}M(G_1,x,y)+(xy)^{n_1}M(G_2,x,y)+x^{n_2}y^{n_1}\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} x^{d_{G_1}(u)}y^{d_{G_2}(v)}\ .$$

■

From Theorems 2.1 and 2.9, we get the following result:

Corollary 2.8 *The Nirmala index of the join graph* G_1+G_2 *is:*

$$N(G_1+G_2)=\sum_{e=uv\in E(G_1)}\sqrt{d_{G_1}(u)+d_{G_1}(v)+2n_2}+\sum_{e=uv\in E(G_2)}\sqrt{d_{G_2}(u)+d_{G_2}(v)+2n_1}$$
$$+\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}\sqrt{d_{G_1}(u)+d_{G_2}(v)+n_1+n_2}\ .$$

■

Theorem 2.10 *Let* G_1 *and* G_2 *be two graphs with vertex sets* $V(G_1)$, $V(G_2)$, *edge sets* $E(G_1)$, $E(G_2)$, *and orders* n_1, n_2 *respectively. Then the M-Polynomial of the corona product of* G_1 *and* G_2 *is:*

$$M(G_1\odot G_2,x,y)=(xy)^{n_2}M(G_1,x,y)+n_1xyM(G_2,x,y)+x^{n_2}y\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} x^{d_{G_1}(u)}y^{d_{G_2}(v)}$$

Proof: Let $G=G_1\odot G_2$, and $e=uv\in E(G)$. Then there are the following cases,

Case 1 If $e=uv\in E(G_1)$, then $d_G(u)=d_{G_1}(u)+n_2$ and $d_G(v)=d_{G_1}(v)+n_2$,

Case 2 If $e=uv\in E(G_2)$ for some copies of G_2, then $d_G(u)=d_{G_2}(u)+1$ and $d_G(v)=d_{G_2}(v)+1$,

Case 3 If $e=uv$ such that $u\in V(G_1)$, and $v\in V(G_2)$ for some copies of G_2, then $d_G(u)=d_{G_1}(u)+n_2$ and $d_G(v)=d_{G_2}(v)+1$.

From the above three cases,

$$M(G,x,y)=\sum_{e=uv\in E(G)} x^{d_G(u)}y^{d_G(v)}$$
$$=\sum_{e=uv\in E(G_1)} x^{d_G(u)}y^{d_G(v)}+n_1\sum_{e=uv\in E(G_2)} x^{d_G(u)}y^{d_G(v)}+\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} x^{d_G(u)}y^{d_G(v)}$$
$$=\sum_{e=uv\in E(G_1)} x^{d_{G_1}(u)+n_2}y^{d_{G_2}(v)+n_2}+n_1\sum_{e=uv\in E(G_2)} x^{d_{G_2}(u)+1}y^{d_{G_2}(v)+1}$$
$$+\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} x^{d_{G_1}(u)+n_2}y^{d_{G_2}(v)+1}$$
$$=(xy)^{n_2}\sum_{e=uv\in E(G_1)} x^{d_{G_1}(u)}y^{d_{G_2}(v)}+n_1xy\sum_{e=uv\in E(G_2)} x^{d_{G_2}(u)}y^{d_{G_2}(v)}$$
$$+x^{n_2}y\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} x^{d_{G_1}(u)}y^{d_{G_2}(v)}$$
$$=(xy)^{n_2}M(G_1,x,y)+n_1xyM(G_2,x,y)+x^{n_2}y\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} x^{d_{G_1}(u)}y^{d_{G_2}(v)}\ .$$

■

From Theorems 2.1 and 2.10, we get the following result:

Corollary 2.9 *The Nirmala index of the corona graph* $G_1\odot G_2$ *is:*

$$N(G_1 \odot G_2) = \sum_{e=uv\in E(G_1)} \sqrt{d_{G_1}(u) + d_{G_1}(v) + 2n_2 + n_1} \sum_{e=uv\in E(G_2)} \sqrt{d_{G_2}(u) + d_{G_2}(v) + 2} + \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \sqrt{d_{G_1}(u) + d_{G_2}(v) + n_2 + 1}\,.$$

■

Conclusions

In conclusion, we studied the M-Polynomial and Nirmala index, in such away computing both concepts of some certain graphs. The exact computational formulas are presented of them. These theoretical results are proved. Our results could be beneficial to compute other topological indices for the same studied graphs.

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متعددة الحدود من النمط M ومؤثر نيرمالا لبيانات مركبة محددة

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الملخص

متعددة الحدود من النمط M واحدة من متعددات الحدود المهمة والجديرة بالاهتمام في نظرية البيان الكيميائية. في هذا البحث قمنا باحتساب متعددة الحدود من النمط M لبيانات مركبة محددة اضافة الى احتساب مؤثر نيرمالا من خلال متعددة الحدود المذكورة. والبيانات المركبة هذه حصلنا عليها في هذا البحث من خلال اجراء عمليات الربط وكورونا والعنقودية لبيانات بسيطة معينة.