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On the Embedding of an Arc into Cubic Curves in a Finite Projective Plane of Order Seven

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Abstract
The main aims of this research are to find the stabilizer groups of cubic curves over a finite field of order 7 and studying the properties of their groups and then constructing the arcs of degree 2 which are embedding in cubic curves of even size which are considering as the arcs of degree 3. Also drawing all these arcs.

Keywords: Stabilizer groups, arcs, Cubic curves.

الخلاصة
الاهداف الرئيسية لهذا البحث هو ايجاد الزمر المثبتة للمنحنيات المكعبة حول الحقل المنتهي من الرتبة 7 ودراسة الخواص لهذه الزمر وكذلك تشكيل الاقواس من الدرجة الثانية والتي تغمر في المنحنيات المكعبة ذات الحجم الزوجي والتي نفسها تعتبر كأقواس من الدرجة الثانية . كذلك رسم كل هذه الأقواس.

Introduction

The subject of this research depends on themes of Projective geometry over a finite field, Group theory, Field theory, Presentation theory. The strategy to construct the stabilizer groups and also to embedded the arcs is given as following:
Constructing the linear transformations group $PGL(3,q)$ of $PG(2,7)$. Which its elements are considering the non-singular matrices $A_n = [a_{ij}]$, $a_{i,j}$ in F_7, $i,j = 1,2,3$ for some n in $\mathbb{N}$ and satisfying $K(tA_n) = K$ for all t in $F_7\backslash\{0\}$ and K be any arc. Also, we have found the arcs which are embedding in cubic curves which are split into two sets, one of them contains the inflection points and the other does not, the set which does not contain the inflection points is considering the arc of degree two.
The brief history of this theme is shown as follows. In 2010, Najm Al-Seraji [2] has been studied the cubic curves over a finite field of order 17. In 2011, Emad Al-Zangana [3] has been shown the cubic curves over a finite field of order 19. In 2013, Emad Al-Zangana [4] has been described the cubic curves over a finite field of orders 2, 3, 5, 7. In 2013, Emad Al-Zangana [5] has been classified the cubic curves over a finite field of order 11, 13.
Now, we recall the definitions which are using in this research as follow:

Definition(1.1)[1]: Denote by S and S^* two subspaces of $P(n,K)$. A projectivity $\beta: S \to S^*$ is a bijection given by a matrix T, necessarily non-singular, where $P(X^*) = P(X)\beta$ if $tX^* = XT$, with $t \in K\backslash\{0\}$. Write $\beta = M(T)$; then $\beta = M(\lambda T)$ for any λ in $K\backslash\{0\}$. The group of projectivities of $PG(n,K)$ is denoted by $PGL(n+1,K)$.

Definition(1.2)[1]: The stabilizer of x in A set Λ under the action of G is the group $G_x = \{g \in G | xg = x\}$.

Definition(1.3)[1]: An (n,r) arc K or arc of degree r in $PG(k,q)$ with $n \geq r+1$ is a set of points with property that every hyperplane meets K in at most r points of K and there is some hyperplane meeting K in exactly r points.

Definition (1.4) [1]: A non-singular point P of $\mathcal{F}$ is a point of *inflexion* of $\mathcal{F}$ if $I(P, \ell_p \cap \mathcal{F}) \geq 3$.
Here, P is also called an inflexion; the tangent ℓ_p at P is the inflexion tangent.

The classification of cubic curves over a finite field of order7:

The polynomial of degree three $g_4(x) = x^3 - x - 2$ is primitive in $F_7 = \{0,1,2,3,4,5,6\}$, since $g_4(0) = -2$, $g_4(1) = -2$, $g_4(2) = 4$, $g_4(3) = 1$, $g_4(4) = 2$, $g_4(5) = 6$, $g_4(6) = 5$, also $g_4(\varepsilon^{284}) = 0$, $g_3(\varepsilon^{236}) = 0$, $g_4(\varepsilon^{278}) = 0$, this means $\varepsilon^{284}, \varepsilon^{236}, \varepsilon^{278}$ are roots of g_4 in F_{7^3}.
The companion matrix of $g_3(x) = x^3 - x - 2$ in $F_7[x]$ generated the points and lines of $PG(2,7)$ as follows:

$$P(k) = [1,0,0]C(g)^k = [1,0,0]\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}^k,$$

k=0,1, … ,56 .
With selecting the points in $PG(2,7)$ which are the third coordinate equal to zero, this means belong to $L_0 = v(z)$, that is $v(z) = tz = z$ for all t in $F_7\backslash\{0\}$ and with $P(k) = k$, we obtain L_0={0,1,3,13,32, 36,43,52}, that is

$$L_{k=}L_0 C(g)^k = L_0 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}^k, k=0,1, \dots ,56 .$$

The number of distinct cubic curves in $PG(2,7)$ is 26 see [4], one of them is given as follows:

$$\beta_1 = x^3 + y^3 + z^3 \qquad (1)$$

The points of $PG(2,7)$ on β_1 in equation (1) are [5,1,0], [0,5,1], [6,1,0], [0,6,1], [5,0,1], [3,0,1], [3,1,0], [0,3,1], [6,0,1].
After the calculation and help the computer, we are obtained that the number of matrices which are stabilizing of β_1 in equation (1) is 216, and we can not write them, because they are too much. Moreover, the stabilizer group of β_1 in equation (1) which is denoted by G_{β_1} which contains:

- 9 matrices of order 2;
- 80 matrix of order 3;
- 54 matrix of order 4;
- 72 matrix of order 6;
- The identity matrix.

Drawing of β_1 in equation (1) is given in Figure 1 as following:

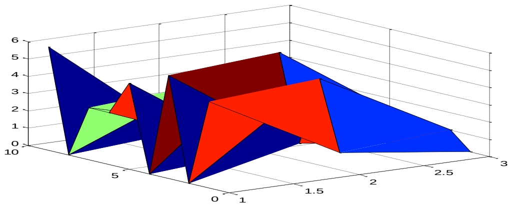

Figure 1: Drawing of β_1.

Another one of cubic curve which is given in [4] is:

$$\beta_2 = xyz - 2(x + y + z)^3 \qquad (2)$$

The points of $PG(2,7)$ on β_2 in equation (2) are [4,4,1], [6,1,0], [0,6,1], [1,2,1], [2,1,1], [6,0,1]. To find the stabilizer group of β_2 in equation(2), we are doing calculations by help the computer. Thus

$$G_{\beta_2} \cong \boldsymbol{S_4} = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 6 & 6 & 6 \end{pmatrix} \right\rangle.$$

β_2 in equation (2) is drawn in Figure 2 below.

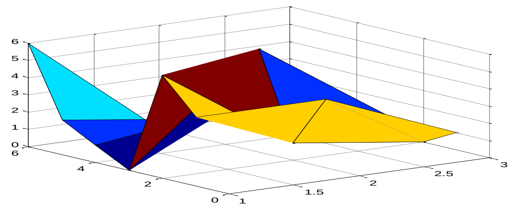

Figure 2: Drawing of β_2 .

Let $\beta_2^* = \{[1,2,1], [2,1,1], [6,0,1]\}$ be a subset of β_2 in equation (2) which is forming by partition the β_2 into two sets such that β_2^* does not contains the inflexion points of β_2, so we note that β_2^* represents an arc of degree two. After the calculation and help the computer, we are obtained that the number of matrices which are stabilizing of β_2^* is 216, and we can not write them, because they are too much.

Moreover, the stabilizer group of β_2^* which is denoted by $G_{\beta_2^*}$ which contains

- 21 matrix of order 2;
- 80 matrix of order 3;
- 18 matrix of order 4;
- 60 matrix of order 6;
- 36 matrix of order 12;
- The identity matrix.

Drawing of β_2^* is given in Figure 3 as following:

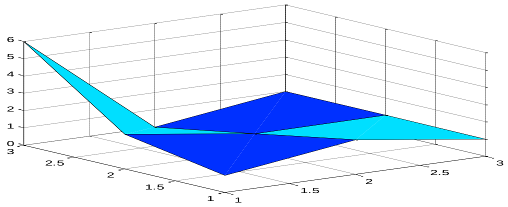

Figure 3: Drawing of β_2^*.

From [4], we obtain:

$$\beta_3 = xyz + 3(x+y+z)^3 \qquad (3)$$

The points of $PG(2,7)$ on β_3 in equation (3) are $[1,3,1], [6,1,0], [0,6,1], [5,5,1], [3,1,1], [6,0,1]$.

To find the stabilizer group of β_3 in equation(3), we are doing calculations by help the computer. Thus

$$G_{\beta_3} \cong \boldsymbol{S_4} = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 6 \\ 2 & 6 & 4 \\ 6 & 2 & 4 \end{pmatrix} \right\rangle.$$

β_3 in equation (3) is drawn in Figure 4

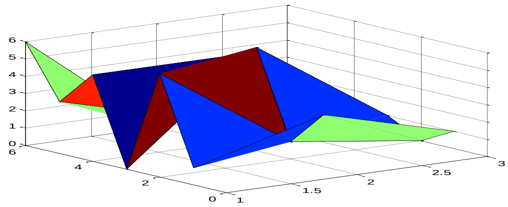

Figure 4: Drawing of β_3.

Let $\beta_3^* = \{[0,6,1],[3,1,1],[6,0,1]\}$ be a subset of β_3 in equation (3) which is forming by partition the β_3 into two sets such that β_3^* does not contains the inflection points of β_3, so we note that β_3^* represents an arc of degree two.

After the calculation and help the computer, we are obtained that the number of matrices which are stabilizing of β_3^* is 216 , and we can not write them, because they are too much. Moreover, the stabilizer group of β_3^* which is denoted by $G_{\beta_3^*}$ which contains

- 21 matrix of order 2;
- 80 matrix of order 3;
- 18 matrix of order 4;
- 60 matrix of order 6;
- 36 matrix of order 12;
- The identity matrix.

Drawing of β_3^* is given in Figure 5 as following:

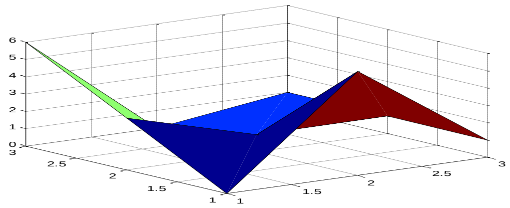

Figure 5: Drawing of β_3^*.

From [4], we obtain:

$$\beta_4 = xyz - 3(x+y+z)^3 \qquad (4)$$

The points of $PG(2,7)$ on β_4 in equation (4)are $[4,6,1], [2,2,1], [6,1,0], [0,6,1], [3,6,1], [4,1,1], [1,4,1], [2,5,1], [6,4,1], [5,2,1], [6,3,1], [6,0,1]$.

To find the stabilizer group of β_4 in equation (4), we are doing calculations by help the computer. Thus

$$G_{\beta_4} \cong \boldsymbol{S_3} = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle.$$

β_4 in equation (36) is drawn in Figure 6

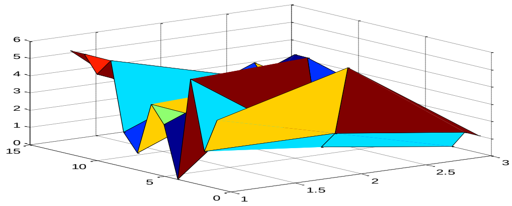

Figure 6: Drawing of β_4.

Let $\beta_4^* = \{[3,6,1],[5,2,1],[6,0,1]\}$ be a subset of β_4 in equation (4) which is forming by partition β_4 into two sets such that β_4^* does not contains the inflection points of β_4, so we note that β_4^* represents an arc of degree two. Also, to find the stabilizer group of β_4^*, by some calculation, we obtain

$$G_{\beta_4^*} \cong \boldsymbol{D_6} = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 3 & 3 & 6 \\ 4 & 2 & 6 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle.$$

Drawing of β_4^* is given in Figure 7

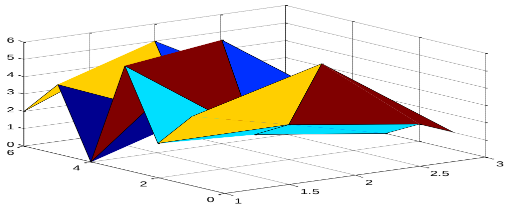

Figure 7: Drawing of β_4^*.

From [4], we obtain:

$$\beta_5 = xyz + (x + y + z)^3 \qquad (5)$$

The points of $PG(2,7)$ on β_5 in equation (5) are [6,1,0], [0,6,1], [1,6,1], [6,1,1], [1,1,1], [6,6,1], [6,0,1].

To find the stabilizer group of β_5 in equation (5), we are doing calculations by help the computer. Thus

$$G_{\beta_5} \cong \boldsymbol{S_3} = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle.$$ β_5 in equation (5) is drawn in Figure 8

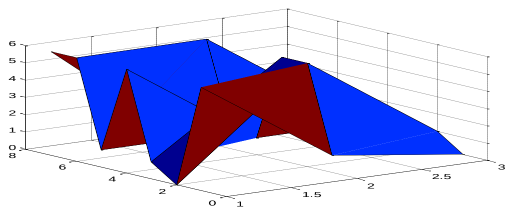

Figure 8: Drawing of β_5.

From [4], we obtain:

$$\beta_6 = xy(x + y) + 3z^3 \qquad (6)$$

The points of $PG(2,7)$ on β_6 in equation (6) are [1,0,0], [0,1,0], [6,1,0].

After the calculation and help the computer, we are obtained that the number of matrices which are stabilizing of β_6 in equation (6) is 1764, and we can not write them, because they are too much. Moreover, the stabilizer group of β_6 in equation (6) which is denoted by G_{β_6}which contains

- 91 matrix of order 2;
- 224 matrix of order 3;
- 980 matrix of order 6;
- 48 matrix of order 7;
- 252 matrix of order 14;
- 168 matrix of order 21;
- The identity matrix.

Drawing of β_6 in equation (6) is given in Figure 9 as following:

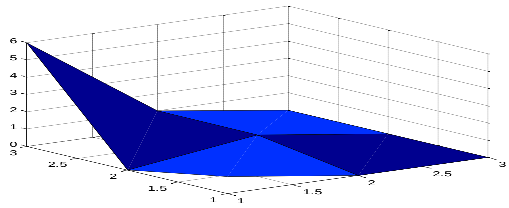

Figure 9: Drawing of β_6.

From [4], we have:

$$\beta_7 = xy(x + y) - 2z^3 \qquad (7)$$

The points of $PG(2,7)$ on β_7 in equation (7) are [1,0,0], [0,1,0], [4,4,1], [4,6,1], [2,2,1], [6,1,0], [2,3,1], [1,5,1], [6,4,1], [1,1,1], [5,1,1], [3,2,1]. To find the stabilizer group of β_7 in equation(7), we are doing calculations by help the computer. Thus

$$G_{\beta_7} \cong \boldsymbol{S_3} \times \boldsymbol{Z_3} = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \right\rangle.$$

β_7 in equation (7) is drawn in Figure 10

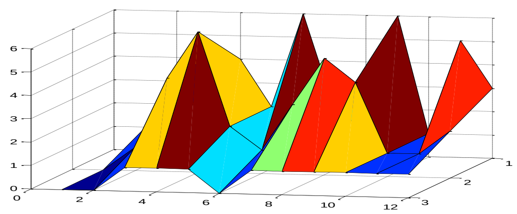

Figure 10: Drawing of β_7.

Let $\beta_7^* = \{[2,2,1], [2,3,1], [1,5,1], [1,1,1], [5,1,1], [3,2,1]\}$ be a subset of β_7 in equation (7) which is forming by partition β_7 into two sets such that β_7^* does not contains the inflection points of β_7, so we note that β_7^* represents an arc of degree

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two. Also, to find the stabilizer group of β_7^*, by some calculation, we obtain $G_{\beta_7^*}$ contains

- 9 matrices of order 2;
- 8 matrices of order 3;
- 18 matrix of order 4;
- The identity matrix.

Drawing of β_7^* is given in Figure 11 as following:

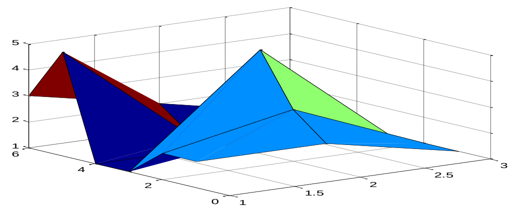

Figure 11: Drawing of β_7^*.

From [4], we have:

$$\beta_8 = yz^2 + x^3 - 3xy^2 - y^3 \qquad (8)$$

The points of $PG(2,7)$ on β_8 in equation (8) are [0,0,1], [0,6,1], [2,3,1], [5,4,1], [0,1,1]. To find the stabilizer group of β_8 in equation (8), we are doing calculations by help the computer. Thus

$$G_{\beta_8} \cong \boldsymbol{D_4} = \left\langle \begin{pmatrix} 1 & 4 & 0 \\ 4 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

β_8 in equation (8) is drawn in Figure 12

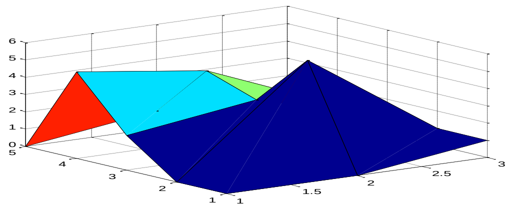

Figure 12: Drawing of β_8.

From [4], we have:

$$\beta_9 = yz^2 + x^3 - 3xy^2 + y^3 \qquad (9)$$

The points of $PG(2,7)$ on β_9 in equation (9) are [0,0,1], [4,3,1], [2,6,1], [1,4,1], [3,4,1], [1,5,1], [1,1,1], [6,3,1], [5,1,1], [6,6,1], [6,2,1].

To find the stabilizer group of β_9 in equation (9), we are doing calculations by help the computer. Thus

$$G_{\beta_9} \cong \boldsymbol{Z_2} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \right\rangle.$$

β_9 in equation (9) is drawn in Figure 13.

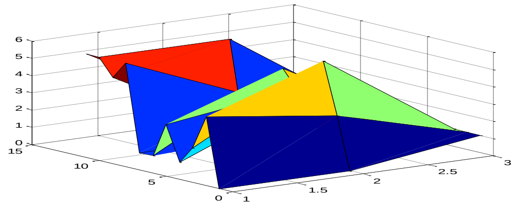

Figure 13: Drawing of β_9.

From [4], we obtain:

$$\beta_{10} = yz^2 + x^3 + xy^2 \qquad (10)$$

The points of $PG(2,7)$ on β_{10} in equation (10) are [0,1,0], [0,0,1], [6,5,1], [1,4,1], [2,5,1], [1,2,1], [5,2,1], [6,3,1]. To find the stabilizer group of β_{10} in equation (10), we are doing calculations by help the computer. Thus

$$G_{\beta_{10}} \cong \boldsymbol{Z_2} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \right\rangle.$$

β_{10} in equation (10) is drawn in Figure 14.

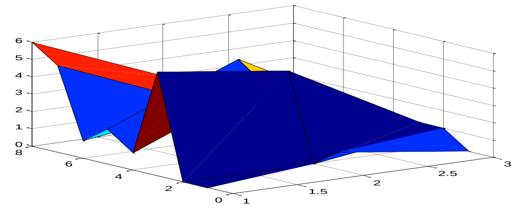

Figure 14: Drawing of β_{10}.

Let $\beta_{10}^* = \{[1,4,1], [1,2,1], [5,2,1], [6,3,1]\}$ be a subset of β_{10} in equation (10) which is forming by partition β_{10} into two sets such that β_{10}^* does not contains the inflection points of β_{10}, so we note that β_{10}^* represents an arc of degree two. Also, to find the stabilizer group of β_{10}^*, by some calculation, we obtain

$$G_{\beta_{10}^*} \cong \boldsymbol{S_4} = \left\langle \begin{pmatrix} 1 & 0 & 5 \\ 4 & 6 & 3 \\ 6 & 1 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ 6 & 3 & 4 \end{pmatrix} \right\rangle.$$

Drawing of β_{10}^* is given in Figure 15.

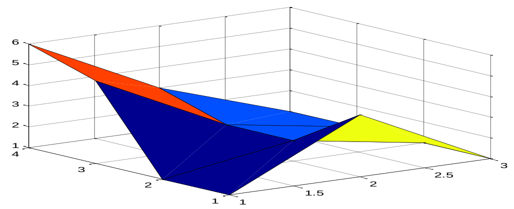

Figure 15: Drawing of β_{10}^*.

From [4], we obtain:

$$\beta_{11} = yz^2 + x^3 + 3xy^2 \qquad (11)$$

The points of $PG(2,7)$ on β_{11} in equation (11) are [0,1,0], [0,0,1], [5,1,0], [4,3,1], [3,6,1], [4,1,1], [3,4,1], [2,1,0]. To find the stabilizer group of β_{11} in equation(11), we are doing calculations by help the computer. Thus

$$G_{\beta_{11}} \cong \boldsymbol{Z_2} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \right\rangle.$$

β_{11} in equation (11) is drawn in Figure 16

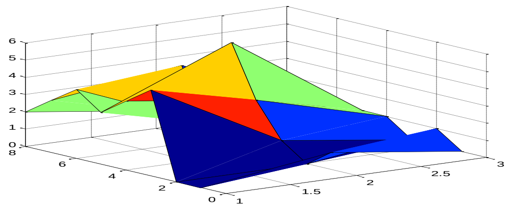

Figure 16: Drawing of β_{11}.

Let $\beta_{11}^* = \{[4,3,1], [3,6,1], [4,1,1], [3,4,1]\}$ be a subset of β_{11} in equation (11) which is forming by partition β_{11} into two sets such that β_{11}^* does not contains the inflection points of β_{11}, so we note that β_{11}^* represents an arc of degree two. Also, to find the stabilizer group of β_{11}^*, by some calculation, we obtain

$$G_{\beta_{11}^*} \cong \boldsymbol{S_4} = \left\langle \begin{pmatrix} 0 & 1 & 3 \\ 0 & 5 & 0 \\ 1 & 4 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 0 \\ 5 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix} \right\rangle.$$

Drawing of β_{11}^* is given in Figure 17.

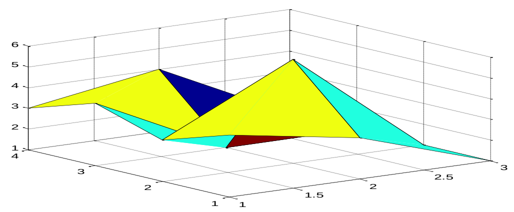

Figure17: Drawing of β_{11}^*.

From [4], we obtain:

$$\beta_{12} = yz^2 + x^3 - 3xy^2 + 3y^3 \qquad (12)$$

The points of $PG(2,7)$ on β_{12} in equation (12) are[0,0,1], [3,6,1], [4,1,1], [3,5,1], [2,5,1], [3,1,0], [0,3,1], [5,2,1], [4,2,1], [0,4,1].To find the stabilizer group of β_{12} in equation (12), we are doing calculations by help the computer. Thus:

$$G_{\beta_{12}} \cong \boldsymbol{Z_2} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \right\rangle.$$

β_{12} in equation (12) is drawn in Figure 18.

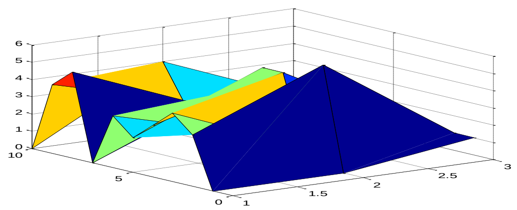

Figure 18: Drawing of β_{12}.

Let

$\beta_{12}^* = \{[2,5,1], [0,3,1], [5,2,1], [4,2,1], [0,4,1]\}$ be a subset of β_{12} in equation (12) which is forming by partition β_{12} into two sets such that β_{12}^* does not contains the inflection points of β_{12}, so we note that β_{12}^* represents an arc of degree two. Also, to find the stabilizer group of β_{12}^*, by some calculation, we obtain

$$G_{\beta_{12}^*} \cong \boldsymbol{S_3} = \left\langle \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 6 \\ 6 & 6 & 3 \\ 2 & 6 & 3 \end{pmatrix} \right\rangle.$$

Drawing of β_{12}^* is given in Figure 19.

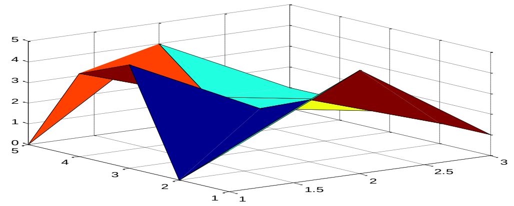

Figure 19: Drawing of β_{12}^*.

From [4], we have:

$$\beta_{13} = yz^2 + x^3 - 2xy^2 + 2y^3 \qquad (13)$$

The points of $PG(2,7)$ on β_{13} in equation (13) are [0,0,1], [4,3,1], [3,4,1], [4,5,1], [5,6,1], [2,1,1], [3,2,1]. To find the stabilizer group of β_{13} in equation (13), we are doing calculations by help the computer. Thus

$$G_{\beta_{13}} \cong \boldsymbol{Z_2} \times \boldsymbol{Z_2} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} 1 & 5 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix} \right\rangle.$$

β_{13} in equation (13) is drawn in Figure 20.

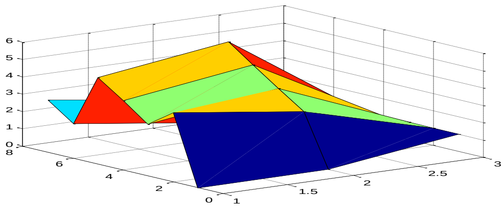

Figure 20: Drawing of β_{13}.

From [4], we have:

$$\beta_{14} = yz^2 + x^3 - 2xy^2 + y^3 \qquad (14)$$

The points of $PG(2,7)$ on β_{14} in equation (14) are [0,0,1], [3,5,1], [1,1,0], [4,2,1]. To find the stabilizer group of β_{14} in equation (14), we are doing calculations by help the computer. Thus $G_{\beta_{14}}$ contains

- 7 matrices of order 2;
- 8 matrices of order 3;
- 20 matrix of order 6;
- The identity matrix.

Drawing of β_{14} in equation (14) is given in figure 21 as following:

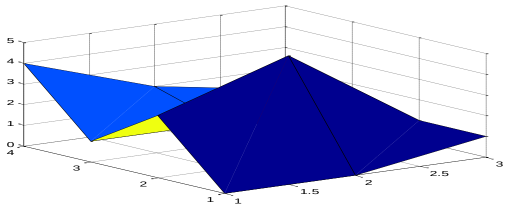

Figure 21: Drawing of β_{14}.

Let $\beta_{14}^* = \{[1,1,0],[4,2,1]\}$ be a subset of β_{14} in equation (14) which is forming by partition the β_{14} into two sets such that β_{14}^* does not contains the inflection points of β_{14}, so we note that β_{14}^* represents an arc of degree two. After the calculation and help the computer, we are obtained that the number of matrices which are stabilizing of β_{14}^* is 3528, and we cannot write them, because they are too much. Moreover, the stabilizer group of β_{14}^* which is denoted by $G_{\beta_{14}^*}$ which contains

- 103 matrix of order 2;
- 222 matrix of order 3;
- 292 matrix of order 4;
- 1602 matrix of order 6;
- 588 matrix of order 12;
- 336 matrix of order 14;
- 168 matrix of order 21;
- 168 matrix of order 42;
- The identity matrix.

Drawing of β_{14}^* is given in Figure 22 as following:

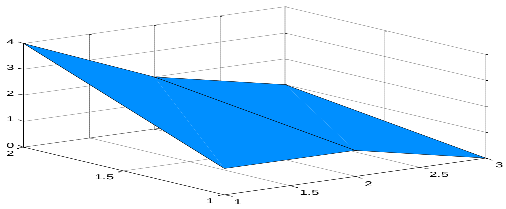

Figure 22: Drawing of β_{14}^*.

From [4], we have:

$$\beta_{15} = yz^2 + x^3 - 2xy^2 + 3y^3 \qquad (15)$$

The points of $PG(2,7)$ on β_{15} in equation (15) are [0,0,1], [2,6,1], [2,4,1], [2,3,1], [5,4,1], [2,1,0], [0,3,1], [5,1,1], [5,3,1], [0,4,1].To find the stabilizer group of β_{15} in equation (15), we are doing calculations by help the computer. Thus:

$$G_{\beta_{15}} \cong \boldsymbol{Z_2} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \right\rangle.$$

β_{15} in equation (15) is drawn in Figure 23.

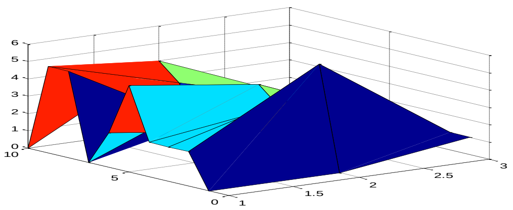

Figure 23: Drawing of β_{15}.

Let $\beta_{15}^* = \{[2,4,1], [0,3,1], [5,1,1], [5,3,1], [0,4,1]\}$ be a subset of β_{15} in equation (15) which is forming by partition β_{15} into two sets such that β_{15}^* does not contains the inflection points of β_{15}, so we note that β_{15}^* represents an arc of degree two. Also, to find the stabilizer group of β_{15}^*, by some calculation, we obtain

$$G_{\beta_{15}^*} \cong \boldsymbol{S_3} = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 5 \\ 6 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 5 \\ 0 & 2 & 0 \\ 2 & 5 & 6 \end{pmatrix} \right\rangle.$$

Drawing of β_{15}^* is given in Figure 24.

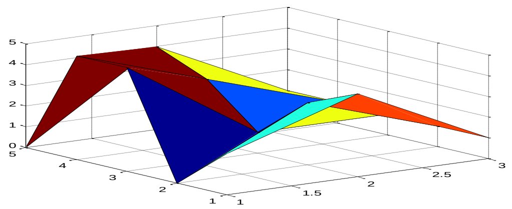

Figure 24: Drawing of β_{15}^*.

From [4], we have:

$$\beta_{16} = yz^2 + x^3 - 3y^3 \qquad (16)$$

The points of $PG(2,7)$ on β_{16} in equation (16) are [0,0,1], [1,3,1], [4,3,1], [2,2,1], [6,5,1], [2,3,1], [5,5,1], [3,4,1], [5,4,1], [3,5,1], [1,2,1], [6,4,1], [4,2,1]. To find the stabilizer group of β_{16} in equation (16), we are doing calculations by help the computer. Thus

$$G_{\beta_{16}} \cong \boldsymbol{Z_6} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \right\rangle.$$

β_{16} in equation (16) is drawn in Figure 25.

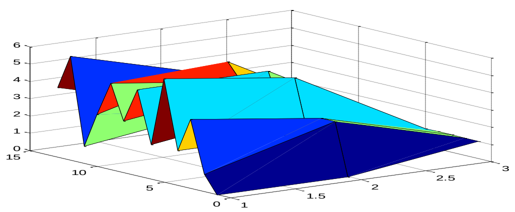

Figure 25: Drawing of β_{16}.

From [4], we obtain:

$$\beta_{17} = yz^2 + x^3 + y^3 \qquad (17)$$

The points of $PG(2,7)$ on β_{17} in equation (17) are [0,0,1], [5,1,0], [6,1,0], [3,1,0]. To find the stabilizer group of β_{17} in equation (17), we are doing calculations by help the computer. Thus $G_{\beta_{17}}$ contains

- 7 matrices of order 2;
- 8 matrices of order 3;
- 20 matrix of order 6;
- The identity matrix.

Drawing of β_{17} in equation (17) is given in Figure 26 as following:

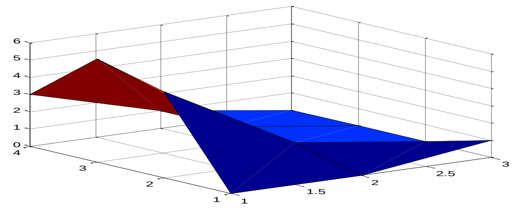

Figure 26: Drawing of β_{17}.

Let $\beta_{17}^* = \{[6,1,0], [3,1,0]\}$ be a subset of β_{17} in equation (17) which is forming by partition the β_{17} into two sets such that β_{17}^* does not contains the inflection points of β_{17}, so we note that β_{17}^* represents an arc of degree two. After the calculation and help the computer, we are obtained that the number of matrices which are stabilizing of β_{17}^* is 3528, and we can not write them, because they are too much. Moreover, the stabilizer group of β_{17}^*which is denoted by $G_{\beta_{17}^*}$ which contains

- 103 matrix of order 2;
- 222 matrix of order 3;
- 292 matrix of order 4;
- 1602 matrix of order 6;
- 588 matrix of order 12;
- 336 matrix of order 14;
- 168 matrix of order 21;
- 168 matrix of order 42;
- The identity matrix.

Drawing of β_{17}^* is given in Figure 27 as following:

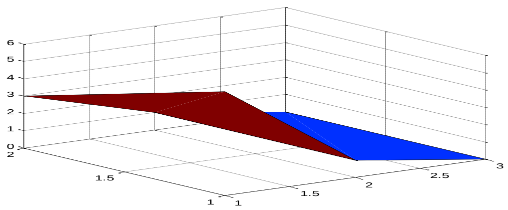

Figure 27: Drawing of β_{17}^{*}.

From [4], we obtain:

$$\beta_{18} = yz^2 + x^3 + 2y^3 \qquad (18)$$

The points of $PG(2,7)$ on β_{18} in equation (18) are [0,0,1], [4,4,1], [3,3,1], [2,4,1], [1,4,1], [6,3,1], [5,3,1]. To find the stabilizer group of β_{18} in equation (18), we are doing calculations by help the computer. Thus

$$G_{\beta_{18}} \cong \boldsymbol{Z_6} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \right\rangle.$$

β_{18} in equation (18) is drawn in Figure 28.

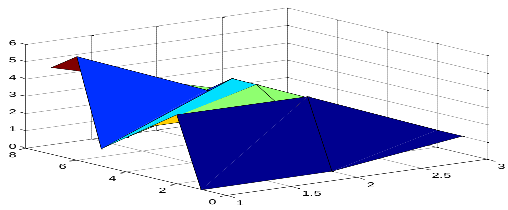

Figure 28: Drawing of β_{18}.

From [4], we obtain:

$$\beta_{19} = xy^2 + x^2z + 3yz^2 - (x^3 + 3y^3 + 2z^3 - 2xyz) \qquad (19)$$

The points of $PG(2,7)$ on β_{19} in equation (19) are [5,5,1], [2,1,1], [4,1,0], [0,4,1], [3,2,1], [6,0,1]. To find the stabilizer group of β_{19} in equation (19), we are doing calculations by help the computer. Thus

$$G_{\beta_{19}} \cong \boldsymbol{Z_3} = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \right\rangle.$$

β_{19} in equation (19) is drawn in Figure 29.

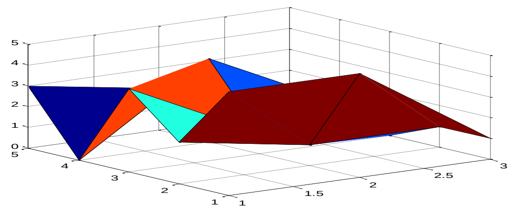

Figure 29: Drawing of β_{19}.

From [4], we have:

$$\beta_{20} = xy^2 + x^2z - 2yz^2 - (x^3 - 2y^3 - 3z^3 - xyz) \qquad (20)$$

The points of $PG(2,7)$ on β_{20} in equation (20) are[1,3,1], [3,5,1], [1,5,1], [1,2,1], [6,4,1], [4,5,1]. To find the stabilizer group of β_{20} in equation (20), we are doing calculations by help the computer. Thus

$$G_{\beta_{20}} \cong \boldsymbol{Z_3} = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix} \right\rangle.$$

β_{20} in equation (20) is drawn in Figure 30.

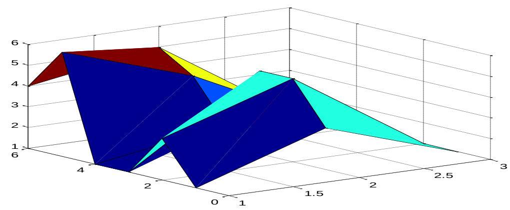

Figure 30: Drawing of β_{20}.

From [4], we have:

$$\beta_{21} = xy^2 + x^2z - 2yz^2 - 3(x^3 - 2y^3 - 3z^3 - xyz) \qquad (21)$$

The points of $PG(2,7)$ on β_{21} in equation (21) are[5,1,0], [0,5,1], [3,3,1], [1,0,1], [4,1,1], [5,4,1], [3,0,1], [4,0,1], [0,4,1]. To find the stabilizer group of β_{21} in equation (21), we are doing calculations by help the computer. Thus

$$G_{\beta_{21}} \cong \boldsymbol{Z_3} = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix} \right\rangle.$$

β_{21} in equation (21) is drawn in Figure 31.

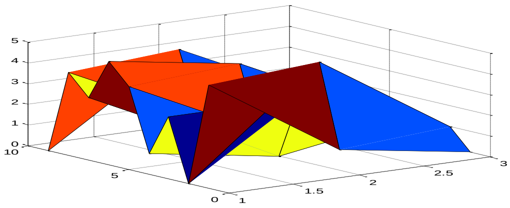

Figure 31: Drawing of β_{21}.

From [4], we have:

$$\beta_{22} = xy^2 + x^2z + 3yz^2 - 3(x^3 + 3y^3 + 2z^3 - 2xyz) \qquad (22)$$

The points of $PG(2,7)$ on β_{22} in equation (22) are [4,3,1], [3,3,1], [4,6,1], [6,1,0], [0,6,1], [1,4,1], [3,1,1], [1,6,1], [1,1,1], [4,0,1], [5,3,1], [6,2,1]. To find the stabilizer group of β_{22} in equation (22), we are doing calculations by help the computer. Thus

$$G_{\beta_{22}} \cong \boldsymbol{Z_3} = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \right\rangle.$$

β_{22} in equation (22) is drawn in Figure 32.

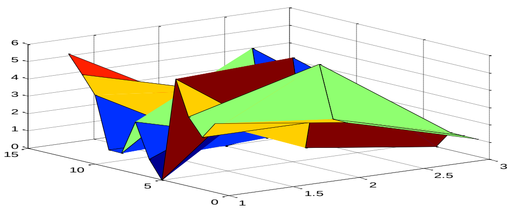

Figure 32: Drawing of β_{22}.

From [4], we obtain:

$$\beta_{23} = xy^2 + x^2z + 3yz^2 \qquad (23)$$

The points of $PG(2,7)$ on β_{23} in equation (23) are [1,0,0], [0,1,0], [0,0,1].
After the calculation and help the computer, we are obtained that the number of matrices which are stabilizing of β_{23} in equation (23) is 216, and we can not write them, because they are too much. Moreover, the stabilizer group of β_{23} in equation (23) which is denoted by $G_{\beta_{23}}$ which contains

- 21 matrix of order 2;
- 80 matrix of order 3;
- 18 matrix of order 4;
- 60 matrix of order 6;
- 36 matrix of order 12;
- The identity matrix.

Drawing of β_{23} in equation (23) is given in Figure 33 as following:

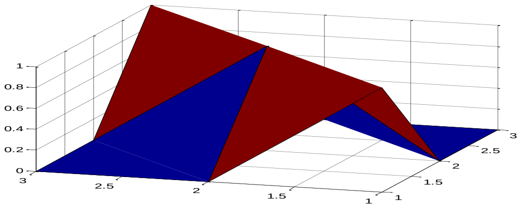

Figure 33: Drawing of β_{23}.

From [4], we obtain:

$$\beta_{24} = xy^2 + x^2z - 2yz^2 \qquad (24)$$

The points of $PG(2,7)$ on β_{24} in equation (24) are [1,0,0], [0,1,0], [0,0,1], [2,4,1], [3,6,1], [5,5,1], [3,4,1], [1,1,1], [6,3,1], [5,1,1], [4,2,1], [6,2,1]. To find the stabilizer group of β_{24} in equation (24), we are doing calculations by help the computer. Thus

$$G_{\beta_{24}} \cong \boldsymbol{Z_3} \times \boldsymbol{Z_3} = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix} \right\rangle.$$

β_{24} in equation (24) is drawn in Figure 34.

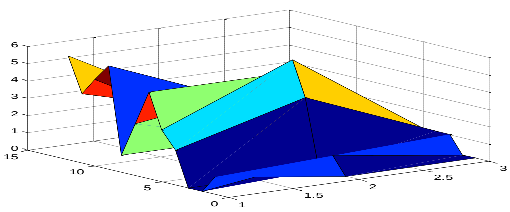

Figure 34: Drawing of β_{24}.

From [4], we obtain:

$$\beta_{25} = x^3 + 3y^3 + 2z^3 - 3xyz \qquad (25)$$

The points of $PG(2,7)$ on β_{25} in equation (25) are [4,4,1], [3,6,1], [5,5,1], [5,4,1], [1,2,1], [6,1,1], [6,3,1], [2,1,1], [3,2,1]. To find the stabilizer group of β_{25} in equation (25), we are doing calculations by help the computer. Thus

$$G_{\beta_{25}} \cong \boldsymbol{Z_3} \times \boldsymbol{Z_3} = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix} \right\rangle.$$

β_{25} in equation (25) is drawn in Figure 35.

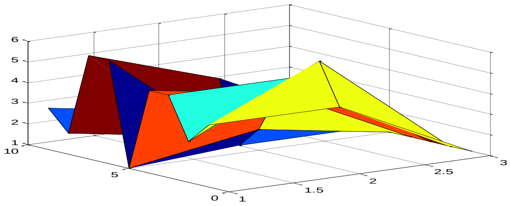

Figure 35: Drawing of β_{25}.

From [4], we obtain:

$$\beta_{26} = x^3 + 3y^3 + 2z^3 \qquad (26)$$

The points of $PG(2,7)$ on β_{26} in equation (26) are [1,3,1], [4,3,1], [4,6,1], [2,6,1], [2,3,1], [1,5,1], [1,6,1], [2,5,1], [4,5,1].

After the calculation and help the computer, we are obtained that the number of matrices which are stabilizing of β_{26} in equation (26) is 54, and we cannot write them, because they are too much. Moreover, the stabilizer group of β_{26} in equation (26) which is denoted by $G_{\beta_{26}}$ which contains

- 9 matrices of order 2;
- 26 matrix of order 3;
- 18 matrix of order 6;
- The identity matrix.

Drawing of β_{26} in equation (26) is given in Figure 36 as following:

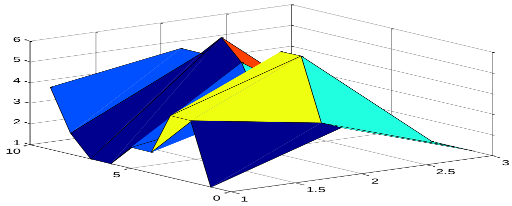

Figure 36: Drawing of β_{26}.

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