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Numerical Solution of Fractional Integro-Differential Equations Via Fourth-Degree Hat Functions

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ABSTRACT: The goal of this paper is to construct new fourth-degree hat functions (FDHFs) and study their properties in order to develop a new numerical method for solving fractional integro-differential equations. The equation under consideration is transformed into a set of algebraic equations by using FDHFs, which makes it simple to solve the system using one of the iterative methods. In fact, this method's advantage was that it was easy to use and had fifth-order convergence, as we showed in the section on error analysis. The numerical results demonstrate that the new technique works well through the presented examples.

Keywords: Fourth-degree hat functions; Fractional integro-differential equations; Caputo derivative; Error analysis

1. INTRODUCTION

In fractional calculus, the notion of derivatives and integrals is generalized to any real and even complex order. The concept of fractional computation arose in 1695 when G.W Leibniz suggested that there was a possibility of fractional differentiation of the order [1]. Many standard properties are broken by fractional differential and integral operators, including the standard product (Leibniz) rule, the standard chain rule, the semi-group property for orders of derivatives, and the semi-group property for dynamic maps [2–9]. The violation of the Leibniz rule's standard form is a characteristic property of non-integer order derivatives [2]. On the other hand, long-term memory and non-local dynamics are two of the most important applications of fractional derivatives and integrals of non-integer order.

Fractional calculus has a long and illustrious history that goes back more than 300 years. Nevertheless, for a long time, it was regarded as a pure mathematical field that lacked real-world applications. In the last few decades, the subject of fractional-order calculus has gotten a lot of attention because it allows you to represent a system more precisely without (or with minimal) approximation. Furthermore, this approach is a good tool for analyzing fractional dimension systems with long-term "memory" and chaotic behavior, and it is advantageous to model the behavior of a process in fractional-order because the response will include many values that would otherwise be ignored by integer-order due to approximations. As a result, fractional calculus has piqued the interest of scientists and engineers alike. For instance, fractional calculus models have been found to be a useful tool for describing the mechanics of viscoelastic materials and anomalous particle transport in groundwater. Signal processing, control of dynamic systems, wave propagation, medicine, economics, and finance are some of the other applications of fractional calculus models [12–20].

The modeling of many phenomena in physics and engineering relies heavily on nonlinear differential equations (DEs) and integro-differential equations (IDEs) [21–25]. Many researchers have given a lot of attention to fractional differential equations, which are a generalization of classical differential equations [26–30]. Several studies have been conducted in the last decade or so to develop numerical schemes to deal with fractional integro-differential equations (FIDEs), both

linear and nonlinear. FIDEs were dealt with using numerical approximation methods like variational iteration technique [31], homotopy perturbation method [32], Runge-Kutta convolution quadrature method [33], Adomian decomposition method [34, 35], finite element method [36], and finite difference method [37]. In dealing with various problems of dynamical systems, estimation methods based on polynomial series and orthogonal functions have attracted a lot of attention. The main feature of using orthogonal functions is that they reduce dynamical system problems to those of solving a system of algebraic equations using differentiation or integration operational matrices. Special attention has been given to applications of the block pulse functions [38, 39], spline wavelets [40], Legendre wavelets [41, 42], Chebyshev polynomials [43, 44], Walsh functions [45], and wavelet collocation method [46].

In addition, hat basis functions are a common and effective method for solving all kinds of differential and integral equations by using a piecewise polynomial approximation. Hat basis functions consist of a set of piecewise continuous functions with the shape of hats when plotted in two-dimensional planes. It is also well known that the generalized hat functions (GHFs) are constructed by using first-degree polynomials (segment lines), the modified hat functions (MHFs) are constructed by using second-degree polynomials, and the adjustment hat functions (AHFs) are constructed by using third-degree polynomials. These functions have many advantages, including the fact that they are easily defined and that the equation in question is converted into a system of algebraic equations that can be solved using these functions without the need for integration. The unknown coefficients of the function's approximation on this basis can also be easily calculated, and the proposed numerical method's computational cost is low. As a result, a number of researchers have used hat basis functions to solve various types of FIDEs; for example, see [47–51]. The basic idea behind this research paper is to extend hat basis functions to FDHFs for solving FIDEs. The numerical study of the following FIDE is the subject of this paper:

$$ {}^{c}D^{\alpha}y(t) = \omega(t) + \int_0^t K(t,s)Q(y(s))ds, \qquad t \in [0,T], \tag{1} $$

with the initial condition,

$$ y^{(i)}(0) = \gamma_i, \qquad i = 0,1,2,\ldots,r-1, \tag{2} $$

where $y(t)$ is the unknown function that must be approximated, $\omega(t)$ and $K(t,s)$ are well-known and continuous on $[0,T]$, the initial state of $y(t)$ is described by the values of γ_i $(i = 0,1,2,\ldots,r-1)$, and r is the highest integer order greater than the fractional derivative, ${}^{c}D^{\alpha}$ is the Caputo fractional derivative of order α, $0 < \alpha < 1$.

The following is how the current article is structured: Section 2 introduces the definition and properties of FDHFs, as well as some fundamental definitions of fractional calculus theory. Section 3 is dedicated to the integer-order and fractional-order operational matrices that are computed. The proposed numerical method for solving the problem under study is discussed in Section 4. Our proposed method's error estimate is proven in Section 5. We present some numerical examples and our numerical results to demonstrate the accuracy of the FDHFs in Section 6. Finally, a summary of this research paper is given in Section 7.

2. FRACTIONAL CALCULUS

Fractional calculus is a branch of mathematics that studies the properties of integrals and derivatives with non-integer orders of integration and differentiation (called fractional integrals and derivatives). The Riemann-Liouville and Caputo definitions are the most widely used for fractional integrals and derivatives. This article is based on the Caputo definition of fractional derivative because it is the only one that has the same form as integer-order differential equations in initial conditions.

Now, the fractional integral of Riemann-Liouville and the Caputo derivative are defined as follows:

Definition 2.1. *[52] Let $y(t)$ be a continuous function with $t > 0$. The Riemann-Liouville fractional integral operator of order α, $\alpha \geq 0$ of the function $y(t)$ is defined as follows:*

$$ J_t^{\alpha}y(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds, & \alpha > 0, \\ y(t), & \alpha = 0, \end{cases} \tag{3} $$

where $\Gamma(\cdot)$ is the fractional-order gamma function.

Definition 2.2. *[53] Let $y(t)$ be a continuous function with $t > 0$. The Caputo fractional derivative of order $\alpha > 0$ of the function $y(t)$ is defined as follows:*

$$ {}^{c}D^{\alpha}y(t) = \frac{1}{\Gamma(q-\alpha)} \int_0^t (t-s)^{q-\alpha-1} y^{(q)}(s)ds, \tag{4} $$

where $y^{(q)}(s) = \frac{d^q y(s)}{ds^q}$, $q \in \mathbb{N}$, *and* $q-1 < \alpha \leq q$.

The following formula establishes the relationship between the Caputo fractional derivative and the Riemann-Liouville fractional integral:

$$({}^cD^{\alpha} J_t^{\alpha} y)(t) = y(t), \tag{5}$$

$$J_t^{\alpha}\, {}^cD^{\alpha} y(t) = y(t) - \sum_{r=0}^{q-1} y^{(r)}(0)\frac{t^r}{r!}, \quad q-1 < \alpha \leq q. \tag{6}$$

3. FOURTH-DEGREE HAT FUNCTIONS AND THEIR PROPERTIES

In order to construct the FDHFs, assume that the interval $\Omega = [0,T]$ is divided into n equidistant subintervals, and then each of these subintervals must be divided again into four equidistant subintervals with a length equal to h, where $h = \frac{T}{4n}$ and $n \in \mathbb{N}$. The FDHFs form a set of $(4n+1)$ linearly independent functions in $L^2[0,T]$. These functions are defined as follows:

$$\xi_0(t) = \begin{cases} \frac{(t-h)(t-2h)(t-3h)(t-4h)}{24h^4}, & 0 \leq t \leq 4h, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

if $k = 1,2,...,n-1$,

$$\xi_{4k}(t) = \begin{cases} \frac{(t-(4k-1)h)(t-(4k-2)h)(t-(4k-3)h)(t-(4k-4)h)}{24h^4}, & 4(k-1)h \leq t \leq 4kh, \\ \frac{(t-(4k+1)h)(t-(4k+2)h)(t-(4k+3)h)(t-(4k+4)h)}{24h^4}, & 4kh \leq t \leq 4(k+1)h, \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

if $k = 1,2,\ldots n$,

$$\xi_{4k-1}(t) = \begin{cases} \frac{-(t-4kh)(t-(4k-2)h)(t-(4k-3)h)(t-(4k-4)h)}{6h^4}, & (4k-4)h \leq t \leq (4k)h, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

$$\xi_{4k-2}(t) = \begin{cases} \frac{(t-4kh)(t-(4k-1)h)(t-(4k-3)h)(t-(4k-4)h)}{4h^4}, & (4k-4)h \leq t \leq 4kh, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

$$\xi_{4k-3}(t) = \begin{cases} \frac{(t-(4k-2)h)(t-(4k-1)h)(t-4kh)(t-(4k-4)h)}{6h^4}, & (4k-4)h \leq t \leq 4kh, \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

and

$$\xi_{4n}(t) = \begin{cases} \frac{(t-(T-h))(t-(T-2h))(t-(T-3h))(t-(T-4h))}{24h^4}, & T-4h \leq t \leq T, \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

According to the definition (3), FDHFs have the following properties:

(i) According to the definition of FDHFs, there is a significant relation:

$$\xi_i(jh) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad i,j = 0,1,2,\ldots,4n. \tag{13}$$

(ii) The total sum of FDHFs is one, implying:

$$\sum_{i=0}^{4n} \xi_i(t) = 1. \tag{14}$$

(iii) The functions $\xi_0(t), \xi_1(t), \ldots, \xi_{4n}(t)$ are linearly independent for all $t \in [0, T]$.

(iv) Any function $y(t) \in L^2[0, T]$ can be approximated in terms of FDHFs as:

$$y(t) \approx y_{4n}(t) = \sum_{\kappa=0}^{4n} y_\kappa \xi_\kappa(t) = Y^T \Xi(t) = \Xi^T(t) Y, \tag{15}$$

where $\Xi(t) = [\xi_0(t), \xi_1(t), \xi_2(t), ..., \xi_{4n}(t)]^{\mathrm{T}}$, and $Y = [y_0, y_1, ..., y_{4n}]^{\mathrm{T}}$.

The use of FDHFs to approximate a function $y(t)$ is significant because the coefficients y_κ in Eq.(15) are given by:

$$y_\kappa = y(\kappa h), \qquad \kappa = 0, 1, \ldots, 4n. \tag{16}$$

(v) Any function $K(t, s) \in L^2([0, T] \times [0, T])$ can be approximated in terms of FDHFs as:

$$K(t, s) \approx K_{4n}(t, s) = \sum_{r=0}^{4n} \sum_{\kappa=0}^{4n} K_{\kappa r} \xi_\kappa(t) \xi_r(s) = \Xi^{\mathrm{T}}(t) D \Xi(s) = \Xi^{\mathrm{T}}(s) D^{\mathrm{T}} \Xi(t), \tag{17}$$

where,

$$K_{\kappa r}(t, s) = L(\kappa h, r h), \qquad \forall \kappa, r = 0, 1, 2, ..., 4n.$$

Lemma 3.1. *For any constant vector $Y^T = [y_0, y_1, ..., y_{4n}]$, we have*

$$\Xi(t) \Xi^T(t) Y \approx \tilde{Y} \Xi(t), \tag{18}$$

where $\tilde{Y}_{(4n+1)\times(4n+1)} = diag(y_0, y_1, ..., y_{4n})$. Also, if $B_{(4n+1)\times(4n+1)}$ be any constant matrix, we have

$$\Xi^T(t) B \Xi(t) \approx \tilde{B}^T \Xi(t) = \Xi^T(t) \tilde{B}, \tag{19}$$

where $\tilde{B}$ is a $(4n+1)$-vector with components equal to the diagonal entries of the matrix B.

Example 3.1. *To clarify the definition of FDHFs on the interval $[0, 1]$ and $n = 2$, one can see Fig.1, which shows the 9-set of FDHFs. Also, one can note that all the above properties of FDHFs are satisfied.*

4. OPERATIONAL MATRICES OF FDHFS

In this section, we derive an integer-order operational matrix, which is symbolized by P, as well as a fractional-order operational matrix of integration, which is symbolized by P_α, for FDHFs in Theorems 4.1 and 4.2, respectively.
To construct an operational matrix P that satisfies

$$\int_0^t \Xi(s) ds \approx P \Xi(t), \tag{20}$$

where $\Xi(t) = [\xi_0(t), \xi_1(t), \ldots, \xi_{4n}(t)]^T$. Now we are attempting to write $\int_0^t \xi_\kappa(s) ds$ as a linear combination of the functions $\xi_0(t), \xi_1(t), \ldots, \xi_{4n}(t)$ as follows:

$$\int_0^t \xi_\kappa(s) ds \approx \sum_{r=0}^{4n} P_{\kappa,r} \xi_r(t), \qquad \forall \kappa = 0, 1, 2, \ldots, 4n. \tag{21}$$

The coefficients $P_{\kappa,r}$ can be calculated as follows:

$$P_{\kappa,r} = \int_0^{rh} \xi_\kappa(s) ds, \qquad \forall\, r, \kappa = 0, 1, 2, \ldots, 4n. \tag{22}$$

As a direct consequence of this, we can state the following theorem:

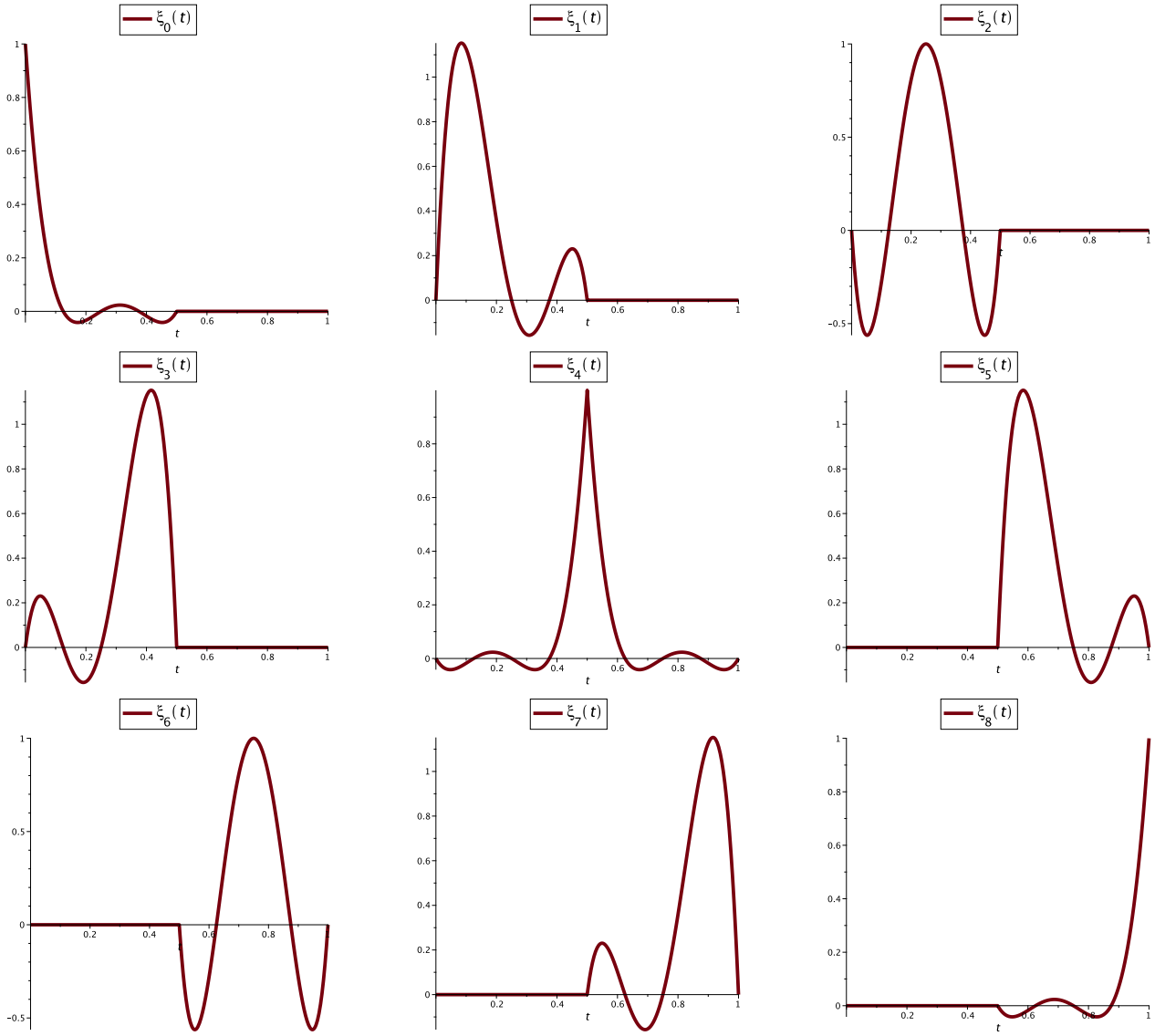

FIGURE 1. Plot of FDHFs with $T = 1$ and $n = 2$.

Theorem 4.1. *The integration of $\Xi(t)$ can be estimated as follows:*

$$\int_0^t \Xi(\tau)d\tau = P\Xi(t), \qquad t \in [0, T], \tag{23}$$

where $P_{(4n+1)\times(4n+1)}$ is the operational matrix given by:

$$P = \frac{h}{720}\begin{pmatrix} 0 & \varphi_1 & \varphi_2 & \varphi_2 & \varphi_2 & \varphi_2 & \cdots & \varphi_2 & \varphi_2 \\ \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6 & \varphi_6 & \varphi_6 & \cdots & \varphi_6 & \varphi_6 \\ \varphi_3 & \varphi_7 & \varphi_4 & \varphi_5 & \varphi_6 & \varphi_6 & \cdots & \varphi_6 & \varphi_6 \\ \varphi_3 & \varphi_7 & \varphi_7 & \varphi_4 & \varphi_5 & \varphi_6 & \cdots & \varphi_6 & \varphi_6 \\ \varphi_3 & \varphi_7 & \varphi_7 & \varphi_7 & \varphi_4 & \varphi_5 & \cdots & \varphi_6 & \varphi_6 \\ \varphi_3 & \varphi_7 & \varphi_7 & \varphi_7 & \varphi_7 & \varphi_4 & \cdots & \varphi_6 & \varphi_6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \varphi_3 & \varphi_7 & \varphi_7 & \varphi_7 & \varphi_7 & \varphi_7 & \cdots & \varphi_4 & \varphi_5 \\ \varphi_3 & \varphi_7 & \varphi_7 & \varphi_7 & \varphi_7 & \varphi_7 & \cdots & \varphi_7 & \varphi_4 \end{pmatrix}, \tag{24}$$

where

$$\varphi_1 = (251, 232, 243, 224) \quad , \quad \varphi_2 = (224, 224, 224, 224) \quad , \quad \varphi_3 = (0, 0, 0, 0)^{\mathrm{T}},$$

$$\varphi_4 = \begin{pmatrix} 646 & 992 & 918 & 1024 \\ -264 & 192 & 648 & 384 \\ 106 & 32 & 378 & 1024 \\ -19 & -8 & -27 & 224 \end{pmatrix} \quad , \quad \varphi_5 = \begin{pmatrix} 1024 & 1024 & 1024 & 1024 \\ 384 & 384 & 384 & 384 \\ 1024 & 1024 & 1024 & 1024 \\ 475 & 456 & 467 & 448 \end{pmatrix},$$

$$\varphi_6 = \begin{pmatrix} 1024 & 1024 & 1024 & 1024 \\ 384 & 384 & 384 & 384 \\ 1024 & 1024 & 1024 & 1024 \\ 448 & 448 & 448 & 448 \end{pmatrix} \quad , \quad \varphi_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 4.2. *Let $\Xi(t)$ be the FDHFs vector, and $\alpha > 0$. Then,*

$$J_t^{\alpha}\Xi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Xi(s)ds \approx P_{\alpha}\Xi(t), \quad t \in [0, T], \tag{25}$$

where P_α is a matrix of dimension $(4n+1)\times(4n+1)$ called the operational matrix of fractional integration of order α of the FDHFs which can be calculated in the following manner.

$$P_{\alpha} = \frac{1}{\alpha\Gamma(\alpha)} \begin{pmatrix} 0 & \varsigma_{0,1}(h) & \varsigma_{0,2}(h) & \varsigma_{0,3}(h) & \varsigma_{0,4}(h) & \varsigma_{0,4}(h) & \cdots & \varsigma_{0,4}(h) \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_4 & \gamma_4 & \cdots & \gamma_4 \\ \gamma_1 & \gamma_5 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_4 & \cdots & \gamma_4 \\ \gamma_1 & \gamma_5 & \gamma_5 & \gamma_2 & \gamma_3 & \gamma_4 & \cdots & \gamma_4 \\ \gamma_1 & \gamma_5 & \gamma_5 & \gamma_5 & \gamma_2 & \gamma_3 & \cdots & \gamma_4 \\ \gamma_1 & \gamma_5 & \gamma_5 & \gamma_5 & \gamma_5 & \gamma_2 & \cdots & \gamma_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_1 & \gamma_5 & \gamma_5 & \gamma_5 & \gamma_5 & \gamma_5 & \cdots & \gamma_2 \end{pmatrix}, \tag{26}$$

where

$$\gamma_2 = \begin{pmatrix} \beta_{4k-3,4k-3} & \beta_{4k-3,4k-2} & \beta_{4k-3,4k-1} & \beta_{4k-3,4k} \\ \phi_{4k-2,4k-3} & \phi_{4k-2,4k-2} & \phi_{4k-2,4k-1} & \phi_{4k-2,4k} \\ \rho_{4k-1,4k-3} & \rho_{4k-1,4k-2} & \rho_{4k-1,4k-1} & \rho_{4k-1,4k} \\ \eta_{4k,4k-3} & \eta_{4k,4k-2} & \eta_{4k,4k-1} & \eta_{4k,4k} \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} \beta_{4k-3,4k} & \beta_{4k-3,4k} & \beta_{4k-3,4k} & \beta_{4k-3,4k} \\ \phi_{4k-2,4k} & \phi_{4k-2,4k} & \phi_{4k-2,4k} & \phi_{4k-2,4k} \\ \rho_{4k-1,4k} & \rho_{4k-1,4k} & \rho_{4k-1,4k} & \rho_{4k-1,4k} \\ \eta_{4k,4k+1} & \eta_{4k,4k+2} & \eta_{4k,4k+3} & \eta_{4k,4k+4} \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} \beta_{4k-3,4k} & \beta_{4k-3,4k} & \beta_{4k-3,4k} & \beta_{4k-3,4k} \\ \phi_{4k-2,4k} & \phi_{4k-2,4k} & \phi_{4k-2,4k} & \phi_{4k-2,4k} \\ \rho_{4k-1,4k} & \rho_{4k-1,4k} & \rho_{4k-1,4k} & \rho_{4k-1,4k} \\ \eta_{4k,4k+4} & \eta_{4k,4k+4} & \eta_{4k,4k+4} & \eta_{4k,4k+4} \end{pmatrix},$$

$$\gamma_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}},$$

and

$$\varsigma_{0,j}(h) = (jh)^{\alpha} + \int_0^{jh} (jh-s)^{\alpha} \xi'_0(s)ds, \quad j = 1, 2, 3, 4,$$

if $k = 1,2,\ldots,n$,

$$\beta_{4k-3,j} = \int_{(4k-4)h}^{jh} (jh-s)^{\alpha} \xi'_{4k-3}(s)ds, \quad j = 4k-3,4k-2,4k-1,4k,$$

$$\phi_{4k-2,j} = \int_{(4k-4)h}^{jh} (jh-s)^{\alpha} \xi'_{4k-2}(s)ds, \quad j = 4k-3,4k-2,4k-1,4k,$$

$$\rho_{4k-1,j} = \int_{(4k-4)h}^{jh} (jh-s)^{\alpha} \xi'_{4k-1}(s)ds, \quad j = 4k-3,4k-2,4k-1,4k,$$

if $k = 1,2,\ldots,n-1$,

$$\eta_{4k,j} = \int_{(4k-4)h}^{jh} (jh-s)^{\alpha} \xi'_{4k}(s)ds, \qquad j = 4k-3,4k-2,4k-1,4k,$$

and

$$\eta_{4k,j} = \int_{(4k-4)h}^{4kh} (jh-s)^{\alpha} \xi'_{4k}(s)ds + \int_{4kh}^{jh} (jh-s)^{\alpha} \xi'_{4k}(s)ds, \qquad j = 4k+1,4k+2,4k+3,4k+4.$$

Proof. First, using the integration by parts formula to compute the $J_t^{\alpha}\xi_0(t)$, we get:

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi_0(s)ds = \frac{t^{\alpha}}{\alpha\Gamma(\alpha)} \xi_0(0) + \frac{1}{\alpha\Gamma(\alpha)} \int_0^t (t-s)^{\alpha} \xi'_0(s)ds = \frac{t^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{1}{\alpha\Gamma(\alpha)} \int_0^t (t-s)^{\alpha} \xi'_0(s)ds. \tag{27}$$

Furthermore, when the relation (27) is expanded in terms of FDHFs, we get:

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi_0(s)ds \approx \sum_{j=0}^{4n} \mu_{0j}^{(\alpha)} \xi_j(t),$$

where

$$\mu_{0j}^{(\alpha)} = \frac{1}{\Gamma(\alpha)} \int_0^{jh} (jh-s)^{\alpha-1} \xi_0(s)ds = \frac{(jh)^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{1}{\alpha\Gamma(\alpha)} \int_0^{jh} (jh-s)^{\alpha} \xi'_0(s)ds, \quad j = 0,1,\ldots,4n. \tag{28}$$

Thence, from the relation (28) and the definition of $\xi_0(t)$, we obtain:

$$\mu_{0j}^{(\alpha)} = \begin{cases} 0, & j = 0, \\ \frac{1}{\alpha\Gamma(\alpha)} \left(h^{\alpha} + \int_0^h (h-s)^{\alpha} \xi'_0(s)ds \right), & j = 1, \\ \frac{1}{\alpha\Gamma(\alpha)} \left(2^{\alpha}h^{\alpha} + \int_0^{2h} (2h-s)^{\alpha} \xi'_0(s)ds \right), & j = 2, \\ \frac{1}{\alpha\Gamma(\alpha)} \left(3^{\alpha}h^{\alpha} + \int_0^{3h} (3h-s)^{\alpha} \xi'_0(s)ds \right), & j = 3, \\ \frac{1}{\alpha\Gamma(\alpha)} \left(4^{\alpha}h^{\alpha} + \int_0^{4h} (4h-s)^{\alpha} \xi'_0(s)ds \right), & j \geq 4, \end{cases}$$

where

$$\xi'_0(s) = \begin{cases} \frac{1}{12h^4} \left(2s^3 - 15hs^2 + 35h^2s - 25h^3 \right), & 0 < s < 4h, \\ 0, & \text{otherwise}, \end{cases}$$

if $k = 1, 2, \ldots, n$. So

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi_{4k-1}(s) ds = \frac{1}{\Gamma(\alpha)} \int_{(4k-4)h}^t (t-s)^{\alpha-1} \xi_{4k-1}(s) ds$$
$$= \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^t (t-s)^{\alpha} \xi'_{4k-1}(s) ds. \tag{29}$$

When the relation (29) is expanded in terms of FDHFs, we get:

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi_{4k-1}(s) ds \approx \sum_{j=0}^{4n} \mu^{(\alpha)}_{(4k-1)j} \xi_j(t),$$

where

$$\mu^{(\alpha)}_{(4k-1)j} = \frac{1}{\Gamma(\alpha)} \int_{(4k-4)h}^{jh} (jh-s)^{\alpha-1} \xi_{4k-1}(s) ds$$
$$= \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{jh} (jh-s)^{\alpha} \xi'_{4k-1}(s) ds. \tag{30}$$

Thence, from the relation (30) and the definition of $\xi_{4k-1}(t)$, we obtain:

$$\mu^{(\alpha)}_{(4k-1)j} = \begin{cases} 0, & j \le 4k-4, \\ \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{(4k-3)h} ((4k-3)h-s)^{\alpha} \xi'_{4k-1}(s) ds, & j = 4k-3, \\ \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{(4k-2)h} ((4k-2)h-s)^{\alpha} \xi'_{4k-1}(s) ds, & j = 4k-2, \\ \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{(4k-1)h} ((4k-1)h-s)^{\alpha} \xi'_{4k-1}(s) ds, & j = 4k-1, \\ \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{4kh} (4kh-s)^{\alpha} \xi'_{4k-1}(s) ds, & j \ge 4k, \end{cases}$$

where

$$\xi'_{4k-1}(s) = \begin{cases} \frac{-\left(4s^3 - 3(16k-9)hs^2 + 2(96k^2-108k+26)h^2 s - (256k^3-432k^2+208k-24)h^3\right)}{6h^4}, & (4k-4)h < s < 4kh, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi_{4k-2}(s) ds = \frac{1}{\Gamma(\alpha)} \int_{(4k-4)h}^t (t-s)^{\alpha-1} \xi_{4k-2}(s) ds$$
$$= \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^t (t-s)^{\alpha} \xi'_{4k-2}(s) ds. \tag{31}$$

When the relation (31) is expanded in terms of FDHFs, we get:

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi_{4k-2}(s) ds \approx \sum_{j=0}^{4n} \mu^{(\alpha)}_{(4k-2)j} \xi_j(t),$$

where

$$\mu^{(\alpha)}_{(4k-2)j} = \frac{1}{\Gamma(\alpha)} \int_{(4k-4)h}^{jh} (jh-s)^{\alpha-1} \xi_{4k-2}(s) ds$$
$$= \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{jh} (jh-s)^{\alpha} \xi'_{4k-2}(s) ds. \tag{32}$$

Thence, from the relation (32) and the definition of $\xi_{4k-2}(t)$, we obtain:

$$\mu^{(\alpha)}_{(4k-2)j} = \begin{cases} 0, & j \le 4k-4, \\ \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{(4k-3)h} ((4k-3)h-s)^{\alpha} \xi'_{4k-2}(s)ds, & j = 4k-3, \\ \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{(4k-2)h} ((4k-2)h-s)^{\alpha} \xi'_{4k-2}(s)ds, & j = 4k-2, \\ \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{(4k-1)h} ((4k-1)h-s)^{\alpha} \xi'_{4k-2}(s)ds, & j = 4k-1, \\ \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{4kh} (4kh-s)^{\alpha} \xi'_{4k-2}(s)ds, & j \ge 4k, \end{cases}$$

where

$$\xi'_{4k-2}(s) = \begin{cases} \frac{\left(2s^3-12(2k-1)hs^2+(96k^2-96k+19)h^2s-(128k^3-192k^2+76k-6)h^3\right)}{2h^4}, & (4k-4)h < s < 4kh, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi_{4k-3}(s)ds &= \frac{1}{\Gamma(\alpha)} \int_{(4k-4)h}^t (t-s)^{\alpha-1} \xi_{4k-3}(s)ds \\ &= \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^t (t-s)^{\alpha} \xi'_{4k-3}(s)ds. \end{aligned} \tag{33}$$

When the relation (33) is expanded in terms of FDHFs, we get:

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi_{4k-3}(s)ds \approx \sum_{j=0}^{4n} \mu^{(\alpha)}_{(4k-3)j} \xi_j(t),$$

where

$$\begin{aligned} \mu^{(\alpha)}_{(4k-3)j} &= \frac{1}{\Gamma(\alpha)} \int_{(4k-4)h}^{jh} (jh-s)^{\alpha-1} \xi_{4k-3}(s)ds \\ &= \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{jh} (jh-s)^{\alpha} \xi'_{4k-3}(s)ds. \end{aligned} \tag{34}$$

Thence, from the relation (34) and the definition of $\xi_{4k-3}(t)$, we obtain:

$$\mu^{(\alpha)}_{(4k-3)j} = \begin{cases} 0, & j \le 4k-4, \\ \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{(4k-3)h} ((4k-3)h-s)^{\alpha} \xi'_{4k-3}(s)ds, & j = 4k-3, \\ \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{(4k-2)h} ((4k-2)h-s)^{\alpha} \xi'_{4k-3}(s)ds, & j = 4k-2, \\ \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{(4k-1)h} ((4k-1)h-s)^{\alpha} \xi'_{4k-3}(s)ds, & j = 4k-1, \\ \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^{4kh} (4kh-s)^{\alpha} \xi'_{4k-3}(s)ds, & j \ge 4k, \end{cases}$$

where

$$\xi'_{4k-3}(s) = \begin{cases} \frac{\left(4s^3-3(16k-7)hs^2+2(96k^2-84k+14)h^2s-(256k^3-336k^2+112k-8)h^3\right)}{6h^4}, & (4k-4)h < s < 4kh, \\ 0, & \text{otherwise}. \end{cases}$$

Now, if $k = 1, 2, ..., n-1$. So

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi_{4k}(s)ds &= \frac{1}{\Gamma(\alpha)} \int_{(4k-4)h}^t (t-s)^{\alpha-1} \xi_{4k}(s)ds \\ &= \frac{1}{\alpha\Gamma(\alpha)} \int_{(4k-4)h}^t (t-s)^{\alpha} \xi'_{4k}(s)ds. \end{aligned} \tag{35}$$

When the relation (35) is expanded in terms of FDHFs, we get:

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\xi_{4k}(s)ds \approx \sum_{j=0}^{4n}\mu_{4kj}^{(\alpha)}\xi_j(t),$$

where

$$\begin{aligned}\mu_{4kj}^{(\alpha)} &= \frac{1}{\Gamma(\alpha)}\int_{(4k-4)h}^{jh}(jh-s)^{\alpha-1}\xi_{4k}(s)ds\\ &= \frac{1}{\alpha\Gamma(\alpha)}\int_{(4k-4)h}^{jh}(jh-s)^{\alpha}\xi'_{4k}(s)ds.\end{aligned} \tag{36}$$

Thence, from the relation (36) and the definition of $\xi_{4k}(t)$, we obtain:

$$\mu_{4kj}^{(\alpha)} = \begin{cases} 0, & j \le 4k-4,\\ \frac{1}{\alpha\Gamma(\alpha)}\int_{(4k-4)h}^{(4k-3)h}((4k-3)h-s)^{\alpha}\xi'_{4k}(s)ds, & j = 4k-3,\\ \frac{1}{\alpha\Gamma(\alpha)}\int_{(4k-4)h}^{(4k-2)h}((4k-2)h-s)^{\alpha}\xi'_{4k}(s)ds, & j = 4k-2,\\ \frac{1}{\alpha\Gamma(\alpha)}\int_{(4k-4)h}^{(4k-1)h}((4k-1)h-s)^{\alpha}\xi'_{4k}(s)ds, & j = 4k-1,\\ \frac{1}{\alpha\Gamma(\alpha)}\int_{(4k-4)h}^{4kh}(4kh-s)^{\alpha}\xi'_{4k}(s)ds, & j \ge 4k,\\ \frac{1}{\alpha\Gamma(\alpha)}\left(\int_{(4k-4)h}^{4kh}((4k+1)h-s)^{\alpha}\xi'_{4k}(s)ds\right. & \\ \quad \left.+\int_{4kh}^{(4k+1)h}((4k+1)h-s)^{\alpha}\xi'_{4k}(s)ds\right), & j = 4k+1,\\ \frac{1}{\alpha\Gamma(\alpha)}\left(\int_{(4k-4)h}^{4kh}((4k+2)h-s)^{\alpha}\xi'_{4k}(s)ds\right. & \\ \quad \left.+\int_{4kh}^{(4k+2)h}((4k+2)h-s)^{\alpha}\xi'_{4k}(s)ds\right), & j = 4k+2,\\ \frac{1}{\alpha\Gamma(\alpha)}\left(\int_{(4k-4)h}^{4kh}((4k+3)h-s)^{\alpha}\xi'_{4k}(s)ds\right. & \\ \quad \left.+\int_{4kh}^{(4k+3)h}((4k+3)h-s)^{\alpha}\xi'_{4k}(s)ds\right), & j = 4k+3,\\ \frac{1}{\alpha\Gamma(\alpha)}\left(\int_{(4k-4)h}^{4kh}((4k+4)h-s)^{\alpha}\xi'_{4k}(s)ds\right. & \\ \quad \left.+\int_{4kh}^{(4k+4)h}((4k+4)h-s)^{\alpha}\xi'_{4k}(s)ds\right), & j \ge 4k+4,\end{cases}$$

where

$$\xi'_{4k}(s) = \begin{cases} \frac{2s^3-3(8k-5)hs^2+(96k^2-120k+35)h^2s-(128k^3-240k^2+140k-25)h^3}{12h^4}, & 4(k-1)h < s < 4kh,\\ \frac{2s^3-3(8k+5)hs^2+(96k^2+120k+35)h^2s-(128k^3+240k^2+140k+25)h^3}{12h^4}, & 4kh < s < 4(k+1)h,\\ 0, & \text{otherwise.}\end{cases}$$

Finally, for $k = 4n$. So

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\xi_{4n}(s)ds &= \frac{1}{\Gamma(\alpha)}\int_{T-4h}^{t}(t-s)^{\alpha-1}\xi_{4n}(s)ds\\ &= \frac{1}{\alpha\Gamma(\alpha)}\int_{T-4h}^{t}(t-s)^{\alpha}\xi'_{4n}(s)ds.\end{aligned} \tag{37}$$

When the relation (37) is expanded in terms of FDHFs, we get:

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\xi_{4n}(s)ds \approx \sum_{j=0}^{4n}\mu_{4nj}^{(\alpha)}\xi_j(t),$$

where

$$\begin{aligned}\mu_{4nj}^{(\alpha)} &= \frac{1}{\Gamma(\alpha)}\int_{T-4h}^{jh}(jh-s)^{\alpha-1}\xi_{4n}(s)ds \\ &= \frac{1}{\alpha\Gamma(\alpha)}\int_{T-4h}^{jh}(jh-s)^{\alpha}\xi'_{4n}(s)ds. \end{aligned} \tag{38}$$

Thence, from the relation (38) and the definition of $\xi_{4n}(t)$, we obtain:

$$\mu_{4nj}^{(\alpha)} = \begin{cases} 0, & j \le 4n-4, \\ \frac{1}{\alpha\Gamma(\alpha)}\int_{T-4h}^{T-3h}((T-3h)-s)^{\alpha}\xi'_{4n}(s)ds, & j = 4n-3, \\ \frac{1}{\alpha\Gamma(\alpha)}\int_{T-4h}^{T-2h}((T-2h)-s)^{\alpha}\xi'_{4n}(s)ds, & j = 4n-2, \\ \frac{1}{\alpha\Gamma(\alpha)}\int_{T-4h}^{T-h}((T-h)-s)^{\alpha}\xi'_{4n}(s)ds, & j = 4n-1, \\ \frac{1}{\alpha\Gamma(\alpha)}\int_{T-4h}^{T}(T-s)^{\alpha}\xi'_{4n}(s)ds, & j = 4n, \end{cases}$$

where

$$\xi'_{4n}(s) = \begin{cases} \frac{(2s^3-3(2T-5h))s^2+(6T^2-30Th+35h^2)s-(2T^3-15T^2h+35Th^2-25h^3)}{12h^4}, & T-4h < s < T, \\ 0, & \text{otherwise.} \end{cases}$$

The proof is finished.

5. NUMERICAL METHOD

The aim of this section is to find a numerical solution to FIDEs based on FDHFs. A numerical method based on the operational matrices based on FDHFs and other concepts defined in the previous section is proposed to solve problems (1)-(2). Firstly, by taking the Riemann-Liouville fractional integration of the equation (1), we obtain:

$$y(t) = x_0(t) + J_t^{\alpha}(\omega(t)) + J_t^{\alpha}\left(\int_0^t K(t,s)Q(y(s))ds\right), \tag{39}$$

where

$$x_0(t) = \sum_{\tau=0}^{q-1} y^{(\tau)}(0)\frac{t^{\tau}}{\tau!}.$$

To solve this equation, we must first approximate the functions $y(t), x_0(t), \omega(t)$ and $K(t,s)$ by the FDHFs as follows:

$$y(t) \approx Y^{\mathrm{T}}\Xi(t) = \Xi^{\mathrm{T}}(t)Y, \tag{40}$$

$$x_0(t) \approx X^{\mathrm{T}}\Xi(t) = \Xi^{\mathrm{T}}(t)X, \tag{41}$$

$$\omega(t) \approx A^{\mathrm{T}}\Xi(t) = \Xi^{\mathrm{T}}(t)A, \tag{42}$$

$$K(t,s) \approx \Xi^{\mathrm{T}}(t)D\Xi(s) = \Xi^{\mathrm{T}}(s)D^{\mathrm{T}}\Xi(t). \tag{43}$$

Also, taking into account the approximations

$$h(t) = Q(y(t)), \tag{44}$$

where

$$h(t) \approx H^{\mathrm{T}}\Xi(t) = \Xi^{\mathrm{T}}(t)H, \tag{45}$$

where Y, X, A, and H are the coefficient vectors of the FDHFs $y(t)$, $x_0(t)$, $\omega(t)$, and $h(t)$, respectively, and D is the coefficient matrix of $K(t,s)$.
Now, by substituting the aforementioned approximations into Eq.(39), we get

$$Y^{\mathrm{T}}\Xi(t) = X^{\mathrm{T}}\Xi(t) + J_t^{\alpha}(A^{\mathrm{T}}\Xi(t)) + J_t^{\alpha}(\int_0^t \Xi^{\mathrm{T}}(t)D\Xi(s)\Xi^{\mathrm{T}}(s)Hds). \tag{46}$$

So we have

$$Y^{\mathrm{T}}\Xi(t) = X^{\mathrm{T}}\Xi(t) + A^{\mathrm{T}}J_t^{\alpha}(\Xi(t)) + J_t^{\alpha}(\Xi^{\mathrm{T}}(t)D\int_0^t \Xi(s)\Xi^{\mathrm{T}}(s)Hds). \tag{47}$$

In addition, from Lemma (3.1), we obtain:

$$Y^{\mathrm{T}}\Xi(t) = X^{\mathrm{T}}\Xi(t) + A^{\mathrm{T}}J_t^{\alpha}(\Xi(t)) + J_t^{\alpha}(\Xi^{\mathrm{T}}(t)D\int_0^t \tilde{H}\Xi(s)ds). \tag{48}$$

Now, using Eq.(23) in Eq.(48), we have

$$Y^{\mathrm{T}}\Xi(t) = X^{\mathrm{T}}\Xi(t) + A^{T}J_t^{\alpha}(\Xi(t)) + J_t^{\alpha}(\Xi^{T}(t)D\tilde{H}P\Xi(t)). \tag{49}$$

Take into account the following presumptions:

$$\Theta = D\tilde{H}P.$$

Therefore

$$Y^{\mathrm{T}}\Xi(t) = X^{\mathrm{T}}\Xi(t) + A^{T}J_t^{\alpha}(\Xi(t)) + J_t^{\alpha}(\Xi^{T}(t)\Theta\Xi(t)), \tag{50}$$

where Θ is an $(4n+1)\times(4n+1)$ matrix. Using Eq.(19), we get:

$$Y^{\mathrm{T}}\Xi(t) = X^{\mathrm{T}}\Xi(t) + A^{T}J_t^{\alpha}(\Xi(t)) + J_t^{\alpha}(\tilde{\Theta}^{\mathrm{T}}\Xi(t)), \tag{51}$$

where $\tilde{\Theta} = diag(\Theta)$ is a $(4n+1)$-vectors whose elements are equal to the diagonal entries of the matrix Θ.
Now, from Eq.(25), we have:

$$Y^{\mathrm{T}}\Xi(t) = X^{\mathrm{T}}\Xi(t) + A^{T}P_{\alpha}\Xi(t) + \tilde{\Theta}^{\mathrm{T}}P_{\alpha}\Xi(t). \tag{52}$$

From Eqs.(44), (45) and (52) we can deduce the following:

$$H^{T}\Xi(t) \approx h(t) = Q(y(t)) \approx (X^{\mathrm{T}}\Xi(t) + A^{T}P_{\alpha}\Xi(t) + \tilde{\Theta}^{\mathrm{T}}P_{\alpha}\Xi(t)).$$

Therefore

$$H^{T}\Xi(t) = (X^{\mathrm{T}}\Xi(t) + A^{T}P_{\alpha}\Xi(t) + \tilde{\Theta}^{\mathrm{T}}P_{\alpha}\Xi(t)). \tag{53}$$

The above equation is a nonlinear system of $(4n+1)$ algebraic equations with $(4n+1)$ unknown coefficients, from which we find the unknown vector H by Newton's iterative method. Afterward, we can get an approximate solution to the problems (1)-(2) as follows:

$$y(t) \approx y_{4n}(t) = Y^{\mathrm{T}}\Xi(t) = X^{\mathrm{T}}\Xi(t) + A^{T}P_{\alpha}\Xi(t) + \tilde{\Theta}^{\mathrm{T}}P_{\alpha}\Xi(t).$$

6. ERROR ANALYSIS

The goal of this section is to determine the convergence rate of the suggested method for solving FIDEs. Indeed, we establish that the proposed method converges at an $O(h^5)$ rate. To do so, we define the norm.

$$\|y\| = \sup_{t\in\Omega} |y(t)|. \tag{54}$$

Theorem 6.1. *Suppose $\omega(t) \in C^5(\Omega)$ and $\omega_{4n}(t) = \sum_{\kappa=0}^{4n} \omega(t_\kappa)\xi_\kappa(t)$, $t_\kappa = \kappa h$ be the FDHFs expansion of $\omega(t)$. Also, suppose $E(t) = \omega(t) - \omega_{4n}(t)$, $t \in \Omega$. Then we have*

$$\|\omega(t)-\omega_{4n}(t)\| \le \psi_1 h^5, \tag{55}$$

where ψ_1 is a constant; consequently, the order of convergence is five. That is:

$$\|E(t)\| \approx O(h^5).$$

Proof. Assume

$$E_i(t) = \begin{cases} \omega(t)-\omega_{4n}(t), & t \in V_i, \\ 0, & t \in \Omega - V_i, \end{cases}$$

where $V_i = \{t | ih \le t \le (i+4)h,\ h = \frac{T}{4n}\}$, $i = 0,4,8,...,4n-4$. Then, we get

$$E_i(t) = \omega(t)-\omega_{4n}(t) = \omega(t) - \sum_{\kappa=0}^{4n} \omega(\kappa h)\xi_\kappa(t),$$

$$E_i(t) = \omega(t) - [\omega(ih)\xi_i(t) + \omega((i+1)h)\xi_{i+1}(t) + \omega((i+2)h)\xi_{i+2}(t) + \omega((i+3)h)\xi_{i+3}(t) + \omega((i+4)h)\xi_{i+4}(t)].$$

By using a fourth-degree interpolation error, we have [54].

$$E_i(t) = \frac{(t-ih)(t-(i+1)h)(t-(i+2)h)(t-(i+3)h)(t-(i+4)h)}{120} \cdot \frac{d^5\omega(\chi_i)}{dt^5},$$

where $\chi_i \in (ih,(i+4)h)$.
Let us suppose $\sigma(t) = (t-ih)(t-(i+1)h)(t-(i+2)h)(t-(i+3)h)(t-(i+4)h)$. Since $\sigma(t)$ is a continuous function and V_i is compacted, we have:

$$\sup_{t\in V_i} |\sigma(t)| = \max_{t\in V_i} |\sigma(t)| = 3.6314h^5.$$

Consequently, we have

$$|E_i(t)| \le \frac{1}{120}|\sigma(t)| \left|\frac{d^5\omega(\chi_i)}{dt^5}\right|.$$

Therefore, we now have

$$\|E(t)\| = \max_{i=0,4,...,4n-4} \sup_{t\in V_i} |E_i(t)| \le \max_{i=0,4,...,4n-4} 0.03026h^5 \left|\frac{d^5\omega(\chi_i)}{dt^5}\right|.$$

After that, there's $\nu \in \{0,4,8,\ldots,4n-4\}$, we get:

$$\|E(t)\| \le \max_{i=0,4,...,4n-4} 0.03026h^5 \left|\frac{d^5\omega(\chi_i)}{dt^5}\right| = 0.03026h^5 \left|\frac{d^5\omega(\chi_\nu)}{dt^5}\right|.$$

Finally, using the relation (54), we get:

$$\|E(t)\| \le 0.03026h^5 \left|\frac{d^5\omega(\chi_\nu)}{dt^5}\right| \le 0.03026h^5 \left\|\frac{d^5\omega(t)}{dt^5}\right\| \le \psi_1 h^5. \tag{56}$$

Based on the relation (56), we obtain:

$$\|E(t)\| \approx O(h^5).$$

The proof was eventually finished.

Theorem 6.2. *Assume* $K(t,s) \in C^5(\Omega \times \Omega)$ *and* $K_{4n}(t,s) = \sum_{i=0}^{4n} \sum_{j=0}^{4n} K(ih, jh)\xi_i(t)\xi_j(s)$ *be the FDHFs expansion of* $K(t,s)$. *Also, assume* $E_{4n}(t,s) = K(t,s) - K_{4n}(t,s)$ *be the truncation error,* $t \in G = (\Omega \times \Omega)$. *Then we have*

$$\|K(t,s) - K_{4n}(t,s)\| \le \psi_2 h^5, \tag{57}$$

where ψ_2 is a constant; consequently, the order of convergence is five. That is:

$$\|E(t,s)\| \approx O(h^5).$$

Proof. Assume

$$E_{\ell\upsilon}(t,s) = \begin{cases} K(t,s) - K_{4n}(t,s), & (t,s) \in V_{\ell\upsilon}, \\ 0, & (t,s) \in G - V_{\ell\upsilon}, \end{cases}$$

where $V_{\ell\upsilon} = \{(t,s) | \ell h \le t \le (\ell+4)h, \upsilon h \le s \le (\upsilon+4)h,\ h = \frac{T}{4n}\}$, $\ell, \upsilon = 0,4,8,...,4n-4$. Then, we get

$$E_{\ell\upsilon}(t,s) = K(t,s) - K_{4n}(t,s) = K(t,s) - \sum_{i=0}^{4n} \sum_{j=0}^{4n} K(ih, jh)\xi_i(t)\xi_j(s),$$

$$\begin{aligned} E_{\ell\upsilon}(t,s) = K(t,s) &- (K(\ell h, \upsilon h)\xi_\ell(t)\xi_\upsilon(s) + K(\ell h, (\upsilon+1)h)\xi_\ell(t)\xi_{\upsilon+1}(s) + \cdots \\ &+ K(\ell h, (\upsilon+4)h)\xi_\ell(t)\xi_{\upsilon+4}(s) + \cdots + K((\ell+4)h, (\upsilon+4)h)\xi_{\ell+4}(t)\xi_{\upsilon+4}(s)). \end{aligned}$$

By using a fourth-degree interpolation error, we have [54].

$$\begin{aligned} E_{\ell\upsilon}(t,s) &= \frac{(t-\ell h)(t-(\ell+1)h)(t-(\ell+2)h)(t-(\ell+3)h)(t-(\ell+4)h)}{120} \cdot \frac{\partial^5 K(\chi_\ell, s)}{\partial t^5} \\ &+ \frac{(s-\upsilon h)(s-(\upsilon+1)h)(s-(\upsilon+2)h)(s-(\upsilon+3)h)(s-(\upsilon+4)h)}{120} \cdot \frac{\partial^5 K(t, \eta_\upsilon)}{\partial s^5} \\ &- \frac{(t-\ell h)(t-(\ell+1)h)\ldots(t-(\ell+4)h)(s-\upsilon h)\ldots(s-(\upsilon+4)h)}{14400} \cdot \frac{\partial^{10} K(\bar{\chi}_q, \bar{\eta}_\upsilon)}{\partial t^5 \partial s^5}, \end{aligned}$$

where $\chi_\ell, \bar{\chi}_\ell \in (\ell h, (\ell+4)h)$ and $\eta_\upsilon, \bar{\eta}_\upsilon \in (\upsilon h, (\upsilon+4)h)$.
We consider $u(t) = (t-\ell h)(t-(\ell+1)h)(t-(\ell+2)h)(t-(\ell+3)h)(t-(\ell+4)h)$ and $\gamma(s) = (s-\upsilon h)(s-(\upsilon+1)h)(s-(\upsilon+2)h)(s-(\upsilon+3)h)(s-(\upsilon+4)h)$.
Consequently, we have

$$\begin{aligned} |E_{\ell\upsilon}(t,s)| \le \frac{1}{120}|u(t)|\left|\frac{\partial^5 K(\chi_\ell, s)}{\partial t^5}\right| + \frac{1}{120}|\gamma(s)|\left|\frac{\partial^5 K(t, \eta_\upsilon)}{\partial s^5}\right| \\ + \frac{1}{14400}|u(t)||\gamma(s)|\left|\frac{\partial^{10} K(\bar{\chi}_\ell, \bar{\eta}_\upsilon)}{\partial t^5 \partial s^5}\right|. \end{aligned}$$

Since $\sup_{t \in (\ell h, (\ell+4)h)} |u(t)| = 3.6314h^5$, and $\sup_{s \in (\upsilon h, (\upsilon+4)h)} |\gamma(s)| = 3.6314h^5$, we obtain

$$\begin{aligned} \|E(t,s)\| &= \max_{\substack{\ell=0,4,...,4n-1 \\ \upsilon=0,4,...,4n-1}} \sup_{(t,s) \in V_{\ell\upsilon}} |E_{\ell\upsilon}(t,s)| \\ &\le 0.03026h^5 \max_{\substack{\ell=0,4,...,4n-1 \\ \upsilon=0,4,...,4n-1}} \sup_{(t,s) \in V_{\ell\upsilon}} \left(\left|\frac{\partial^5 K(\chi_\ell, s)}{\partial t^5}\right| + \left|\frac{\partial^5 K(t, \eta_\upsilon)}{\partial s^5}\right| \right. \\ &\qquad \left. + 0.03026h^5 \left|\frac{\partial^{10} K(\bar{\chi}_\ell, \bar{\eta}_\upsilon)}{\partial t^5 \partial s^5}\right| \right). \end{aligned}$$

After that, there are $\beta, \iota \in \{0,4,...,4n-4\}$, we get:

$$\|E(t,s)\| \leq 0.03026h^5 \sup_{(t,s)\in V_{\ell\upsilon}} \left(\left| \frac{\partial^5 K(\chi_\beta, s)}{\partial t^5} \right| + \left| \frac{\partial^5 K(t, \eta_\iota)}{\partial s^5} \right| + 0.03026h^5 \left| \frac{\partial^{10} K(\bar{\chi}_\beta, \bar{\eta}_\iota)}{\partial t^5 \partial s^5} \right| \right).$$

Finally, using the relation (54), we get:

$$\begin{aligned} \|E(t,s)\| &\leq 0.03026h^5 \left(\left\| \frac{\partial^5 K(t,s)}{\partial t^5} \right\| + \left\| \frac{\partial^5 K(t,s)}{\partial s^5} \right\| + 0.03026h^5 \left\| \frac{\partial^{10} K(t,s)}{\partial t^5 \partial s^5} \right\| \right) \\ &\leq \psi_2 h^5. \end{aligned} \tag{58}$$

Based on the relation (58), we obtain:

$$\|E(t,s)\| \approx O(h^5).$$

The proof was eventually finished.

Theorem 6.3. *Assume that $y(t)$ and $y_{4n}(t)$ are the exact and approximate solutions of (1), respectively. Furthermore, assume that the following assumptions are met:*
(i) $\|Q(y(t))\| \leq \Upsilon, \qquad t \in \Omega,$
(ii) $\|K(t,s)\| \leq N, \qquad t \in \Omega \times \Omega,$
(iii) $1 - \frac{T^{\alpha+1} N \varpi}{\Gamma(\alpha)} - \frac{T^{\alpha+1} \psi_2 \varpi h^5}{\Gamma(\alpha)} > 0.$
(iv) The nonlinear term $Q(y(t))$ satisfies the Lipschitz condition:

$$\|Q(y(t)) - Q(y_{4n}(t))\| \leq \varpi \|y(t) - y_{4n}(t)\|, \quad t \in \Omega.$$

Furthermore, we can deduce from Theorem 6.1 that

$$\left\| \sum_{\tau=0}^{q-1} y^{(\tau)}(0) \frac{t^\tau}{\tau!} - \sum_{\tau=0}^{q-1} y_{4n}^{(\tau)}(0) \frac{t^\tau}{\tau!} \right\| \leq \psi h^5. \tag{59}$$

Following that, we have

$$\|y(t) - y_{4n}(t)\| \leq \frac{(\psi + \frac{T^\alpha}{\Gamma(\alpha)} \psi_1 + \frac{T^{\alpha+1}}{\Gamma(\alpha)} \psi_2 \Upsilon) h^5}{1 - \frac{T^{\alpha+1} N \varpi}{\Gamma(\alpha)} - \frac{T^{\alpha+1} \psi_2 \varpi h^5}{\Gamma(\alpha)}},$$

and $\|y(t) - y_{4n}(t)\| \approx O(h^5)$.

Proof. The system (39) can be rewritten as follows:

$$\begin{aligned} y(t) = \sum_{\tau=0}^{q-1} y^{(\tau)}(0) \frac{t^\tau}{\tau!} &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} \omega(x) dx \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} \left(\int_0^x K(x,s) Q(y(s)) ds \right) dx. \end{aligned} \tag{60}$$

Now, we can approximate Eq.(60) using the FDHFs as follows:

$$\begin{aligned} y_{4n}(t) = \sum_{\tau=0}^{q-1} y_{4n}^{(\tau)}(0) \frac{t^\tau}{\tau!} &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} \omega_{4n}(x) dx \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} \left(\int_0^x K_{4n}(x,s) Q(y_{4n}(s)) ds \right) dx. \end{aligned} \tag{61}$$

Now we can deduce the following equation from Eqs.(60) and (61):

$$\begin{aligned} y(t) - y_{4n}(t) = \sum_{\tau=0}^{q-1} y^{(\tau)}(0) \frac{t^\tau}{\tau!} - \sum_{\tau=0}^{q-1} y_{4n}^{(\tau)}(0) \frac{t^\tau}{\tau!} &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} (\omega(x) - \omega_{4n}(x)) dx \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} \left(\int_0^x (K(x,s) Q(y(s)) - K_{4n}(x,s) Q(y_{4n}(s))) ds \right) dx. \end{aligned}$$

Hence

$$\begin{aligned}\|y(t)-y_{4n}(t)\| &= \left\|\sum_{\tau=0}^{q-1} y^{(\tau)}(0)\frac{t^\tau}{\tau!} - \sum_{\tau=0}^{q-1} y_{4n}^{(\tau)}(0)\frac{t^\tau}{\tau!}\right\| \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t \left\|(t-x)^{\alpha-1}\right\| \|\omega(x)-\omega_{4n}(x)\|\,dx \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t \left\|(t-x)^{\alpha-1}\right\| \left(\int_0^x \|K(x,s)Q(y(s)) - K_{4n}(x,s)Q(y_{4n}(s))\|ds\right)dx.\end{aligned}$$

Since $\max\{|t-x|^{\alpha-1}, 0\le t\le T, 0\le x\le T\} = T^{\alpha-1}$, where $T\ge 1$ and $0<\alpha<1$, so

$$\begin{aligned}\|y(t)-y_{4n}(t)\| \le \left\|\sum_{\tau=0}^{q-1} y^{(\tau)}(0)\frac{t^\tau}{\tau!} - \sum_{\tau=0}^{q-1} y_{4n}^{(\tau)}(0)\frac{t^\tau}{\tau!}\right\| + \frac{T^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \|\omega(x)-\omega_{4n}(x)\|dx \\ + \frac{T^{\alpha-1}}{\Gamma(\alpha)}\int_0^t\int_0^x \|K(x,s)Q(y(s)) - K_{4n}(x,s)Q(y_{4n}(s))\|dsdx.\end{aligned}$$

Because $t<T$, $x<T$, then

$$\begin{aligned}\|y(t)-y_{4n}(t)\| \le \left\|\sum_{\tau=0}^{q-1} y^{(\tau)}(0)\frac{t^\tau}{\tau!} - \sum_{\tau=0}^{q-1} y_{4n}^{(\tau)}(0)\frac{t^\tau}{\tau!}\right\| + \frac{T^{\alpha}}{\Gamma(\alpha)}\|\omega(t)-\omega_{4n}(t)\| \\ + \frac{T^{\alpha+1}}{\Gamma(\alpha)}\|K(x,s)Q(y(s)) - K_{4n}(x,s)Q(y_{4n}(s))\|.\end{aligned} \tag{62}$$

Based on Theorem 6.2 and assumptions (i), (ii), and (iv), we arrive at the following conclusion:

$$\begin{aligned}&\|K(x,s)Q(y(s)) - K_{4n}(x,s)Q(y_{4n}(s))\| \\ &\le \|K(t,s)\|\,\|Q(y(t)) - Q(y_{4n}(t))\| + \|K(t,s)-K_{4n}(t,s)\|\,\|Q(y(t)) - Q(y_{4n}(t))\| \\ &\qquad + \|K(t,s)-K_{4n}(t,s)\|\,\|Q(y(t))\| \\ &\le N\varpi\|y(t)-y_{4n}(t)\| + \psi_2\varpi h^5\|y(t)-y_{4n}(t)\| + \psi_2\Upsilon h^5.\end{aligned} \tag{63}$$

Using Eqs.(63), (62), and (59), as well as Theorem 6.1 and assumption (iii), we have

$$\|y(t)-y_{4n}(t)\| \le \frac{(\psi + \frac{T^\alpha}{\Gamma(\alpha)}\psi_1 + \frac{T^{\alpha+1}}{\Gamma(\alpha)}\psi_2\Upsilon)h^5}{1-\frac{T^{\alpha+1}N\varpi}{\Gamma(\alpha)} - \frac{T^{\alpha+1}\psi_2\varpi h^5}{\Gamma(\alpha)}}. \tag{64}$$

Based on the relation (64), we obtain:

$$\|y(t)-y_{4n}(t)\| \approx O(h^5).$$

The proof was eventually finished.

7. NUMERICAL EXAMPLES

This section tests the proposed method on several examples to ensure its applicability, efficiency, and accuracy.

The goal of this study is to see if the numerical technique presented here can be used to solve nonlinear fractional integro-differential equations with initial conditions such as Eqs.(1)-(2). The figures show the exact and approximate solutions for different values of n and α at different intervals. Indeed, five examples have been solved by using four base functions, including GHFs, MHFs, AHFs, and FDHFs. We use the same length of subintervals (same $h=\frac{T}{mn}$, where m is the degree of polynomials used in the definition of the basis function in each method) to ensure a fair comparison between these base functions; that is, we use the same number of basis functions in each method. Also, all the calculations were done using Maple 2020 on a laptop Windows and 11^{th} Gen Intel(R) Core(TM) i9-11900H 2.5HzG with RAM 40Gb.

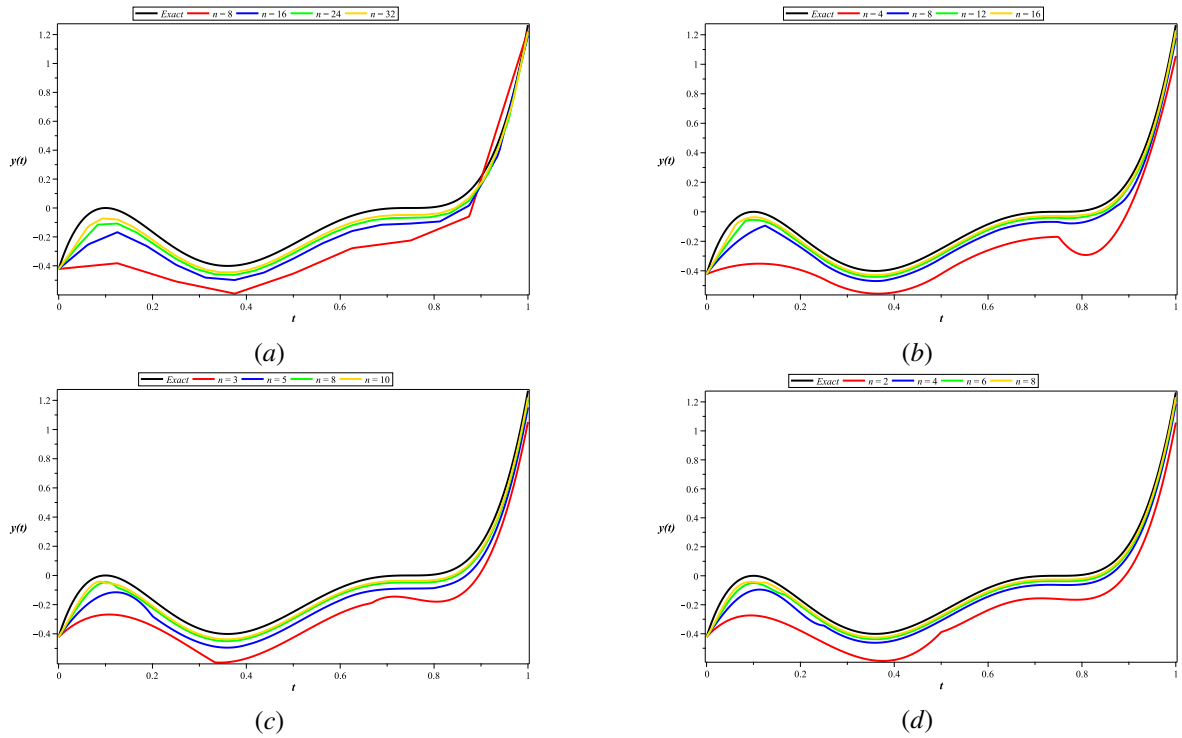

FIGURE 2. The comparison between the numerical solutions of (*a*) GHFs, (*b*) MHFs, (*c*) AHFs, and (*d*) the proposed method for different values of n and the exact solution in Example 7.1.

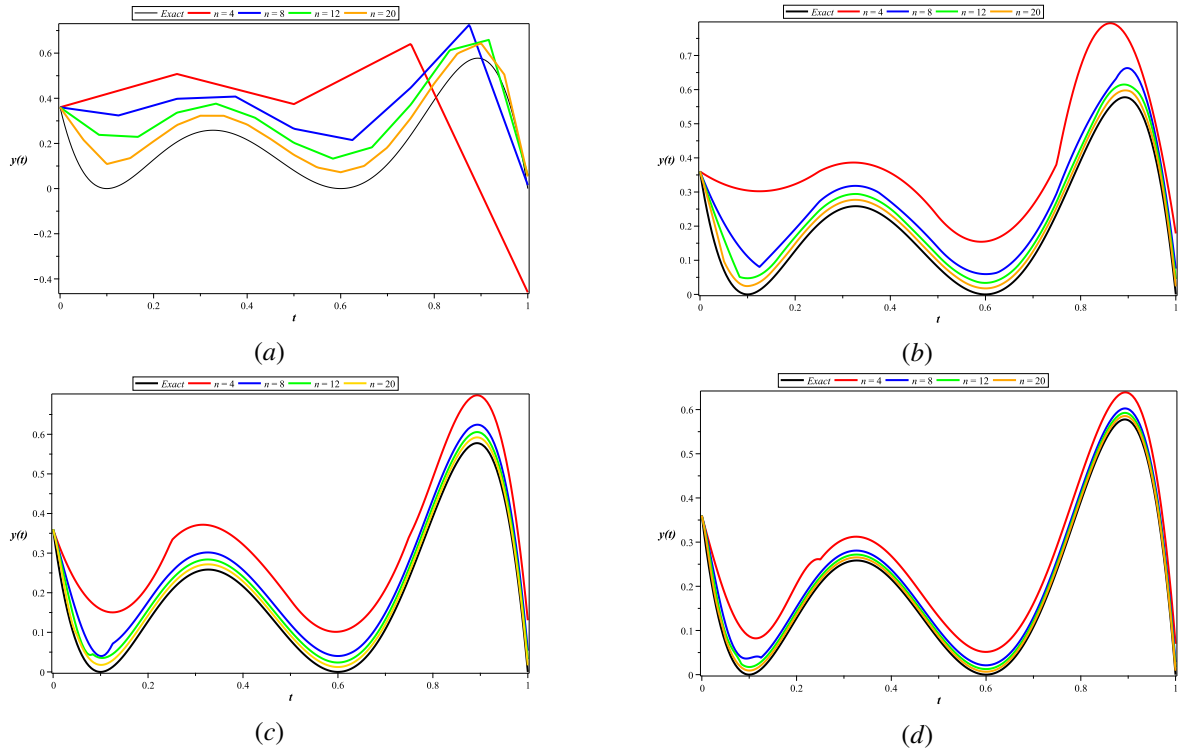

FIGURE 3. The comparison between the numerical solutions of (*a*) GHFs, (*b*) MHFs, (*c*) AHFs, and (*d*) the proposed method for different values of n and the exact solution in Example 7.2.

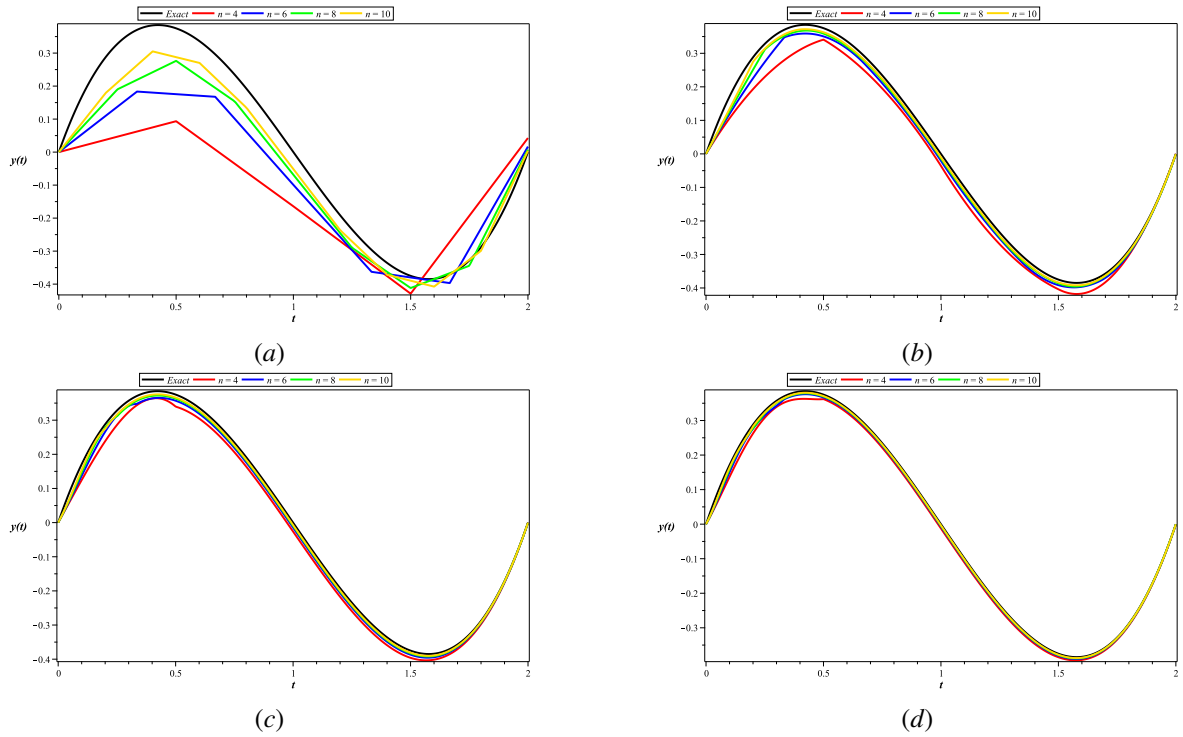

FIGURE 4. The comparison between the numerical solutions of (*a*) GHFs, (*b*) MHFs, (*c*) AHFs, and (*d*) the proposed method for different values of n and the exact solution in Example 7.3.

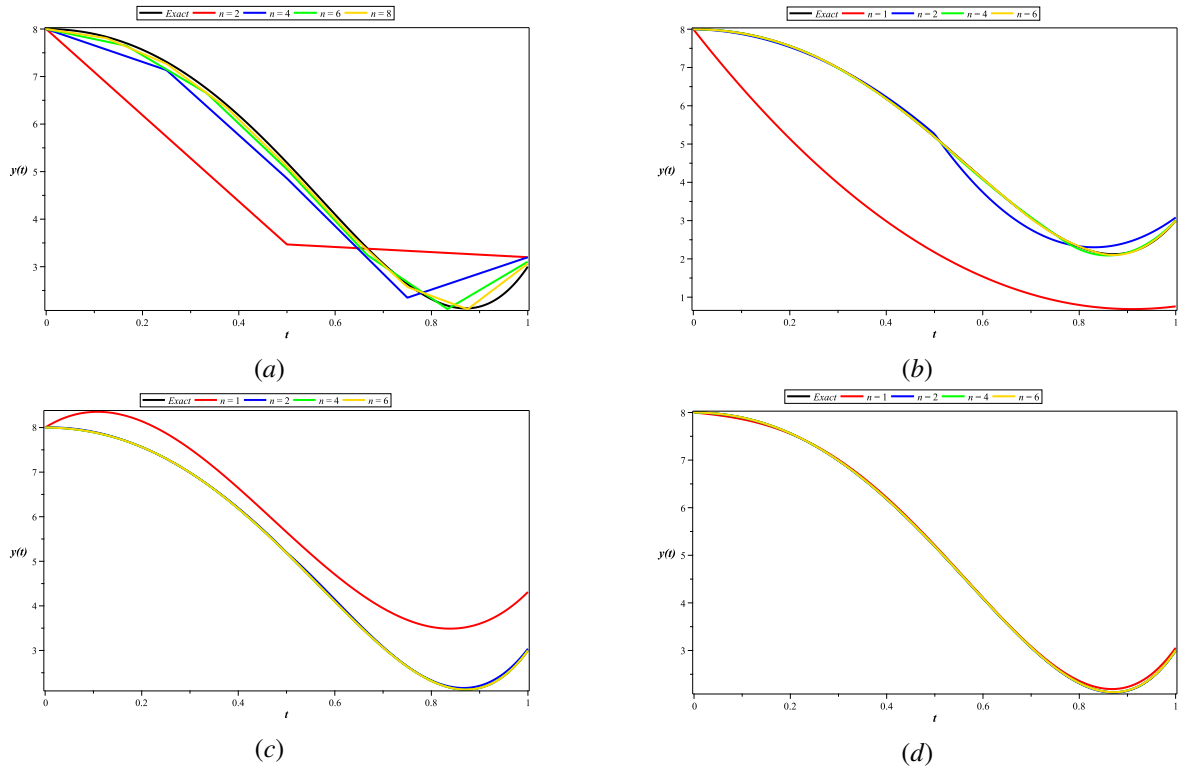

FIGURE 5. The comparison between the numerical solutions of (*a*) GHFs, (*b*) MHFs, (*c*) AHFs, and (*d*) the proposed method for different values of n and the exact solution in Example 7.4.

Example 7.1. *Consider the following FIDE:*

$$ {}^{c}D^{\alpha}y(t) = \omega(t) + \int_0^t sy(s)e^t ds, \qquad t \in [0,1], 0 < \alpha < 1, \tag{65} $$

where

$$ \omega(t) = \frac{\sqrt{2}\Gamma(\frac{3}{4})}{\pi}(\frac{409600}{663}t^{\frac{17}{4}} - \frac{50176}{39}t^{\frac{13}{4}} + \frac{13744}{15}t^{\frac{9}{4}} - \frac{1251}{5}t^{\frac{5}{4}} + \frac{81}{4}t^{\frac{1}{4}}) + e^t(\frac{-100}{7}t^7 + \frac{245}{6}t^6 - \frac{859}{20}t^5 + \frac{1251}{64}t^4 - \frac{27}{8}t^3 + \frac{27}{128}t^2). $$

The problem has an exact solution $y(t) = \frac{1}{64}(10t-1)^2(4t-3)^3$ with the initial condition $y(0) = \frac{-27}{64}$. Figure 2 displays the exact solution in Example 7.1 compared with the numerical solutions of GHFs, MHFs, AHFs and the proposed method for different values of n and with $\alpha = 0.75$. As shown in this figure, the FDHFs technique provides an accurate estimate solution that is in good agreement with the exact solution for all values of t in the interval $[0,1]$. Additionally, we can see that there is an apparent convergence as the value of n is increased.

Example 7.2. *Consider the following FIDE:*

$$ {}^{c}D^{\alpha}y(t) = \omega(t) + \int_0^t sy(s)e^t ds, \qquad t \in [0,1], 0 < \alpha < 1, \tag{66} $$

where

$$ \omega(t) = \frac{-2\sqrt{2}\Gamma(\frac{3}{4})}{16575\pi}t^{\frac{1}{4}}(5120000t^4 - 10444800t^3 + 7107360t^2 - 1840488t + 145197) + \frac{100}{7}e^t t^7 - 40e^t t^6 + \frac{201}{5}e^t t^5 - \frac{347}{20}e^t t^4 + \frac{73}{25}e^t t^3 - \frac{9}{50}e^t t^2. $$

The problem has an exact solution $y(t) = \frac{-1}{25}(10t-1)^2(5t-3)^2(t-1)$ with the initial condition $y(0) = \frac{9}{25}$. Figure 3 displays the exact solution in Example 7.2 compared with the numerical solutions of GHFs, MHFs, AHFs and the proposed method for different values of n and with $\alpha = 0.75$. As shown in this figure, the FDHFs technique provides an accurate estimate solution that is in good agreement with the exact solution for all values of t in the interval $[0,1]$. Additionally, we can see that there is an apparent convergence as the value of n is increased.

Example 7.3. *Consider the following FIDE:*

$$ {}^{c}D^{\alpha}y(t) = \omega(t) + \int_0^t ts^2y^2(s)ds, \qquad t \in [0,2], 0 < \alpha < 1, \tag{67} $$

where

$$ \omega(t) = \frac{11318}{5333}t^{\frac{47}{20}} - \frac{57359}{11501}t^{\frac{27}{20}} + \frac{46064}{20525}t^{\frac{7}{20}} - \frac{1}{9}t^{10} + \frac{3}{4}t^9 - \frac{13}{7}t^8 + 2t^7 - \frac{4}{5}t^6. $$

The problem has an exact solution $y(t) = t^3 - 3t^2 + 2t$ with the initial condition $y(0) = 0$. Figure 4 displays the exact solution in Example 7.3 compared with the numerical solutions of GHFs, MHFs, AHFs and the proposed method for different values of n and with $\alpha = 0.65$. As shown in this figure, the FDHFs technique provides an accurate estimate solution that is in good agreement with the exact solution for all values of t in the interval $[0,2]$. Additionally, we can see that there is an apparent convergence as the value of n is increased.

Example 7.4. *Consider the following FIDE:*

$$ {}^{c}D^{\alpha}y(t) = \omega(t) + \int_0^t sy^2(s)\cos^2(t)ds, \qquad t \in [0,1], 0 < \alpha < 1, \tag{68} $$

where

$$ \omega(t) = (\frac{-25}{3}t^{12} + 10t^{10} + \frac{200}{9}t^9 - \frac{25}{8}t^8 - \frac{260}{7}t^7 - \frac{50}{3}t^6 + 16t^5 + 40t^4 - 32t^2)cos^2(t) + \frac{42889}{1594}t^{\frac{22}{5}} - \frac{19794}{1967}t^{\frac{12}{5}} - \frac{152765}{9488}t^{\frac{7}{5}}. $$

The problem has an exact solution $y(t) = 10t^5 - 5t^3 - 10t^2 + 8$ with the initial condition $y(0) = 8$. Figure 5 displays the exact solution in Example 7.4 compared with the numerical solutions of GHFs, MHFs, AHFs and the proposed method for different values of n and with $\alpha = 0.6$. As shown in this figure, the FDHFs technique provides an accurate estimate solution that is in good agreement with the exact solution for all values of t in the interval $[0, 1]$. Additionally, we can see that there is an apparent convergence as the value of n is increased.

8. CONCLUSION

This study proposed a different approach depending on the FDHFs that have been built and implemented to provide a computational solution for solving FIDEs. Numerical estimations were performed on four test examples using the proposed method with different values of α and different intervals, and the obtained results indicate that the FDHFs work well and achieve the required accuracy. This method converts the given equation into a system of algebraic equations that can be solved using Newton's iterative method. The FDHFs method is also demonstrated to be convergent, with a convergence order of $O(h^5)$.

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CONFLICTS OF INTEREST

The authors declare no conflict of interest.

REFERENCES

[1] I. Petras, "Fractional-order memristor-based Chua's circuit," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 57, no. 12, pp. 975–979, 2010.
[2] V. E. Tarasov, "No violation of the leibniz rule. no fractional derivative," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 11, pp. 2945–2948, 2013.
[3] A. R. Khudair, "On solving non-homogeneous fractional differential equations of euler type," *Computational and Applied Mathematics*, vol. 32, pp. 577–584, jun 2013.
[4] M. D. Ortigueira and J. T. Machado, "What is a fractional derivative?," *Journal of computational Physics*, vol. 293, pp. 4–13, 2015.
[5] V. E. Tarasov, "On chain rule for fractional derivatives," *Communications in Nonlinear Science and Numerical Simulation*, vol. 30, no. 1-3, pp. 1–4, 2016.
[6] S. L. Khalaf and A. R. Khudair, "Particular solution of linear sequential fractional differential equation with constant coefficients by inverse fractional differential operators," *Differential Equations and Dynamical Systems*, vol. 25, pp. 373–383, may 2017.
[7] A. R. Khudair, S. Haddad, and S. L. khalaf, "Restricted fractional differential transform for solving irrational order fractional differential equations," *Chaos, Solitons & Fractals*, vol. 101, pp. 81–85, aug 2017.
[8] K. Sayevand, J. T. Machado, and D. Baleanu, "A new glance on the leibniz rule for fractional derivatives," *Communications in Nonlinear Science and Numerical Simulation*, vol. 62, pp. 244–249, 2018.
[9] A. F. A. Jalil and A. R. Khudair, "Toward solving fractional differential equations via solving ordinary differential equations," *Computational and Applied Mathematics*, vol. 41, jan 2022.
[10] R. E. Gutierrez, J. M. Rosario, and J. Tenreiro Machado, "Fractional order calculus: basic concepts and engineering applications," *Mathematical problems in engineering*, vol. 2010, 2010.
[11] I. Podlubny, "The laplace transform method for linear differential equations of the fractional order," *arXiv preprint funct-an/9710005*, 1997.
[12] J. G. T. Ribeiro, J. T. P. de Castro, and M. A. Meggiolaro, "Using fractional calculus to model viscoelastic behavior in concrete and polymers,"
[13] H. Mesgarani, J. Rashidinia, Y. E. Aghdam, and O. Nikan, "Numerical treatment of the space fractional advection–dispersion model arising in groundwater hydrology," *Computational and Applied Mathematics*, vol. 40, no. 1, pp. 1–17, 2021.
[14] P. Shah and R. Sekhar, "Predictive modeling and control of clamp load loss in bolted joints based on fractional calculus," in *Advances in Computing and Network Communications*, pp. 15–32, Springer, 2021.
[15] X. Pan, J. Zhu, H. Yu, L. Chen, Y. Liu, and L. Li, "Robust corner detection with fractional calculus for magnetic resonance imaging," *Biomedical Signal Processing and Control*, vol. 63, p. 102112, 2021.
[16] Z. E. A. Fellah, M. Fellah, E. Ogam, A. Berbiche, and C. Depollier, "Reflection and transmission of transient ultrasonic wave in fractal porous material: Application of fractional calculus," *Wave Motion*, vol. 106, p. 102804, 2021.
[17] M. Hassouna, A. Ouhadan, *et al.*, "Fractional calculus: applications in rheology," in *Fractional Order Systems*, pp. 513–549, Elsevier, 2022.
[18] A. Vo and A. Ekpenyong, "Fractional calculus modeling of cell viscoelasticity quantifies drug response and maturation more robustly than integer order models," *arXiv preprint arXiv:2201.02589*, 2022.
[19] H. Jahanshahi, S. S. Sajjadi, S. Bekiros, and A. A. Aly, "On the development of variable-order fractional hyperchaotic economic system with a nonlinear model predictive controller," *Chaos, Solitons & Fractals*, vol. 144, p. 110698, 2021.

[20] N. Ö. Ö. TUĞRUL, E. ERGÜN, D. C. KÖSEOĞLU, K. Karacuha, K. ŞİMŞEK, and E. Karacuha, "Modeling of telecommunication revenue as a percentage of gross domestic product's for countries with fractional calculus," *The Journal of Cognitive Systems*, vol. 6, no. 1, pp. 28–34, 2021.
[21] A. R. Alharbi, M. Almatrafi, and K. Lotfy, "Constructions of solitary travelling wave solutions for ito integro-differential equation arising in plasma physics," *Results in Physics*, vol. 19, p. 103533, 2020.
[22] R. Indiaminov, R. Butaev, N. Isayev, K. Ismayilov, B. Yuldoshev, and A. Numonov, "Nonlinear integro-differential equations of bending of physically nonlinear viscoelastic plates," in *IOP Conference Series: Materials Science and Engineering*, vol. 869, p. 052048, IOP Publishing, 2020.
[23] S. Ray, *Nonlinear differential equations in physics*. Springer, 2020.
[24] Z. A. Lazima and S. L. Khalaf, "Optimal control design of the in-vivo HIV fractional model," *Iraqi Journal of Science*, pp. 3877–3888, sep 2022.
[25] S. L. Khalaf, M. S. Kadhim, and A. R. Khudair, "Studying of COVID-19 fractional model: Stability analysis," *Partial Differential Equations in Applied Mathematics*, vol. 7, p. 100470, jun 2023.
[26] N. Adjimi, A. Boutiara, M. S. Abdo, and M. Benbachir, "Existence results for nonlinear neutral generalized caputo fractional differential equations," *Journal of Pseudo-Differential Operators and Applications*, vol. 12, no. 2, pp. 1–17, 2021.
[27] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations, Volume 204 (North-Holland Mathematics Studies)*. USA: Elsevier Science Inc., 2006.
[28] M. S. Abdo, T. Abdeljawad, K. D. Kucche, M. A. Alqudah, S. M. Ali, and M. B. Jeelani, "On nonlinear pantograph fractional differential equations with Atangana–Baleanu–Caputo derivative," *Advances in Difference Equations*, vol. 2021, no. 1, pp. 1–17, 2021.
[29] K. D. Kucche and S. T. Sutar, "Analysis of nonlinear fractional differential equations involving Atangana-Baleanu-Caputo derivative," *Chaos, Solitons & Fractals*, vol. 143, p. 110556, 2021.
[30] M. Manigandan, M. Subramanian, P. Duraisamy, and T. N. Gopal, "On Caputo-Hadamard type fractional differential equations with nonlocal discrete boundary conditions," *Discontinuity, Nonlinearity, and Complexity*, vol. 10, no. 02, pp. 185–194, 2021.
[31] Y. Nawaz, "Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations," *Computers & Mathematics with Applications*, vol. 61, no. 8, pp. 2330–2341, 2011.
[32] P. Das, S. Rana, and H. Ramos, "Homotopy perturbation method for solving caputo-type fractional-order Volterra-Fredholm integro-differential equations," *Computational and Mathematical Methods*, vol. 1, no. 5, p. e1047, 2019.
[33] G. Zhang and R. Zhu, "Runge–kutta convolution quadrature methods with convergence and stability analysis for nonlinear singular fractional integro-differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 84, p. 105132, 2020.
[34] S. Momani and M. A. Noor, "Numerical methods for fourth-order fractional integro-differential equations," *Applied Mathematics and Computation*, vol. 182, no. 1, pp. 754–760, 2006.
[35] A. Panda, S. Santra, and J. Mohapatra, "Adomian decomposition and homotopy perturbation method for the solution of time fractional partial integro-differential equations," *Journal of Applied Mathematics and Computing*, pp. 1–18, 2021.
[36] L. Kumar, S. G. Sista, and K. Sreenadh, "Finite element analysis of time fractional integro-differential equations of kirchhoff type for non-homogeneous materials," *arXiv preprint arXiv:2111.06708*, 2021.
[37] S. Santra and J. Mohapatra, "A novel finite difference technique with error estimate for time fractional partial integro-differential equation of Volterra type," *Journal of Computational and Applied Mathematics*, vol. 400, p. 113746, 2022.
[38] F. Mohammadi, "Fractional integro-differential equation with a weakly singular kernel by using block pulse functions," *UPB Sci. Bull., Ser. A*, vol. 79, no. 1, pp. 57–66, 2017.
[39] S. Najafalizadeh and R. Ezzati, "A block pulse operational matrix method for solving two-dimensional nonlinear integro-differential equations of fractional order," *Journal of Computational and Applied Mathematics*, vol. 326, pp. 159–170, 2017.
[40] H. Mesgarani, H. Safdarı, A. Ghasemian, and Y. Esmaeelzade, "The cubic b-spline operational matrix based on haar scaling functions for solving varieties of the fractional integro-differential equations," *Journal of Mathematics*, vol. 51, no. 8, pp. 45–65, 2019.
[41] M. R. M. Shabestari, R. Ezzati, and T. Allahviranloo, "Numerical solution of fuzzy fractional integro-differential equation via two-dimensional legendre wavelet method," *Journal of Intelligent & Fuzzy Systems*, vol. 34, no. 4, pp. 2453–2465, 2018.
[42] L. Wu, Z. Chen, and X. Ding, "A minimal search method for solving fractional integro-differential equations based on modified legendre multiwavelets," *Journal of Applied Mathematics and Computing*, pp. 1–17, 2021.
[43] K. K. Ali, M. A. Abd El Salam, E. M. Mohamed, B. Samet, S. Kumar, and M. Osman, "Numerical solution for generalized nonlinear fractional integro-differential equations with linear functional arguments using chebyshev series," *Advances in Difference Equations*, vol. 2020, no. 1, pp. 1–23, 2020.
[44] K. Sadri, K. Hosseini, D. Baleanu, A. Ahmadian, and S. Salahshour, "Bivariate chebyshev polynomials of the fifth kind for variable-order time-fractional partial integro-differential equations with weakly singular kernel," *Advances in Difference Equations*, vol. 2021, no. 1, pp. 1–26, 2021.
[45] A. Khajehnasiri, R. Ezzati, and A. Jafari Shaerlar, "Walsh functions and their applications to solving nonlinear fractional Volterra integro-differential equation," *International Journal of Nonlinear Analysis and Applications*, vol. 12, no. 2, pp. 1577–1589, 2021.
[46] M. Faheem and A. Khan, "A wavelet collocation method based on gegenbauer scaling function for solving fourth-order time-fractional integro-differential equations with a weakly singular kernel," *Applied Numerical Mathematics*, vol. 184, pp. 197–218, feb 2023.
[47] S. Nemati and P. M. Lima, "Numerical solution of nonlinear fractional integro-differential equations with weakly singular kernels via a modification of hat functions," *Applied Mathematics and Computation*, vol. 327, pp. 79–92, 2018.
[48] K. Amirahmad, M. A. Kermani, and R. Ezzati, "Fractional order operational matrix method for solving two-dimensional nonlinear fractional Volterra integro-differential equations," *Kragujevac Journal of Mathematics*, vol. 45, no. 4, pp. 571–585, 2021.
[49] E. Aryani, A. Babaei, and A. Valinejad, "An accurate approach based on modified hat functions for solving a system of fractional stochastic integro-differential equations," *JOURNAL OF MATHEMATICAL EXTENSION*, vol. 15, 2021.
[50] F. Mirzaee and N. Samadyar, "Application of hat basis functions for solving two-dimensional stochastic fractional integral equations," *Computational and Applied Mathematics*, vol. 37, no. 4, pp. 4899–4916, 2018.
[51] S. Nemati and D. F. Torres, "A new spectral method based on two classes of hat functions for solving systems of fractional differential equations and an application to respiratory syncytial virus infection," *Soft Computing*, vol. 25, no. 9, pp. 6745–6757, 2021.
[52] Y. Wang and L. Zhu, "Solving nonlinear Volterra integro-differential equations of fractional order by using euler wavelet method," *Advances in difference equations*, vol. 2017, no. 1, pp. 1–16, 2017.
[53] A. E. Aboanber, A. A. Nahla, and A. M. Edress, "Developed mathematical technique for fractional stochastic point kinetics model in nuclear reactor dynamics," *Nuclear Science and Techniques*, vol. 29, jul 2018.
[54] S. Nemati, P. Lima, and Y. Ordokhani, "Numerical solution of a class of two-dimensional nonlinear Volterra integral equations using Legendre polynomials," *Journal of Computational and Applied Mathematics*, vol. 242, pp. 53–69, apr 2013.