Topologically Transitive Property of Markov Chain

Payman M. Mahmood
Dep. of Statistics, College of Administration and Economics, Kirkuk University, Kirkuk, Iraq
Email: paymanmajeed@yahoo.com

(**Received:** 23 / 11 / 2010 ---- **Accepted:** 16 / 3 / 2011)

Abstract

In this paper, we give topologically transitive property to a dynamical system in ergodic theory for there more we study their effects on Markov chain. We show that the Markov chain is topologically transitive if and only if (if) its directed graph is irreducible or its transition matrix is irreducible (primitive).

Key words: Dynamical system, Markov chain, irreducibility, primitive, topologically transitive, directed graph, transition matrix.

1. Introduction

Let X be a topological space. The smallest σ-algebra containing all of open subsets of X is called the Borel σ-algebra of X. For any compact set $B \in \beta$, μ is its finite measure on β (i.e. $\mu(X)=1$), so that the triplet (X,β,μ) is a probability space. A transformation $T: X \to X$ is said to be:

1. measurable if $T^{-1}(\beta) \subseteq \beta$ (i.e. T is surjective).
2. measure-preserving if T is measurable and $\mu(B) = \mu(T^{-1}(B))\ \forall B \in \beta$.

In [10], we say that μ is T-invariant or that T preserves μ.

3. ergodic (with respect to μ) if $T^{-1}(B) = B$ for some $B \in \beta$, then $\mu(B)=0$ or $\mu(B)=1$.

If T is a transformation from X to itself, it is measure-preserving then $((X,\beta,\mu),T)$ is said to be a dynamical system in ergodic theory [9].

The direct product space $\prod_{i=-\infty}^{\infty}(X,\beta,\mu)_i$ is said to be Markov chain if $P(x_{i+1} \mid x_0 x_1 \ldots x_i) = P(x_{i+1} \mid x_i) = P_{x_i x_{i+1}}$,$\forall i \in Z$ and $x_0, x_1, \ldots x_{i+1} \in X$, Markov chain is denoted by X_P and defined as follows:

$$X_P = \{x = (P_{x_i x_{i+1}})_{i \in Z} : x_i x_{i+1} \in X\ , \forall i \in Z\}$$

The elements of X are called the states of the Markov chain, therefore X is called state space that have positive probability, it is assemble into a row vector called the probability vector and is denoted by α. If $X = \{1,2,\ldots,m\}$ then $\alpha = (p_1, p_2, \ldots, p_m)$ such that $p_i > 0\ \&\ \sum p_i = 1$, $i = 1,2,\ldots m$.

Recall that $P_{x_i x_{i+1}}$ is actually a conditional probability and that is called the transition probability for going from the state x_i to the state x_{i+1}. It is convenient to collect the transition probabilities $P_{x_i x_{i+1}}$ into a square matrix, called the transition matrix and is denoted by $M = (P_{ij})$, $1 \le i, j \le m$ such that $P_{ij} \ge 0\ \&\ \sum_{j=1}^{m} P_{ij} = 1$, $\forall 1 \le i \le m$. A probability vector α on X is invariant under transition probabilities M if $\alpha M = \alpha$ (i.e. $\sum_{i=1}^{m} p_i P_{ij} = p_j$) [6], [3].

The Markov probability measure of a cylinder set may then be defined by

$$\mu_P([x_0, x_1, \ldots, x_k]) = p_{x_0} P_{x_0 x_1} \ldots P_{x_{k-1} x_k}$$
$$= p_{x_0} \prod_{i=0}^{k-1} P_{x_i x_{i+1}} \quad [3].$$

This is enough to define the measure on the entire Borel σ-algebra.

Let $M = (P_{ij})$ be a $m \times m$ transition matrix and $G = G_M$ be its associated directed graph with vertex of G are the states of X that have positive probability $V = \{i \in X : p_i > 0\}$ and the edges of G are the transitions from one state to another that have positive conditional probability $\varepsilon = \{(i,j) : P_{ij} > o\}$ [7].

The Markov chain determined by G is defined by $X_P = \{x = (P_{x_i x_{i+1}})_{i \in Z} : P_{x_i x_{i+1}} > 0\ \forall i \in Z, x_i \in V\}$.

In this paper, we assume from now on that a transition matrix $M = (P_{ij})$ and a directed graph G are essential.

In the paper, we define that an ergodic measure-preserving transformation T with respect to μ for a dynamical system in ergodic theory to be topologically transitive and show its equivalent conditions. We prove that the Markov shift σ_P is topologically transitive if and only if (if) transition matrix M or directed graph G or Markov chain X_P is irreducible (M is primitive).

2. Preliminaries

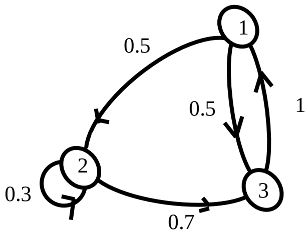
First we present the fundamental definition

Definition 2.1: [7] A directed graph G is irreducible if for every ordered pair of vertices i and j there is a path in G starting at i and terminating at j.

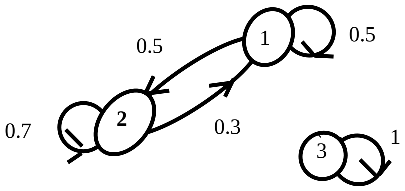
Irreducible directed graph are sometimes called strongly connected in graph theory. Note that for a directed graph to be irreducible you need to check that, for any two vertices i and j, there exist a path from i to j and a path from j to i.

Definition 2.2: [7] [4] Let M be a $m \times m$ transition matrix. We call the transition matrix irreducible if $\forall 1 \le i, j \le m, \exists n > 0$ such that $P^n{}_{ij} > 0$ where $(P^n{}_{ij})$ is the matrix M^n. Otherwise M is reducible.

Examples 2.3: When $m = 3$ the transition matrix $M = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0 & 0.3 & 0.7 \\ 1 & 0 & 0 \end{pmatrix}$ is irreducible and its directed graph is :



However, the transition matrix $M' = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is reducible because $P^n{}_{i3} = 0$ & $P^n{}_{3j} = 0$, $\forall n > 0$, $i, j = 1,2$ and its directed graph is :



Definition 2.4: [2] The transition matrix M is a primitive if there exists $n > 0$ such that M^n has all entries strictly positive (i.e. $M^n > 0$).

Remark 2.5: Every primitive transition matrix is irreducible matrix.

The opposite is not always true as we show in the following example:

Example 2.6: The transition matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is irreducible but is not primitive because $M^n = M$ or I .

Theorem 2.7: A directed graph G is irreducible if and only if its transition matrix M is irreducible.

Proof: It follows from the definition

Corollary 2.8: If transition matrix M is primitive then its directed graph G is irreducible.

Proof: The proof is from remark 2.5 and theorem 2.7.

Definition 2.9: [5] A Markov chain is said to be irreducible if, for any pair of states i and j there exists $t > 0$ such that $P^t{}_{ij} > o$.

The following theorem is introduced to show the relation between irreducible Markov chain and irreducible graph.

Theorem 2.10: A Markov chain X_P is irreducible if and only if its directed graph G is irreducible.

Proof: It follows from the definition.

Corollary 2.11: A Markov chain X_P is irreducible if and only if its transition matrix M is irreducible.

Proof: This is clear.

Corollary 2.12: If M is a primitive transition $m \times m$ matrix, then the Markov chain X_P is irreducible.

Proof: The proof is from corollary 2.8 and theorem 2.10.

3. Transitivity

First we present definition and equivalent conditions for topologically transitive on dynamical system in ergodic. Let (X_P, μ_P, σ_P) as defined in the below be a subsystem of $\prod_{i=-\infty}^{\infty}((X, \beta, \mu), T)_i$, where X_P is sequences of states that have positive conditional probability. Such a sequence is naturally endowed with a topology, the product topology. The open sets of the topology are called cylinder sets. These cylinder sets generate σ -algebra, the Borel σ -algebra; it is the smallest (coarsest) σ -algebra that contains the topology. In this section, we show that a Markov shift $\sigma_P : X_P \to X_p$ is topologically transitive if and only if M is irreducible.

Definition 3.1: [10] [4] Let (X, T) be a dynamical system. We say that a homeomorphism $T : X \to X$ of a compact metric space X is topologically transitive if there exists some $x \in X$ such that its orbit $O_T(x) = \{T^n x : n \in Z\} = \{\ldots, T^{-2}x, T^{-1}x, x, T x, T^2 x, \ldots\}$ is dense in X , i.e. $\overline{O_T(x)} = X$.

Definition 3.2: Let $((X, \beta, \mu), T)$ be a dynamical system in ergodic theory and let $T : X \to X$ be a homeomorphism of the compact metric space X with T is an ergodic measure-preserving transformation with respect to μ where μ is a probability measure on the Borel subsets of X giving non-zero measure to every non-empty open set. We say T is topologically transitive if there exists a point $x \in X$ such that $\mu(\overline{O_T(x)}) = \mu(X) = 1$.

In [8] and [10] topologically transitive is called topologically ergodic.

The following theorem gives equivalent conditions for ergodic measure-preserving transformation T to be topologically transitive.

Theorem 3.3: The following are equivalent.

i. T is topologically transitive**.**

ii. If $E \in \beta$ is a closed set with $TE = E$ then $\mu(E) = \mu(X) = 1$ or $\mu(E^{\circ}) = \mu(\phi) = 0$ (0r, equivalently, if $U \in \beta$ is an open set with $TU = U$ then $\mu(U) = \mu(\phi) = 0$ or $\mu(\overline{U}) = \mu(X) = 1$).

iii. $\forall U, V \in \beta$ Open sets with $\mu(U), \mu(V) > o, \exists n \in Z$ such that $\mu(T^n U \cap V) > 0$.

Proof: $(i) \Rightarrow (ii)$. Suppose that $\mu(\overline{O_T(x)}) = \mu(X) = 1$ and let $E \neq \phi$, E closed (*i.e.* $\overline{E} = E$) and $TE = E$. Suppose W is open and $W \subset E, W \neq \phi$. Then there exists p with $T^p(x) \in W \subset E$ so that $O_T(x) \subset E$, $X = E$ and by ergodicity has measure 0 or 1. Therefore $\mu(E) = \mu(X) = 1$ or $\mu(E^{\circ}) = \mu(\phi) = 0$.

(0r, equivalently: Suppose that $\mu(\overline{O_T(x)}) = \mu(X) = 1$ and let $TU = U \neq \phi$. Then there exists $p \in Z$ with $T^p(x) \in U$. Moreover, for any $m \in Z$ we have that $T^m(x) \in T^{m-p}U = U$. By ergodicity and $\mu(\overline{O_T(x)}) = \mu(X) = 1$ (*i.e.* $\bigcup_{m \in z} T^m(x) \subset X$ is dense). Therefore $\mu(\overline{U}) = \mu(X) = 1$ or $\mu(U) = 0$.

$(ii) \Rightarrow (iii)$. Suppose that $U, V \in \beta$, $U, V \neq \phi$ are open sets with $\mu(U), \mu(V) > 0$. Then $\bigcup_{n \in Z} T^n U$ is open $T-$invariant set, so $\mu(\bigcup_{n \in Z} T^n U) = 1$ by condition (ii) (*i.e.* $\bigcup_{n \in Z} T^n U$ is dense). Thus $\mu(\bigcup_{n \in Z} T^n U \cap V) > 0$ and so $\exists n \in z$ with $\mu(T^n U \cap V) > 0$.

$(iii) \Rightarrow (i)$. This is clear.

Let X_P be a Markov chain and μ_P be a unique Markov probability measure on X_P with transition probabilities P_{ij} then a measure-preserving transformation $\sigma_P : X_P \to X_p$ (i.e. μ_P is σ_P-invariant) is called the Markov shift [1], and the (X_P, μ_P, σ_P) is called the Markov shift system determined by $M = (P_{ij})$.

The following theorem gives necessary and sufficient conditions for $\sigma_P : X_P \to X_p$ to be topologically transitive.

Theorem 3.4: A Markov shift σ_P is topologically transitive if and only if $M = (P_{ij}) \ \forall 1 \le i, j \le m$ is irreducible**.**

Proof: Suppose that σ_P is topologically transitive and let $C = [x_0, x_1, \ldots, x_{k-1}]$, $D = [x_l, x_{l+1}, \ldots, x_n]$ be two cylinder sets with

$$\mu(C) = \mu_P([x_0, x_1, \ldots, x_{k-1}]) = p_{x_0} P_{x_0 x_1} \ldots P_{x_{k-2} x_{k-1}} ,$$

$$\mu(D) = \mu_P([x_l, x_{l+1}, \ldots, x_n]) = p_{x_l} P_{x_l x_{l+1}} \ldots P_{x_{n-1} x_n} .$$

Observe that $\exists n > 0$ such that $\sigma_P{}^n(C) \cap D = \bigcup_{x_k, \ldots x_{l-1}} [x_0, x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{l-1}, x_l, x_{l+1}, \ldots, x_n]$ since $\mu_P(\sigma_P{}^n(C) \cap D) > 0$ and $\mu_P(\sigma_P{}^n(C) \cap D) = \sum_{x_k, \ldots x_{l-1}} p_{x_0} P_{x_0 x_1} \ldots P_{x_{k-1} x_k} \ldots P_{x_{l-1} x_l} P_{x_l x_{l+1}} \ldots P_{x_{n-1} x_n} > 0$. Suppose $x_0 = i$ and $x_n = j$ such that $1 \le i, j \le m$. Notice that

$$P^n{}_{ij} = \sum_{r_1=1}^{m} \ldots \sum_{r_{n-1}=1}^{m} P_{i r_1} P_{r_1 r_2} \ldots P_{r_{n-2} r_{n-1}} P_{r_{n-1} j} .$$

But since $p_i, P_{i x_1}, \ldots, P_{x_{n-1} j} > 0$ we see that $P^n{}_{ij} > 0$, $\forall \ 1 \le i, j \le m$, $n > o$. Then M is irreducible.

Conversely, suppose that for $1 \le i, j \le m$, $\exists n > 0$ such that $P^n{}_{ij} > 0$. Given $U, V \neq 0$ open sets with $\mu_P(U), \mu_P(V) > 0$. We can choose $[i_{-m}, i_{-(m-1)}, \ldots, i_m] \subset U$ such that $\mu_P([i_{-m}, i_{-(m-1)}, \ldots, i_m]) = p_{i_{-m}} P_{i_{-m} i_{-(m-1)}} \ldots P_{i_{m-1} i_m} \subset \mu_P(U) > 0,$ for $m > 0$ and $[j_{-m}, j_{-(m-1)}, \ldots, j_m] \subset V$ such that $\mu_P([j_{-m}, j_{-(m-1)}, \ldots, j_m]) = p_{j_{-m}} P_{j_{-m} j_{-(m-1)}} \ldots P_{j_{m-1} j_m} \subset \mu_P(V) > 0,$ for $m > 0$ by hypothesis we can find $n > 0$ such that $P^n{}_{i_m j_{-m}} > 0$. This means that we can find $x_1, \ldots, x_{n-1}$ such that $P_{i_m x_1}, P_{x_1 x_2}, P_{x_2 x_3}, \ldots, P_{x_{n-1} j_{-m}} > 0$ and then define

$$\mu_P(\sigma_P{}^n(U) \cap V) = \sum p_{i_{-m}} P_{i_{-m} i_{-(m-1)}} \ldots P_{i_m x_1} \ldots P_{x_{n-1} j_{-m}} \ldots P_{j_{m-1} j_m} > 0$$

. Then we have that $\mu_P(\sigma_P{}^n(U) \cap V) > 0$, i.e. σ_P is topologically transitive.

Corollary 3.5: If M is primitive transition $m \times m$ matrix, then the Markov shift σ_P is topologically transitive.

Proof: Since that every primitive matrix is an irreducible matrix and by theorem 3.4 then σ_P is topologically transitive.

Theorem 3.6: A Markov shift σ_P is topologically transitive if and only if its satisfies one of two following conditions:

1. G is irreducible directed graph.

2. A Markov chain X_P is irreducible.

Proof: 1. let σ_P be topologically transitive**.** By theorem 3.4 and theorem 2.7, then G is irreducible directed graph.

Conversely, let G be an irreducible directed graph. By theorem 2.7 and theorem 3.4, then σ_P is topologically transitive.

2. Let σ_P be a topologically transitive**.** By theorem 3.4 and corollary 2.11, then Markov chain X_P is irreducible.

Conversely, let Markov chain X_P be an irreducible. By corollary 2.11 and theorem 3.4, then σ_P is topologically transitive.

References

[1] O. E. Barndorff-Nielsen, J. L. Jensen & W. S. Kendall, Networks and Chaos: Statistical and Probabilistic Aspects, Chapman &Hall/CRC Monographs on Statistics & Applied Probability; 1st edition (1993).
[2] M. Boyle, Algebraic Aspects of Symbolic Dynamics, (Lectures), 1998.
[3] M. Brin & G. Stuck, Introduction to Dynamical Systems, Cambridge University Press (2002).
[4] B. P. kitchens, Symbolic Dynamics one-sided, two-sided and Countable State Markov Shifts, Springer-Verlag, New York (1998).
[5] G. P. Klimov, Probability Theory and Mathematical Statistics, English translation, Mir publisher (1986) .
[6] L. B. Koralov & Y. G. Sinai, Theory of Probability and Random Processes, Springer-Verlag Berlin Heidelberg (2007).
[7] D. Lind & B. Marcus, An Introduction to Symbolic Dynamics and Coding, Cambridge University Press (1995).
[8] K. Petersen, Symbolic Dynamics, Mathematics 261, University of North Carolina at Chapel Hill Spring (1998) .
[9] K. Schmidt, Dynamical Systems of Algebraic Origin, Progress in mathematics 128, Birkhauser Verlag (1995).
[10] P. Walters, An Introduction to Ergodic Theory, first ed., Springer-Verlag (2000).

خاصية التعدي التبولوجي لسلسلة ماركوف

بيمان مجيد محمود

قسم الإحصاء ، كلية الإدارة والاقتصاد ، جامعة كركوك ، كركوك ، العراق

(**تاريخ الاستلام:** ٢٣ / ١١ / ٢٠١٠ ---- **تاريخ القبول:** ١٦ / ٣ / ٢٠١١)

الملخص

في هذا البحث، قدمنا خاصية التعدي التبولوجي للنظام الديناميكا في نظرية ارجوديك بالإضافة إلى ذلك درسنا تأثيرها على سلسلة ماركوف . أثبتنا ان سلسلة ماركوف تكون متعدية تبولوجياً اذا وفقط اذا كانت (اذا كانت) مصفوفتها الانتقالية غير قابلة للاختزال أو بيانها المتجه غير قابل للاختزال (مصفوفتها الانتقالية أولية).