Spectral Method and B- Spline Functions for Approximate Solution of Optimal Control Problem

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Abstract

In this paper an efficient algorithm is proposed, which is based on applying the idea of spectral method using the B-spline polynomials to find an approximate solution of finite quadratic optimal control problems (QOCP), which are governed by ordinary differential equations, represent the constraints.

Keywords: Optimal control, Spectrai method, B-spline polynomials

طريقة الطيف والدوال التوصيلية للحل التقريبي لمسألة السيطرة المثلى

الخلاصة

هذا البحث يقدم خوارزمية استندت على تطبيق فكرة الطيف باستخدام متعددة حدود الثلمة والمتضمنة تعددات الثلمة التوصيلية لايجاد الحل التقريبي لمسألة السيطرة المثلى التربيعية المنتهية الغير المقيدة والتي تحكم بالمعادلات التفاضلية الاعتيادية التي تمثل القيود.

Introduction

Spectral methods encompass a large class of numerical techniques that have long proven their great ability to solve partial differential equations [7]. If those methods have extensively been used since the 70's in fluid dynamics, they have been introduced in the 90's in the field of general Relativity by the Meudon group and are employed by other groups since then. They have yielded to a lot of scientific advances, like the computation of rapidly rotating neutron stars or binary black holes, for example. More recently, a new direction of research is been investigated with the application of spectral techniques to Gauge field theory [7].

Spectral method has the advantage of being able to deal easily with quadratic optimal control problems. In this method, the solution has assumed a finite linear combination of same set of global analytic basis functions. However, as the order of approximation increases one should be able to represent the QOC problems with better accuracy. The most

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important practical issue, regarding the method here is the choice of the basis functions $\{f_i\}$. Throughout the work in this paper, the basis functions that will be used is: B-spline polynomials.

It is necessary to require that $\{f_i\}$ are either orthogonal or linearly independent. In fact if $\{f_i\}_l^N$ are orthogonal or linearly independent and $\{f_i\}_l^{N+1}$ are neither orthogonal nor linearly independent. Therefore, from a practical point of view it is desirable that $\{f_i\}_l^N$ are at least linearly independent [1].

2. *B-spline Curves:*

The properties and capabilities of B-spline curves make them widely used in computer aided geometric design. B-spline curves are polynomial curves defined as linear combination of the control points by some polynomial functions called basis functions. These basis functions are defined in a piecewise way over a closed interval and the subdivision values of this interval are called knots [6].

A B-spline curve is defined everywhere on R and can be written in the following form:

$$P(t)=\sum_{i=-\infty}^{\infty} a_i B_{i,n}(t)$$

where a_i are controls points and $B_{i,n}(t)$ are the basis functions associated with control points a_i. Each basis function can be thought of as the variable weight which determines how the control point a_i influences the curve at parametric value t. For uniform splines, the basis functions satisfy

$$B_{in}(t)=B_n(t-i) \qquad i=-\infty,\ldots,\infty$$

for a fixed function B.

3. *Definitions of B-spline*

There are many different ways to define B-splines; we will consider two equivalent definitions: using convolution and using blossoming [4].

3.1 *Convolution in B-spline*

We start with the simplest functions, which already meet some of the requirements above: piecewise constant coordinate functions. Any piecewise constant function can be written as

$$P(t)=\sum_{i=-\infty}^{\infty} a_i B_{i0}(t)$$

where $B_0(t)$ is the box function defined as

$$B_0(t)=\begin{cases}1 & if\ 0\le t<1\\ 0 & \text{otherwise}\end{cases}$$

We define the convolution of two functions $f(t)$ and $g(t)$ as

$$(f*g)(t)=\int_{-\infty}^{\infty} f(s)g(t-s)ds$$

The remarkable property of convolution is that each time we convolve a function with a box its smoothness increases. We will see that convolution can be seen as “moving average” operation.

A B-spline basis function of degree n can be obtained by convolving a B-spline basis function of degree n−1 with the box $B_0(t)$. For example, the basis function of degree 1 is defined as the convolution of $B_0(t)$ with itself. We need to compute

$$\int_{-\infty}^{\infty} B_{i,0}(s) B_{i,0}(t-s) ds$$

Now take the convolution of $B_{i0}(t)$ with $B_{i1}(t)$ and we obtain $B_{i2}(t)$, the degree 2 basis function. We can continue further and obtain degree 3 B-spline, $B_{i3}(t)$ by convolving $B_{i2}(t)$ with $B_{i0}(t)$.

3.2 *Blossoming in B-splines* [4]

For a cubic B-spline, if $t \in [2,3]$, then we can obtain the value of $P(t) = P(t,t,t)$ from the values of $P(0,1,2)$, $P(1,2,3)$, $P(2,3,4)$, $P(3,4,5)$ as shown in Figure (1).

For $t \in [i, i+1]$ and cubic B-splines, we need four control points $P(i-2, i-1, i)$, $P(i-1, i, i+1)$, $P(i, i+1, i+2)$ $P(i+1, i+2, i+3)$

4. *Properties of B-splines*

- $B_{i,n}(t)$ is a piecewise polynomial of degree n (each convolution increases the degree by 1).
- $B_{i,n}(t)$ has a support of length n + 1. We have seen that $B_{i,0}(t)$ has a support 1 and each convolution increases the support by 1.
- $B_{i,n}(t)$ is C^{n-1}-continuous. $B_{i,0}(t)$ is C^0 continuous and each convolution increases smoothness by 1.
- The set of functions , i = −∞. . .∞ is affine invariant. This comes from the observation that

$$\sum_{i=-\infty}^{\infty} B_n(t-i) = 1$$

which can be proved by in induction observing that this property holds for the box function and is preserved by the convolution.

5. *Spectral Method for LQOCP* [1]

The idea of spectral method is applied to solve finite linear quadratic optimal control problem (LQOCP) with B-spline .

The LQOC problem can be stated as follows
minimizes

$$J = \int_{t_0}^{t_f} \left(x^T Q x + u^T R u \right) dt \quad \ldots(1)$$

subject to the linear system state equations
satisfying the initial conditions

$$\left. \begin{aligned} \dot{x} &= Ax(t) + Cu(t) \\ x(t_0) &= 0 \end{aligned} \right\} \quad \ldots(2)$$

where $A \in R^n \times R^n$, $C \in R^n \times R^m$, $x \in R^n$, $u \in R^m$, Q is n×n positive semi definite matrix , $x^T Q x \geq 0$ and R is m×m positive definite matrix $u^T R u > 0$ unless u=0.

The spectral methods for finite (LQOCP) are described by the following steps:

<u>*Step 1*</u>:
Write the necessary conditions [5] to determine the optimal control solution of the finite LQOCP, eqs (1)and(2)

$$\dot{x} = Ax - \frac{1}{2} C R^{-1} C^T l \quad \ldots.(3)$$

$$\dot{l} = -2Qx - A^T l \quad \ldots.(4)$$

$$u = -\frac{1}{2} R^{-1} C^T l \quad \ldots..(5)$$

with the initial conditions $x(0)=x_o$ and the final conditions $l(t_f)=0$. The use of adjoint equations (4) with the final conditions $l(t_f)=0$ besides the state equations (3) with the initial conditions $x(0)=x_o$ is essential to get a square set of equations.

Step 2:

Select a set of state and adjoint variables which enable us find the others, say $x_1, x_2,....,x_q, l_1, l_2,...., l_q$ and approximate them by a finite length of B-spline $B_{i,n}(t)$, i.e.,

$$x_j(t)\approx x_j^N(t)=\sum_{i=0}^{N} a_{ij}B_{i,n}(t) \quad .....(6)$$

$$l_j(t)\approx l_j^N(t)=\sum_{i=0}^{N} b_{ij}B_{i,n}(t)$$

j=1,2,…q ….(7)

Where $B_{i,n}(t)$,

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

for i=0,1,…,n where $\binom{n}{i}=\frac{n!}{i!(n-i)!}$ There are n+1 n^{th} degree B-spline polynomials for mathematical convenience.

The remaining 2(n-q) state and adjoint variables are obtained from the system state and the system adjoint equations.

Assume that the system state and the system adjoint equations, which are used to find the remaining 2(n-q) state and adjoint variables are:

$$\dot{x}_j(t)=Ax_j-\frac{1}{2}CR^{-1}C^T l_j \quad ….(8)$$

$$\dot{l}_j(t)=-2Qx_j-A^T l_j \quad \text{j=1,2,…,q} \quad …..(9)$$

Therefore, by substituting (6)-(7) into (8)-(9) yields

$$x_j(t)\approx x_j^N(t)=\sum_{i=0}^{N} a_{ij}B_{i,n}(t) \quad ….(10)$$

$$l_j(t)\approx l_j^N(t)=\sum_{i=0}^{N} b_{ij}B_{i,n}(t)$$

j=q+1,q+2,…,n …. (11)

where a_{ij} and b_{ij}; i=1,2,…,N ; j=q+1,q+2,…,n are function of the parameters a_{ij} and b_{ij}; i=1,2,…,n ; j=1,2,…,q.

Step 3:

Form the 2q(N×N) system of algebraic equations of the unknown parameter a_{ij} and b_{ij}; i=1,2,…,N , j=1,2,…,q, from the unused state and adjoint equation in step 2 as well as from the initial and final conditions. That is the 2q(N×N) system of equations can be formed the equations

$$\dot{x}_j(t)=Ax_i-\frac{1}{2}CR^{-1}C^T l_i$$

$$\dot{l}_i(t)=-2Qx_i-A^T l_i \quad ,$$

j=q+1.q+2,…, n ; i=1,2,…,n

And the conditions

$$x_j(0)=x_0$$

$$l_j(t_f)=0 \; ;$$

j=1,2,…,n

The approximations for the state variables $x_j^N(t)$ and the adjoint variables $l_j^N(t)$;j=1,2,…,n , can be written in a matrix form

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{01} & a_{11} & \cdots & a_{N1} \\ a_{02} & a_{12} & \cdots & a_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{0n} & a_{1n} & \cdots & a_{Nn} \end{pmatrix} \begin{pmatrix} B_{1n} \\ B_{2n} \\ \vdots \\ B_{Nn} \end{pmatrix}$$

$$\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{pmatrix} = \begin{pmatrix} b_{01} & b_{11} & \cdots & b_{N1} \\ b_{02} & b_{12} & \cdots & b_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{0n} & b_{1n} & \cdots & b_{Nn} \end{pmatrix} \begin{pmatrix} B_{1n} \\ B_{2n} \\ \vdots \\ B_{Nn} \end{pmatrix}$$

The two matrices can be written in the form

$$x = aB \qquad \ldots(12)$$

$$l = bB \qquad \ldots(13)$$

where

$$B(t) = \left(B_{0,n}(t)\, B_{1,n}(t)\, \ldots\, B_{n,n}(t)\right)^T$$

Differentiating (12)and (13) to obtain

$$\dot{x} = a\dot{B} \qquad \ldots(14)$$

$$\dot{l} = b\dot{B} \qquad \ldots(15)$$

Rewrite $\dot{B}(t)$ in terms of $B(t)$, then equn. (14) and (15) becomes

$$\dot{x} = aD_B B \qquad \ldots(16)$$

$$\dot{l} = bD_B B \qquad \ldots(17)$$

where the matrix D_B is the Differentiation operational matrix of the B-spline functions given as follows[5]

$$\begin{bmatrix} -n & -1 & 0 & 0 & \ldots & 0 & 0 \\ n & -(n-2) & -2 & 0 & & & \\ 0 & (n-1) & -(n-4) & -3 & & & \\ 0 & 0 & (n-k+1) & -(n-2k) & & \vdots & \vdots \\ & & & (n-k+1) & & & \\ \vdots & \vdots & & & & & \\ 0 & \ldots & \ldots & & & 1 & -(n-2n) \end{bmatrix}$$

Now the approximations (12) and (13) and their derivatives (14) and (15) are inserted into equn.(3)-(4) to yield

$$aD_B B = \left(Aa - \frac{1}{2} CR^{-1} C^T b\right) B \;.(18)$$

$$bD_B B = \left(-2Qa - A^T b\right) B \qquad \ldots(19)$$

Or

$$aD_B B = YB \qquad \ldots.(20)$$

$$bD_B B = ZB \qquad \ldots(21)$$

Since $Y = \left(Aa - \frac{1}{2} CR^{-1} C^T b\right)$ and $Z = \left(-2Qa - A^T b\right)$

The coefficients of the B-spline polynomials up to N-1 will be equaled to get:

$$a\hat{D}_B = \hat{Y} \qquad \ldots.(23)$$

$$b\hat{D}_B = \hat{Z} \qquad \ldots.(24)$$

where $\hat{D}_B, \hat{Y}$ and $\hat{D}_B, \hat{Z}$ are D_B, Y and Z respectively, but with the last row discarded in the three matrices.

Both the initial and final conditions also can be expressed using the basis functions $B_{i,n}(t)$ t=0 and t_f respectively to obtain the equations

$$\sum_{i=0}^{N} a_{ij} B_{i,n}(0) = x_0 \text{ or}$$

$$a_{0j} B_{0n}(0) + a_{1j} B_{1n}(0) + \ldots + a_{N,j} B_{N,n}(0) = x_0 \qquad \ldots..(25)$$

$$\sum_{i=0}^{N} b_{ij} B_{i,n}(t_f) = 0 \text{ or}$$

$$b_{0j} B_{0n}(1) + b_{1j} B_{1n}(1) + \ldots + b_{N,j} B_{N,n}(1) = 0$$

j=1,2,…,n …..(26)

Step 4:

Solve the system of equations obtained from (23)-(24) using Gauss elimination procedure to find the unknown parameter a_{ij} and b_{ij} ; i=1,2,…,n; j=1,2,…,q.

Step 5:

Substitute the values of the parameters a_{ij} and b_{ij} into (6)-(7)and (10)-(11) to get the approximate trajectories $x_j^N(t)$ and approximate adjoints $l_j^N(t)$, the control variables $u_i^N(t)$; i=1,2,…,m can be formed from (5); therefore

$$u_i^N(t) = -\frac{1}{2}R^{-1}C^T l_j^N(t) \quad i=1,2,\dots,m;\ j=1,2,\dots,n \qquad \dots(27)$$

or $$u = -\frac{1}{2}R^{-1}C^T bB \qquad \dots(28)$$

Let $g = -\frac{1}{2}R^{-1}C^T bB$ then

Becomes

$$u = gB \qquad \dots.(29)$$

Step 6:

Approximate the performance index. By substituting equs. (12) and (29) in the performance index (1) yields

$$J^* = \int_0^{t_f} (B^T a^T QaB + B^T g^T RgB)dt$$

Where J^* is the approximate value of J.

Let $a^T Qa = M$ and $g^T Rg = S$, then

$$J^* = \int_0^{t_f} \left(B^T MB + B^T SB\right)dt \qquad \dots.(30)$$

6. *Numerical Example:*

Consider the following finite LQOC problem [3]

Minimize

$$J = \frac{1}{2}\int_0^1 \left(u^2 + 2x^2\right)dt$$

Subject to

$$\dot{x} = \frac{1}{2}x + u \qquad x(0) = 1$$

The exact trajectory and the control are:

$$x(t) = \frac{2e^{3t} - e^3}{e^{\frac{3t}{2}}\left(2 + e^3\right)} \quad \text{and}$$

$$u(t) = \frac{2\left(e^{3t} - e^3\right)}{e^{\frac{3t}{2}}\left(2 + e^3\right)}$$

and the exact value of J is 0.64164498

The results obtained by using the algorithm of spectral method using the B-spline functions In order to apply the spectral method, one first finds:

- The Hamittonian:

$$H = \frac{1}{2}u^2 + x^2 + \frac{1}{2}lx + lu$$

- The adjoint equation:

$$\dot{l} = -2x - \frac{1}{2}l$$

- The sufficient condition for optimality:

$$\frac{\partial H}{\partial u} = 0 \quad \Rightarrow \quad u + l = 0$$

Therefore

$$u = -l \qquad \dots.(31)$$

The final system is:

$$\dot{x} = \frac{1}{2}x - l \qquad \dots.(32)$$

$$\dot{l} = -2x - \frac{1}{2}l \qquad \dots.(33)$$

With the boundary conditions:

$$x(0) = 1 \qquad \dots(34)$$

$$l(1) = 0 \qquad \dots.(35)$$

Then the approximate trajectory $x(t)$, the approximate control $u(t)$ and the approximate value J^* are found as follows:

- $x(t)$ and $l(t)$ are approximated by 3rd order B-spline of unknown parameters,

$$x(\mathrm{t}) \approx \mathrm{a}_0\mathrm{B}_{03} + \mathrm{a}_1\mathrm{B}_{13} + \mathrm{a}_2\mathrm{B}_{23} + \mathrm{a}_3\mathrm{B}_{33} \quad ...(36)$$

$$l(t) \approx b_0B_{03} + b_1B_{13} + b_2B_{23} + b_3B_{33} \quad ....(37)$$

eqs.(32) and (33) with the approximate trajectory and adjoint variables (36) and (37) respectively, to get

$$a_0\dot{B}_{03} + a_1\dot{B}_{13} + a_2\dot{B}_{23} + a_3\dot{B}_{33} = \frac{1}{2}(a_0B_{03} + a_1B_{13} + a_2B_{23} + a_3B_{33}) - (b_0B_{03} + b_1B_{13} + b_2B_{23} + b_3B_{33})$$

$$b_0\dot{B}_{03} + b_1\dot{B}_{13} + b_2\dot{B}_{23} + b_3\dot{B}_{33} = -2(a_0B_{03} + a_1B_{13} + a_2B_{23} + a_3B_{33}) - \frac{1}{2}(b_0B_{03} + b_1B_{13} + b_2B_{23} + b_3B_{33})$$

Equating the coefficients to $\mathrm{B}_{\mathrm{i,n}}$ $;\mathrm{i} = 1,2,3$ yield

$$\frac{-7}{2}a_{03} + 3a_{13} = -b_{03} \quad ...(38)$$

$$-a_{03} - \frac{3}{2}a_{13} + 2a_{23} = -b_{13} \quad ...(39)$$

$$-2a_{13} + \frac{1}{2}a_{23} + a_3 = -b_{23} \quad ....(40)$$

$$\frac{-5}{2}b_{03} + 3b_{13} = -2a_{03} \quad ....(41)$$

$$-b_{03} - \frac{1}{2}b_{13} + 2b_{23} = -2a_{13} \quad .....(42)$$

$$-2b_{13} + \frac{3}{2}b_{23} + b_3 = -2a_{23} \quad ....(43)$$

Additional equations are obtained from the conditions (34)and (35)

$$x(0) = 1 \Rightarrow a_0 = 1 \quad ...(44)$$

$$l(1) = 0 \Rightarrow b_3 = 0 \quad ....(45)$$

- Solve the above system of equations(34)–(43)to find the parameters

a_{i3} b_{i3} ; $\mathrm{i} = 1,2,3$

- Obtain the approximate control using (31).therefore, the approximate trajectory and control are

$$x_B(t) = B_{03} + \frac{73}{138}B_{13} + \frac{239}{552}B_{23} + \frac{67}{368}B_{33}$$

$$u_B(t) = \frac{44}{23}B_{03} + \frac{64}{69}B_{13} + \frac{91}{138}B_{23}$$

and the approximate performance index is $J^* = 0.85579239$

Table (1) shows a comparison between the computed optimal value obtained by using the proposed algorithm with B-spline polynomials for different orders.

7. Conclusions

The spectral method can be employed with using different types of basis functions. It can be convert the optimal control problem into a system of square algebraic equation, which can be solved using Gauss elimination technique, with pivoting.

The spectral technique with the aid of B-spline polynomials of order n

$B_{i,n}$; $i=1,2,...,n$ provided a very convenient and useful procedure to evaluate the approximate performance value J^*. The example was applied for illustration and acceptable results were achieved.

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Table (1)

N	SMB
2	0.82439236
3	0.85579239
4	0.86062932
5	0.86514046

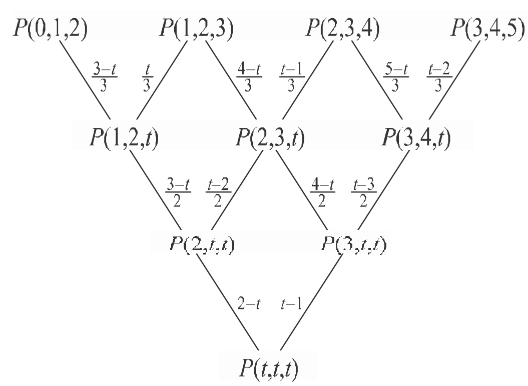

Figure (1)