Classification of The Projective Line of Order Nineteen and its Application to Error-Correcting Codes

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Abstract

In this paper, the k-sets in the projective line of order nineteen up to $k=10$ are classified and their stabilizer groups also computed. Also, the projective line spilt into five tetrads of type harmonic, equianharmonic and neither harmonic nor equianharmonic. Finally, the applications of these sets into error-correcting codes are given.

Key words and phrases: Projective line, coding theory.

1- Introduction

Let $V(n+1,q)$ be the $n+1$ dimensional vector space over the Galois field F_q. Consider the equivalence relation on the elements of $V\setminus\{0\}$ whose equivalence classes are the one-dimensional subspaces of V with zero removed. Thus, if

$X=\langle x_0,\dots,x_n\rangle, Y=\langle y_0,\dots,y_n\rangle\in V\setminus\{0\}$,

then X is equivalent to Y if $X=tY$ for some t in $F_q\setminus\{0\}$; that is, $y_i=tx_i$ for all i. Then the set of equivalenceclasses is the n-dimensional projective space over F_q and is denoted by $PG(n,q)$. The elements of $PG(n,q)$ are called points; the equivalence class of the vector X is the point $P(X)$. Two projective spaces Ω_1,Ω_2 of dimension n they are equivalent $\Omega_1\cong\Omega_2$ if there exist a bijection map τ given by a non-singular $(n+1)\times(n+1)$ matrix A such that

$P(X')=P(X)\tau\Leftrightarrow tX'=XA$,

where $t\in F_q\setminus\{0\}$. This map called *projectivity* and denoted by $\tau=M(A)$.

Definition 1.1[(1)]: A $(k;r)$-arc with $k\geq r+1$ is a set of k points of a projective geometry $PG(n,q)$ such that some r, but no $r+1$ of them are collinear. On $PG(1,q)$, a $(k;1)$-arc is just an unordered set of k distinct points simply called a k-set. A 3-set is called a *triad*, a 4-set a *tetrad*, a 5-set a *pentad*, a 6-set a *hexad*, a 7-set a *heptad*, an 8-set a *octad*, a 9-set a *nonad*, a 10-set a *decad*.

Theorem1.2[(1)]**: (The Fundamental Theorem of Projective Geometry)**

(i) If $\{P_0,\ldots,P_{n+1},\}$ and $\{P'_0,\ldots,P'_{n+1},\}$ are both subsets of $PG(n,q)$ of cardinality $n+2$ such that no $n+1$ points chosen from the same set lie in a hyperplane, then there exists a unique projectivity $\Im$ such that $P'_i = P_i\Im$ for $i = 0,1,\ldots,n+1$. For $n+1$, simplifies: there is a unique projectivity of $PG(1,q)$ transforming any three distinct points on a line to any other three.

Definition 1.3[2, 3] **:**A *projective linear code* C, $[n,k,d]_q$-code, is a subspace of $V(n+1,q)$, where $k=\dim C$, $d = d(C) = \min\{|\{i \mid x_i \neq y_i\}|\}$ and any two columns of a generator matrix G are linearly independent, where G is a $k\times n$ matrix having as rows the vectors of a basis of C. If $d = n-k+1$, then C is called *maximum distance separable codes* (*MDS code)*.

A code C with minimum distance at least $d = 2e+1$ can correct up to e errors. This type of code is called an *e-error correcting code*.

Some groups that occur in this work are listed below[4].

Z_n = cyclic group of order n;

V_4= Klein 4-group which is the direct product of two copies of the cyclic group of order 2;

S_n = symmetric group of degree n;

A_n = alternating group of degree n;

D_n = dihedral group of order $2n$ $=\left\langle r,s \middle| r^n = s^2 = (rs)^2 = 1\right\rangle$.

2-The Cross-Ratio and Stabilizer Group of a Tetrad

Definition 2.1[1]: The cross-ratio $\lambda = \{P_1,P_2,P_3,P_4\}$ of four ordered points $P_1,P_2,P_3,P_4 \in PG(1,q)$ with coordinates t_1,t_2,t_3,t_3 is

$$\lambda = \{P_1,P_2;P_3,P_4\} = \{t_1,t_2;t_3,t_4,\} = \frac{(t_1-t_3)(t_2-t_4)}{(t_1-t_4)(t_2-t_3)}$$

The cross-ratio is invariant under a projective group of order four, given by $\{I,(P_1P_2)(P_3P_4),(P_1P_3)(P_2P_4),(P_1P_4)(P_2P_3)\} \cong \mathrm{V}_4$.

Thus, under all 24 permutations of $\{P_1,P_2;P_3,P_4\}$ the cross-ratio takes just the six values $\lambda, 1/\lambda, (1-\lambda), 1/(1-\lambda), (\lambda-1)/\lambda, \lambda/(\lambda-1)$.

Lemma 1.2[1]**:** On $PG(1,q)$, a projectivity between any two tetrads is determined by the images of three points. Therefore there exists a projectivity $\tau = M(A)$ such that $Q_i = P_iA$, $i=1,2,3,4$ if and only if the cross-ratios of the two sets of four points in the corresponding order are equal.

Definition 2.3[1]: Let T be a tetrad with cross-ratio λ. Then T is called

(1) *harmonic*, denoted by H, if $\lambda = 1/\lambda$ or $\lambda = \lambda/(\lambda-1)$ or $\lambda = (1-\lambda)$;

(2) *equianharmonic*, denoted by E, if $\lambda = 1/(1-\lambda)$ or equivalently, $\lambda = (\lambda-1)/\lambda$;

(3) *neither harmonic nor equianharmonic*, denoted by N, if the cross-ratio is another value.

The cross-ratio of any harmonic tetrad has the values $-1,2,1/2$. The cross-ratio of a tetrad of type E satisfies the equation

$\lambda^2 - \lambda + 1 = 0$. (1)

So, equianharmonic tetrads exist if and only if $\lambda^3 + 1 = 0$ has three solutions in F_q or $\lambda = -1$ is a unique solution of (1) in F_q.

Lemma 2.4[(1)] : On $PG(1,q), q = p^h$, where p is a prime integer and h positive integer,

(i) the number of harmonic tetrads n_H and the stabilizer group G of each one are as in the following table:

	n_H	G
$p=3$	$q(q^2-1)/24$	S_4
$p>3$	$q(q^2-1)/8$	D_4

(ii) the number of harmonic tetrads n_E and the stabilizer group G of each one are as in the following table:

	n_E	G
$p=3$	$q(q^2-1)/24$	S_4
$p \equiv 3 \pmod 3$	$q(q^2-1)/12$	A_4

When the tetrad T of type N, there only four permutations of T amongst the 24 permutations which are projectively equivalent as follows

$I, (P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3)$.

These permutations form a group isomorphic to the Klein 4-group V_4[(3)].

The aims of this research are to answer the following questions in $PG(1,19)$:

1- How many projectively inequivalent k-sets in $PG(1,19)$ are there and what is the stabilizer group of each one?

2- Are D_i and D_i^c equivalent? What is the group of projectivities of $PG(1,19)$ of the partition? where D_i is 10-set and D_i^c its complement.

3- Does the projective line $PG(1,19)$ split into five disjoint harmonic tetrads, five equianharmonic tetrads, and five tetrads of type N_1 or five tetrads of type N_2?

Also, the relation between a projective MDS codes of dimension two and k-sets on $PG(1,19)$ is given.

3-The Algorithm For Classification of The K-Sets in $PG(1,q)$

On $PG(1,q)$, a k-set can be constructed by adding to any $(k-1)$-set one point from the other $q-k+1$ points. According to the Fundamental Theorem of Projective Geometry, any three distinct points on a line are projectively equivalent; so choose a fixed triad $\Re$. A 4-set is formed by adding to $\Re$ one point from the other $q-2$ points on $PG(1,q)$; that is, from $PG(1,q) \setminus \Re = \Re^c$. From Lemma 2.1, there is a unique tetrad of type H and unique tetrad of type E but the tetrad of type N might be divided into subclasses. A 5-set is formed byadding to any tetrad T in χ_i one point from the other $q-3$ points on $PG(1,q)$.The group G_T fixes T and splits the other $q-3$ points into a number of orbits; so, different 5-sets are formed by adding one point from each different orbit.

Theprocedure can be extended to construct $6,7,\ldots,\frac{(q+1)}{2}$- sets or $\frac{(q+1)}{2}+1$-sets in $PG(1,q)$. The $(n-1)$-subsets of an n-set are classified according to their projective type.

Let K and K'. be two pentads. To check they are equivalent the following stepsare used.

(1) Classify tetrads in both pentads.

(2) If the classifications of K and K'. are different then they are projectively inequivalent.

(3) If the classifications of K and K'. are the same, then transformation matrices A_α are constructed from a tetrad T with highest recurrence in the algebraicstructure of K to tetrads T_α in K' with same types of T.

(4) If the action of one A_α on the remaining points of K are equal to the remainingpoints of K' then K and K' are projectively equivalent. If not, it means theyare projectively inequivalent.

This procedure can be extended to check the equivalence between k-sets, $k=6,7,\ldots,\frac{(q+1)}{2}$- sets or $\frac{(q+1)}{2}+1$-sets, and also can be used to calculate the stabilizer group of each k-sets[1].

During this research a k-sets $K=\{a_1,a_2,\ldots,a_k\}$ are partitions by the following way: $\{a_1,a_2,\ldots,a_{k-1}\}$, $\{a_1,a_2,\ldots,a_{k-2},a_k\},\ldots,\{a_2,a_3,\ldots,a_{k-1},a_k\}$.

4-Classification of The Projective Line $PG(1,19)$

On $PG(1,19)$, the projective line over Galois field of order 19, there are 20 points. The points of $PG(1,19)$ are the elements of the set

$$F_{19}\cup\{\infty\}=\{\infty,0,\pm1,\pm2,\pm3,\pm4,\pm5,\pm6,\pm7,\pm8,\pm9\}$$

A tetrad is of type H if the cross-ratio is $-1,2$ or $1/2=-9$. It is of type E if the cross-ratio is -7 or 8, and it is of type N if the cross-ratio is $-2;\ 3;-3;\ -2,3,-3,4,-4,5,-5,6,-6,7,-8$ or 9. As a tetrad of type N has six possible values of its cross-ratios so, there are two tetrads of type N, one with cross-ratios $-2,3,-6,7,-8,9$ denoted by N_1 and the other with $-3,4,-4,5,-5,6$ denoted by N_2. Hence there are four classes of tetrads:

$\chi_1=\{$the class of H tetrads$\}\ni\{\infty,0,1,a\}$ for $a=-1,2,-9$;

$\chi_2=\{$the class of E tetrads$\}\ni\{\infty,0,1,b\}$ for $b=-7,8$;

$\chi_3=\{$the class of N_1 tetrads$\}\ni\{\infty,0,1,c\}$ for $c=-2,3,-6,7,-8,9$;

$\chi_4=\{$the class of N_2 tetrads$\}\ni\{\infty,0,1,d\}$ for $d=-3,4,-4,5,-5,6$.

From Lemma 2.4, $|\chi_1|=855$, $|\chi_2|=570$, $|\chi_3|=|\chi_4|=1710$.

Theorem4.1[5] : On $PG(1,19)$, the number of inequivalent k-sets are

(1) Four projectively distinct tetrads.

(2) Five projectively distinct pentads

(3) Thirteen projectively distinct hexads

(4) Eighteen projectively distinct heptads.
(5) Thirty one projectively distinct octads.
(5) Thirty three projectively distinct heptads.
(6) Forty four projectively distinct decads.
These k-sets, $k = 1,\ldots,10$ and their algebraic structures with its stabilizer group types are given in Tables 1, 2, 3, 4, 5, 6, 7.

Table 1: Distinct tetrads on $PG(1,19)$

Type	The tetrad	Stabilizer
H	$\{\infty,0,1,-1\}$	$D_4 = \langle(1+t)/(1-t),(t+1)/(t-1)\rangle$
E	$\{\infty,0,1,-7\}$	$A_4 = \langle(t+7)/(8t),1/(8t)\rangle$
N_1	$\{\infty,0,1,-2\}$	$V_4 = \langle -2/t,(t-1)/(9t-1)\rangle$
N_2	$\{\infty,0,1,-3\}$	$V_4 = \langle -3/t,(t-1)/(6t-1)\rangle$

Table 2: Distinct heptads on $PG(1,19)$

Type	The tetrad	Types of tetrads	Stabilizer
P_1	$\{\infty,0,1,-1,2\}$	$HHN_1N_1N_2$	$Z_2 = \langle(1-t)\rangle$
P_2	$\{\infty,0,1,-1,4\}$	$HN_2N_2EN_1$	$I = \langle t\rangle$
P_3	$\{\infty,0,1,-7,-2\}$	$EN_1N_1N_1E$	$S_3 = \langle(1-8t),t/(9t-1)\rangle$
P_4	$\{\infty,0,1,-2,-3\}$	$N_1N_2N_1N_2N_1$	$Z_2 = \langle-(t+2)\rangle$
P_5	$\{\infty,0,1,-3,4\}$	$N_2N_2N_2N_2N_2$	$D_5 = \langle(t-4)/(5t-4),(t+3)/(5t-1)\rangle$

Table 3: Distinct hexads on $PG(1,19)$

Type	The hexads	Types of pentads	Stabilizer
ξ_1	$\{\infty,0,1,-1,2,-2\}$	$P_1P_1P_1P_1P_4P_4$	$V_4 = \langle -t,-2/t\rangle$
ξ_2	$\{\infty,0,1,-1,2,-3\}$	$P_1P_1P_2P_3P_2P_4$	$I = \langle t\rangle$
ξ_3	$\{\infty,0,1,-1,2,-4\}$	$P_1P_2P_2P_1P_2P_2$	$Z_2 = \langle(2t+2)/(t-2)\rangle$
ξ_4	$\{\infty,0,1,-1,2,-5\}$	$P_1P_2P_1P_4P_4P_2$	$Z_2 = \langle(1-t)/(1+9t)\rangle$
ξ_5	$\{\infty,0,1,-1,2,-7\}$	$P_1P_2P_2P_4P_4P_5$	$I = \langle t\rangle$
ξ_6	$\{\infty,0,1,-1,2,-8\}$	$P_1P_2P_1P_2P_2P_2$	$Z_2 = \langle(t-2)/(t-1)\rangle$
ξ_7	$\{\infty,0,1,-1,2,-9\}$	$P_1P_1P_1P_1P_1P_1$	$D_6 = \langle(1+t)/(2-t),(2t-1)/(t-2)\rangle$
ξ_8	$\{\infty,0,1,-1,4,-4\}$	$P_2P_2P_2P_2P_3P_3$	$V_4 = \langle -t,4/t\rangle$
ξ_9	$\{\infty,0,1,-1,4,5\}$	$P_2P_2P_5P_5P_2P_2$	$V_4 = \langle(t-4)/(4t-1),(t-5)/(5t-1)\rangle$

ξ_{10}	{∞,0,1,−1,4,−5}	$P_2P_2P_4P_4P_3P_3$	$Z_2 = \langle -1/t \rangle$
ξ_{11}	{∞,0,1,−1,4,7}	$P_2P_2P_2P_2P_2P_2$	$S_3 = \langle (4t+3)/t, -(t+1)/(8t+1) \rangle$
ξ_{12}	{∞,0,1,−1,4,−7}	$P_2P_2P_2P_4P_2P_4$	$Z_2 = \langle (t-4)/(t-1) \rangle$
ξ_{13}	{∞,0,1,−2,−3,6}	$P_4P_4P_4P_4P_4P_4$	$S_3 = \langle (t+2)/(6t+2), (t+3)/(t-1) \rangle$

Table 4: Distinct heptads on $PG(1,19)$

Type	The heptads	Types of hexads	Stabilizer
T_1	{∞,0,1,−1,2,−2,−3}	$\xi_1\xi_2\xi_1\xi_5\xi_2\xi_5\xi_{13}$	$Z_2 = \langle -(t+1) \rangle$
T_2	{∞,0,1,−1,2,−2,6}	$\xi_1\xi_4\xi_2\xi_2\xi_4\xi_{10}\xi_{10}$	$Z_2 = \langle -2/t \rangle$
T_3	{∞,0,1,−1,2,−2,−4}	$\xi_1\xi_3\xi_2\xi_6\xi_7\xi_2\xi_4$	$I = \langle t \rangle$
T_4	{∞,0,1,−1,2,−2,−5}	$\xi_1\xi_4\xi_3\xi_6\xi_5\xi_5\xi_{12}$	$I = \langle t \rangle$
T_5	{∞,0,1,−1,2,−3,9}	$\xi_2\xi_6\xi_2\xi_{12}\xi_8\xi_{11}\xi_{10}$	$I = \langle t \rangle$
T_6	{∞,0,1,−1,2,−3,8}	$\xi_2\xi_5\xi_3\xi_{12}\xi_2\xi_{12}\xi_5$	$Z_2 = \langle (t+1)/(6t-1) \rangle$
T_7	{∞,0,1,−1,2,−3,4}	$\xi_2\xi_2\xi_5\xi_5\xi_{10}\xi_{10}\xi_{13}$	$Z_2 = \langle 1-t \rangle$
T_8	{∞,0,1,−1,2,−3,5}	$\xi_2\xi_3\xi_2\xi_8\xi_{10}\xi_3\xi_{10}$	$Z_2 = \langle (t+3)/(t-1) \rangle$
T_9	{∞,0,1,−1,2,−3,7}	$\xi_2\xi_2\xi_3\xi_6\xi_8\xi_{12}\xi_4$	$I = \langle t \rangle$
T_{10}	{∞,0,1,−1,2,−3,−7}	$\xi_2\xi_5\xi_4\xi_6\xi_{10}\xi_2\xi_5$	$I = \langle t \rangle$
T_{11}	{∞,0,1,−1,2,−3,−4}	$\xi_2\xi_3\xi_5\xi_9\xi_2\xi_9\xi_5$	$Z_2 = \langle -(t+3)/(9t+1) \rangle$
T_{12}	{∞,0,1,−1,2,−4,−8}	$\xi_3\xi_6\xi_6\xi_3\xi_6\xi_{11}\xi_3$	$Z_3 = \langle t/(7t+7) \rangle$
T_{13}	{∞,0,1,−1,2,−4,−5}	$\xi_3\xi_4\xi_9\xi_5\xi_5\xi_{12}\xi_{11}$	$I = \langle t \rangle$
T_{14}	{∞,0,1,−1,2,−5,−7}	$\xi_4\xi_5\xi_{12}\xi_4\xi_{13}\xi_{13}\xi_5$	$Z_2 = \langle (1-t)/(4t+1) \rangle$
T_{15}	{∞,0,1,−1,2,−7,9}	$\xi_5\xi_6\xi_6\xi_9\xi_5\xi_{12}\xi_9$	$Z_2 = \langle -t/(t+1) \rangle$
T_{16}	{∞,0,1,−1,2,−7,8}	$\xi_5\xi_5\xi_8\xi_8\xi_{10}\xi_{10}\xi_9$	$Z_2 = \langle 1-t \rangle$
T_{17}	{∞,0,1,−1,2,−7,−9}	$\xi_5\xi_7\xi_5\xi_5\xi_5\xi_5\xi_5$	$Z_6 = \langle (2t-1)/(t+1) \rangle$
T_{18}	{∞,0,1,−1,4,−5,−7}	$\xi_{10}\xi_{12}\xi_{12}\xi_{12}\xi_{13}\xi_{10}\xi_{10}$	$Z_3 = \langle 8/(t+7) \rangle$

Table 5: Distinct octads on $PG(1,19)$

Type	The octads	Types of heptads	Stabilizer

O_1	$\{\infty,0,1,-1,2,-2,-3,-4\}$	$T_1T_3T_{11}T_1T_{15}T_3T_{11}T_{14}$	$Z_2=\langle -(t+2)\rangle$
O_2	$\{\infty,0,1,-1,2,-2,-3,4\}$	$T_1T_3T_7T_4T_{17}T_{10}T_{10}T_{14}$	$I=\langle t\rangle$
O_3	$\{\infty,0,1,-1,2,-2,-3,5\}$	$T_1T_4T_8T_2T_{16}T_5T_{13}T_{18}$	$I=\langle t\rangle$
O_4	$\{\infty,0,1,-1,2,-2,-3,6\}$	$T_1T_2T_2T_1T_7T_{10}T_{10}T_7$	$Z_2=\langle (1-t)/(1+t)\rangle$
O_5	$\{\infty,0,1,-1,2,-2,-3,7\}$	$T_1T_1T_9T_4T_4T_9T_{14}T_{14}$	$Z_2=\langle -2/t\rangle$
O_6	$\{\infty,0,1,-1,2,-2,-3,8\}$	$T_1T_4T_6T_3T_4T_3T_6T_1$	$Z_2=\langle -(t+2)/(7t+1)\rangle$
O_7	$\{\infty,0,1,-1,2,-2,-3,9\}$	$T_1T_3T_5T_3T_{13}T_5T_{13}T_7$	$Z_2=\langle -(t+1)\rangle$
O_8	$\{\infty,0,1,-1,2,-2,6,-4\}$	$T_2T_3T_5T_3T_{13}T_5T_{13}T_7$	$Z_2=\langle -(t+1)/(5t+1)\rangle$
O_9	$\{\infty,0,1,-1,2,-2,6,4\}$	$T_2T_3T_{10}T_9T_3T_4T_8T_{10}$	$I=\langle t\rangle$
O_{10}	$\{\infty,0,1,-1,2,-2,6,5\}$	$T_2T_4T_{13}T_{10}T_{11}T_9T_{16}T_5$	$I=\langle t\rangle$
O_{11}	$\{\infty,0,1,-1,2,-2,6,-5\}$	$T_2T_4T_{14}T_6T_{10}T_{14}T_7T_{18}$	$I=\langle t\rangle$
O_{12}	$\{\infty,0,1,-1,2,-2,6,-6\}$	$T_2T_2T_3T_3T_3T_3T_8T_8$	$V_4=\langle 2/t,-2/t\rangle$
O_{13}	$\{\infty,0,1,-1,2,-2,-4,-5\}$	$T_3T_4T_{13}T_{11}T_{15}T_{17}T_6T_{13}$	$I=\langle t\rangle$
O_{14}	$\{\infty,0,1,-1,2,-2,-4,8\}$	$T_3T_4T_4T_5T_{12}T_3T_9T_{10}$	$I=\langle t\rangle$
O_{15}	$\{\infty,0,1,-1,2,-2,-4,9\}$	$T_3T_3T_{12}T_2T_{12}T_3T_3T_3$	$S_3=\langle (t-2)/(t+4),2/t\rangle$
O_{16}	$\{\infty,0,1,-1,2,-2,-4,-9\}$	$T_3T_3T_3T_9T_9T_3T_3T_3$	$V_4=\langle -2/t,(2-t)/(1+t)\rangle$
O_{17}	$\{\infty,0,1,-1,2,-2,-5,5\}$	$T_4T_4T_9T_9T_{15}T_{15}T_{11}T_6$	$Z_2=\langle -t\rangle$
O_{18}	$\{\infty,0,1,-1,2,-2,-5,8\}$	$T_4T_4T_{13}T_{12}T_{12}T_{13}T_{15}T_{15}$	$Z_2=\langle -2/t\rangle$
O_{19}	$\{\infty,0,1,-1,2,-2,-5,-8\}$	$T_4T_4T_4T_{13}T_4T_{13}T_{13}T_{13}$	$V_4=\langle (t-2)/(t-1),(2t-2)/(t-2)\rangle$
O_{20}	$\{\infty,0,1,-1,2,-3,9,4\}$	$T_5T_7T_9T_{10}T_6T_8T_5T_{18}$	$I=\langle t\rangle$
O_{21}	$\{\infty,0,1,-1,2,-3,9,-4\}$	$T_5T_{11}T_{12}T_6T_{13}T_9T_{13}T_{10}$	$I=\langle t\rangle$
O_{22}	$\{\infty,0,1,-1,2,-3,9,5\}$	$T_5T_8T_{12}T_9T_9T_5T_{12}T_8$	$Z_2=\langle -(t+3)/(2t+1)\rangle$
O_{23}	$\{\infty,0,1,-1,2,-3,9,-5\}$	$T_5T_{10}T_9T_9T_6T_8T_{13}T_{16}$	$I=\langle t\rangle$
O_{24}	$\{\infty,0,1,-1,2,-3,9,-6\}$	$T_5T_5T_5T_5T_5T_5T_5T_5$	$D_5=\langle (t-1)/(t+1),(2t+1)/(t-2)\rangle$
O_{25}	$\{\infty,0,1,-1,2,-3,9,-7\}$	$T_5T_{10}T_{15}T_{10}T_{15}T_{16}T_5T_{16}$	$Z_2=\langle (t+7)/(t-1)\rangle$
O_{26}	$\{\infty,0,1,-1,2,-3,9,-8\}$	$T_5T_9T_{15}T_7T_{14}T_{16}T_{13}T_{18}$	$I=\langle t\rangle$
O_{27}	$\{\infty,0,1,-1,2,-3,4,5\}$	$T_7T_8T_{11}T_{11}T_{16}T_{16}T_8T_7$	$Z_2=\langle (t-4)/(4t-1)\rangle$
O_{28}	$\{\infty,0,1,-1,2,-3,-7,-4\}$	$T_{10}T_{11}T_{13}T_{13}T_{15}T_{10}T_{11}T_{15}$	$Z_2=\langle -(t+7)/(t+1)\rangle$

O_{29}	$\{\infty,0,1,-1,2,-4,-5,-7\}$	$T_{13}T_{13}T_{14}T_{13}T_{14}T_{14}T_{14}T_{13}$	$V_4=\langle (t-1)/(8t-1),(2t+2)/(t-2)\rangle$
O_{30}	$\{\infty,0,1,-1,2,-7,8,-9\}$	$T_{16}T_{17}T_{17}T_{16}T_{16}T_{16}T_{16}T_{16}$	$D_6=\langle (1+t)/(2-t),t/(t-1)\rangle$
O_{31}	$\{\infty,0,1,-1,4,-5,-7,-8\}$	$T_{18}T_{18}T_{18}T_{18}T_{18}T_{18}T_{18}T_{18}$	$S_3=\langle (1+t)/(1-t),(t-4)/(t-1)\rangle$

Table 6: Distinct nonads on $PG(1,19)$

Type	The nonads	Types of octads	Stabilizer
γ_1	$\{\infty,0,1,-1,2,-2,-3,-4,3\}$	$O_1O_1O_2O_{13}O_2O_{28}O_{13}O_{28}O_{29}$	$Z_2=\langle -(t+1)\rangle$
γ_2	$\{\infty,0,1,-1,2,-2,-3,-4,4\}$	$O_1O_2O_{12}O_{27}O_3O_{13}O_9O_{10}O_{11}$	$I=\langle t\rangle$
γ_3	$\{\infty,0,1,-1,2,-2,-3,-4,5\}$	$O_1O_3O_9O_{27}O_4O_{25}O_7O_{28}O_{26}$	$I=\langle t\rangle$
γ_4	$\{\infty,0,1,-1,2,-2,-3,-4,6\}$	$O_1O_4O_8O_{10}O_5O_{26}O_{14}O_{21}O_{11}$	$I=\langle t\rangle$
γ_5	$\{\infty,0,1,-1,2,-2,-3,-4,7\}$	$O_1O_5O_6O_{17}O_6O_{17}O_{16}O_1O_5$	$Z_2=\langle (7-t)/(1+5t)\rangle$
γ_6	$\{\infty,0,1,-1,2,-2,-3,-4,8\}$	$O_1O_6O_{14}O_{13}O_7O_{18}O_{15}O_{12}O_2$	$I=\langle t\rangle$
γ_7	$\{\infty,0,1,-1,2,-2,-3,4,5\}$	$O_2O_3O_{13}O_{27}O_{10}O_{30}O_{25}O_{23}O_{26}$	$I=\langle t\rangle$
γ_8	$\{\infty,0,1,-1,2,-2,-3,4,-5\}$	$O_2O_2O_9O_4O_9O_2O_4O_2O_5$	$Z_2=\langle -(t+1)\rangle$
γ_9	$\{\infty,0,1,-1,2,-2,-3,4,-6\}$	$O_2O_3O_8O_{26}O_{17}O_{13}O_{20}O_{25}O_{11}$	$I=\langle t\rangle$
γ_{10}	$\{\infty,0,1,-1,2,-2,-3,4,7\}$	$O_2O_5O_7O_{26}O_{19}O_{13}O_{10}O_{11}O_{29}$	$I=\langle t\rangle$
γ_{11}	$\{\infty,0,1,-1,2,-2,-3,4,-7\}$	$O_2O_4O_6O_{11}O_{11}O_2O_{11}O_4O_{11}$	$I=\langle t\rangle$
γ_{12}	$\{\infty,0,1,-1,2,-2,-3,4,8\}$	$O_2O_6O_{14}O_{11}O_{14}O_2O_9O_{20}O_5$	$I=\langle t\rangle$
γ_{13}	$\{\infty,0,1,-1,2,-2,-3,4,9\}$	$O_2O_7O_{16}O_{20}O_{14}O_{13}O_{23}O_{21}O_{26}$	$I=\langle t\rangle$
γ_{14}	$\{\infty,0,1,-1,2,-2,-3,5,6\}$	$O_3O_4O_{10}O_3O_4O_{27}O_{20}O_{10}O_{20}$	$Z_2=\langle (t+3)/(t-1)\rangle$
γ_{15}	$\{\infty,0,1,-1,2,-2,-3,5,-6\}$	$O_3O_3O_{11}O_{20}O_{11}O_{26}O_{20}O_{26}O_{31}$	$Z_2=\langle -(t+1)\rangle$
γ_{16}	$\{\infty,0,1,-1,2,-2,-3,5,7\}$	$O_3O_5O_5O_{23}O_{11}O_3O_{23}O_{29}O_{11}$	$Z_2=\langle (2t)/(t-2)\rangle$
γ_{17}	$\{\infty,0,1,-1,2,-2,-3,5,-7\}$	$O_3O_4O_6O_9O_{12}O_{23}O_8O_7O_{20}$	$I=\langle t\rangle$
γ_{18}	$\{\infty,0,1,-1,2,-2,-3,5,8\}$	$O_3O_6O_{19}O_{23}O_9O_{10}O_{14}O_{21}O_3$	$I=\langle t\rangle$
γ_{19}	$\{\infty,0,1,-1,2,-2,-3,5,-8\}$	$O_3O_5O_{18}O_{22}O_3O_{26}O_{22}O_{28}O_{26}$	$Z_2=\langle -(t+3)/(t+1)\rangle$
γ_{20}	$\{\infty,0,1,-1,2,-2,-3,5,9\}$	$O_3O_7O_{14}O_{22}O_8O_{10}O_{24}O_{21}O_{20}$	$I=\langle t\rangle$
γ_{21}	$\{\infty,0,1,-1,2,-2,-3,8,-9\}$	$O_6O_6O_{13}O_{13}O_{13}O_{13}O_{13}O_{13}O_6$	$S_3=\langle -(t+1),(t+3)/(2t+3)\rangle$
γ_{22}	$\{\infty,0,1,-1,2,-2,6,-4,5\}$	$O_8O_{10}O_9O_{23}O_9O_{10}O_{16}O_{23}O_8$	$Z_2=\langle (t+2)/(4t-1)\rangle$

γ_{23}	$\{\infty,0,1,-1,2,-2,6,-4,-6\}$	$O_8O_{12}O_9O_{16}O_{14}O_{14}O_{15}O_{22}O_9$	$I=\langle t\rangle$
γ_{24}	$\{\infty,0,1,-1,2,-2,6,4,5\}$	$O_9O_{10}O_{13}O_{28}O_{21}O_{13}O_{17}O_{23}O_{21}$	$I=\langle t\rangle$
γ_{25}	$\{\infty,0,1,-1,2,-2,6,4,-5\}$	$O_9O_{11}O_9O_{11}O_{20}O_9O_{11}O_{20}O_{20}$	$Z_3=\langle 4/(2-t)\rangle$
γ_{26}	$\{\infty,0,1,-1,2,-2,6,4,-8\}$	$O_9O_{10}O_{14}O_{23}O_{17}O_{14}O_{18}O_{22}O_{25}$	$I=\langle t\rangle$
γ_{27}	$\{\infty,0,1,-1,2,-2,6,5,-5\}$	$O_{10}O_{11}O_{17}O_{26}O_{23}O_{28}O_{26}O_{27}O_{20}$	$I=\langle t\rangle$
γ_{28}	$\{\infty,0,1,-1,2,-2,6,5,-8\}$	$O_{10}O_{10}O_{18}O_{21}O_{28}O_{28}O_{21}O_{25}O_{25}$	$Z_2=\langle 1/9t\rangle$
γ_{29}	$\{\infty,0,1,-1,2,-2,6,-5,8\}$	$O_{11}O_{11}O_{18}O_{29}O_{21}O_{21}O_{29}O_{26}O_{26}$	$Z_2=\langle 1/9t\rangle$
γ_{30}	$\{\infty,0,1,-1,2,-2,-4,-5,8\}$	$O_{13}O_{14}O_{18}O_{19}O_{21}O_{18}O_{13}O_{27}O_{28}$	$I=\langle t\rangle$
γ_{31}	$\{\infty,0,1,-1,2,-3,9,4,-4\}$	$O_{20}O_{21}O_{27}O_{22}O_{23}O_{21}O_{22}O_{23}O_{20}$	$I=\langle t\rangle$
γ_{32}	$\{\infty,0,1,-1,2,-3,9,4,-6\}$	$O_{20}O_{24}O_{26}O_{23}O_{25}O_{23}O_{20}O_{25}O_{26}$	$Z_2=\langle (2-t)/(1+2t)\rangle$
γ_{33}	$\{\infty,0,1,-1,2,-3,4,5,-4\}$	$O_{27}O_{27}O_{27}O_{27}O_{27}O_{27}O_{27}O_{27}O_{27}$	$D_9=\langle (4+t)/(2-t),(1-t)\rangle$

Table 7: Distinct decads on $PG(1,19)$

Type	The decads	Types of nonads	Stabilizer
D_1	$\{\infty,0,1,-1,2,-2,-3,-4,3,4\}$	$\gamma_1\gamma_2\gamma_1\gamma_2\gamma_7\gamma_7\gamma_{24}\gamma_{24}\gamma_{28}\gamma_{29}$	$Z_2=\langle -t\rangle$
D_2	$\{\infty,0,1,-1,2,-2,-3,-4,3,5\}$	$\gamma_1\gamma_3\gamma_2\gamma_8\gamma_2\gamma_8\gamma_3\gamma_{10}\gamma_1\gamma_{10}$	$Z_2=\langle (t-5)/(t-1)\rangle$
D_3	$\{\infty,0,1,-1,2,-2,-3,-4,3,6\}$	$\gamma_1\gamma_4\gamma_3\gamma_9\gamma_{24}\gamma_{10}\gamma_{27}\gamma_{30}\gamma_{28}\gamma_{29}$	$I=\langle t\rangle$
D_4	$\{\infty,0,1,-1,2,-2,-3,-4,3,7\}$	$\gamma_1\gamma_5\gamma_4\gamma_{11}\gamma_9\gamma_{12}\gamma_{27}\gamma_{13}\gamma_3\gamma_{16}$	$I=\langle t\rangle$
D_5	$\{\infty,0,1,-1,2,-2,-3,-4,3,8\}$	$\gamma_1\gamma_6\gamma_5\gamma_{12}\gamma_{21}\gamma_{13}\gamma_{30}\gamma_6\gamma_{24}\gamma_{10}$	$I=\langle t\rangle$
D_6	$\{\infty,0,1,-1,2,-2,-3,-4,3,9\}$	$\gamma_1\gamma_6\gamma_6\gamma_6\gamma_{30}\gamma_6\gamma_{30}\gamma_{30}\gamma_{30}\gamma_1$	$V_4=\langle -(t+1),(1-t)/1+2t\rangle$
D_7	$\{\infty,0,1,-1,2,-2,-3,-4,4,5\}$	$\gamma_2\gamma_3\gamma_7\gamma_2\gamma_{33}\gamma_{14}\gamma_7\gamma_3\gamma_{27}\gamma_{27}$	$Z_2=\langle 4/t\rangle$
D_8	$\{\infty,0,1,-1,2,-2,-3,-4,4,6\}$	$\gamma_2\gamma_4\gamma_8\gamma_{17}\gamma_{14}\gamma_6\gamma_{10}\gamma_{12}\gamma_{18}\gamma_{10}$	$I=\langle t\rangle$
D_9	$\{\infty,0,1,-1,2,-2,-3,-4,4,-6\}$	$\gamma_2\gamma_2\gamma_9\gamma_{17}\gamma_7\gamma_9\gamma_{21}\gamma_{17}\gamma_7\gamma_{11}$	$Z_2=\langle -(t+2)\rangle$
D_{10}	$\{\infty,0,1,-1,2,-2,-3,-4,4,7\}$	$\gamma_2\gamma_5\gamma_{10}\gamma_{17}\gamma_{27}\gamma_{18}\gamma_{24}\gamma_{22}\gamma_4\gamma_{16}$	$I=\langle t\rangle$
D_{11}	$\{\infty,0,1,-1,2,-2,-3,-4,4,-7\}$	$\gamma_2\gamma_3\gamma_{11}\gamma_{17}\gamma_{27}\gamma_{15}\gamma_9\gamma_{25}\gamma_{14}\gamma_{15}$	$I=\langle t\rangle$
D_{12}	$\{\infty,0,1,-1,2,-2,-3,-4,4,8\}$	$\gamma_2\gamma_6\gamma_{12}\gamma_{23}\gamma_2\gamma_{20}\gamma_6\gamma_{23}\gamma_{20}\gamma_{12}$	$Z_2=\langle (2-t)/(1+5t)\rangle$
D_{13}	$\{\infty,0,1,-1,2,-2,-3,-4,4,-8\}$	$\gamma_2\gamma_4\gamma_{12}\gamma_{23}\gamma_{31}\gamma_{17}\gamma_{13}\gamma_{23}\gamma_{22}\gamma_{25}$	$I=\langle t\rangle$

D_{14}	$\{\infty,0,1,-1,2,-2,-3,-4,4,9\}$	$\gamma_2\gamma_6\gamma_{13}\gamma_{23}\gamma_{31}\gamma_{18}\gamma_{30}\gamma_{26}\gamma_{24}\gamma_9$	$\mathrm{I}=\langle t\rangle$
D_{15}	$\{\infty,0,1,-1,2,-2,-3,-4,4,-9\}$	$\gamma_2\gamma_5\gamma_6\gamma_{24}\gamma_3\gamma_{19}\gamma_{30}\gamma_{18}\gamma_{26}\gamma_4$	$\mathrm{I}=\langle t\rangle$
D_{16}	$\{\infty,0,1,-1,2,-2,-3,-4,5,6\}$	$\gamma_3\gamma_4\gamma_{14}\gamma_{22}\gamma_7\gamma_8\gamma_7\gamma_{13}\gamma_{24}\gamma_9$	$\mathrm{I}=\langle t\rangle$
D_{17}	$\{\infty,0,1,-1,2,-2,-3,-4,5,-7\}$	$\gamma_3\gamma_3\gamma_3\gamma_3\gamma_{17}\gamma_{17}\gamma_{17}\gamma_{17}\gamma_{32}\gamma_{32}$	$\mathrm{V}_4=\langle -(\mathrm{t}+1),(5-\mathrm{t})/(1+\mathrm{t})\rangle$
D_{18}	$\{\infty,0,1,-1,2,-2,-3,-4,5,8\}$	$\gamma_3\gamma_6\gamma_{18}\gamma_{18}\gamma_7\gamma_3\gamma_{28}\gamma_6\gamma_{28}\gamma_7$	$\mathrm{Z}_2=\langle (t+1)/(9t-1)\rangle$
D_{19}	$\{\infty,0,1,-1,2,-2,-3,-4,5,-8\}$	$\gamma_3\gamma_4\gamma_{19}\gamma_{26}\gamma_{31}\gamma_{14}\gamma_{32}\gamma_{20}\gamma_{28}\gamma_{27}$	$\mathrm{I}=\langle t\rangle$
D_{20}	$\{\infty,0,1,-1,2,-2,-3,-4,5,9\}$	$\gamma_3\gamma_6\gamma_{20}\gamma_{23}\gamma_{31}\gamma_{17}\gamma_{26}\gamma\ \gamma\ \gamma$	$\mathrm{I}=\langle t\rangle$
D_{21}	$\{\infty,0,1,-1,2,-2,-3,-4,6,7\}$	$\gamma_4\gamma_5\gamma_8\gamma_{17}\gamma_{26}\gamma_{12}\gamma_9\gamma_{23}\gamma_6\gamma_{12}$	$\mathrm{I}=\langle t\rangle$
D_{22}	$\{\infty,0,1,-1,2,-2,-3,-4,6,8\}$	$\gamma_4\gamma_6\gamma_{11}\gamma_4\gamma_{10}\gamma_{10}\gamma_{29}\gamma_6\gamma_{29}\gamma_{11}$	$\mathrm{Z}_2=\langle -(t+1)/(5t+1)\rangle$
D_{23}	$\{\infty,0,1,-1,2,-2,-3,-4,6,-8\}$	$\gamma_4\gamma_4\gamma_4\gamma_{20}\gamma_{20}\gamma_4\gamma_{15}\gamma_{20}\gamma_{20}\gamma_{15}$	$\mathrm{V}_4=\langle -(t+2),(1-t)/(1+t)\rangle$
D_{24}	$\{\infty,0,1,-1,2,-2,-3,-4,7,-9\}$	$\gamma_5\gamma_5\gamma_5\gamma_5\gamma_5\gamma_5\gamma_5\gamma_5\gamma_5\gamma_5$	$\mathrm{D}_{10}=\langle 2/(t+2),-2/t\rangle$
D_{25}	$\{\infty,0,1,-1,2,-2,-3,-4,8,9\}$	$\gamma_6\gamma_6\gamma_{17}\gamma_{23}\gamma_{13}\gamma_{17}\gamma_{19}\gamma_{23}\gamma_{13}\gamma_8$	$\mathrm{Z}_2=\langle -(t+2)\rangle$
D_{26}	$\{\infty,0,1,-1,2,-2,-3,-4,5,-6\}$	$\gamma_7\gamma_9\gamma_{15}\gamma_9\gamma_{27}\gamma_{27}\gamma_7\gamma_{32}\gamma_{32}\gamma_{15}$	$\mathrm{Z}_2=\langle (t-1)/(5t-1)\rangle$
D_{27}	$\{\infty,0,1,-1,2,-2,-3,-4,5,7\}$	$\gamma_7\gamma_{10}\gamma_{16}\gamma_{10}\gamma_7\gamma_{10}\gamma_7\gamma_7\gamma_{16}\gamma_{10}$	$\mathrm{V}_4=\langle (t-4)/(4t-1),2t/(t-2)\rangle$
D_{28}	$\{\infty,0,1,-1,2,-2,-3,-4,5,8\}$	$\gamma_7\gamma_{12}\gamma_{18}\gamma_{30}\gamma_{27}\gamma_{26}\gamma_{27}\gamma_{26}\gamma_{31}\gamma_{19}$	$\mathrm{I}=\langle t\rangle$
D_{29}	$\{\infty,0,1,-1,2,-2,-3,-4,5,9\}$	$\gamma_7\gamma_{13}\gamma_{20}\gamma_{13}\gamma_{31}\gamma_{20}\gamma_7\gamma_{32}\gamma_{31}\gamma_{32}$	$\mathrm{Z}_2=\langle (t+2)/(t-1)\rangle$
D_{30}	$\{\infty,0,1,-1,2,-2,-3,-4,-5,-7\}$	$\gamma_8\gamma_{11}\gamma_8\gamma_{12}\gamma_{11}\gamma_{25}\gamma_{12}\gamma_{11}\gamma_8\gamma_{12}$	$\mathrm{Z}_3=\langle (t-4)/(6t+6)\rangle$
D_{31}	$\{\infty,0,1,-1,2,-2,-3,-4,-6,7\}$	$\gamma_9\gamma_{10}\gamma_{19}\gamma_{20}\gamma_{19}\gamma_{30}\gamma_{30}\gamma_{20}\gamma_9\gamma_{10}$	$\mathrm{Z}_2=\langle (t-7)/(3t-1)\rangle$
D_{32}	$\{\infty,0,1,-1,2,-2,-3,-4,-6,8\}$	$\gamma_9\gamma_{12}\gamma_{18}\gamma_{20}\gamma_{29}\gamma_{26}\gamma_{10}\gamma_{25}\gamma_{32}\gamma_6$	$\mathrm{I}=\langle t\rangle$
D_{33}	$\{\infty,0,1,-1,2,-2,-3,-4,-6,9\}$	$\gamma_9\gamma_{13}\gamma_{20}\gamma_{22}\gamma_{32}\gamma_{26}\gamma_{24}\gamma_{31}\gamma_{28}\gamma_{27}$	$\mathrm{I}=\langle t\rangle$
D_{34}	$\{\infty,0,1,-1,2,-2,-3,-4,7,8\}$	$\gamma_{10}\gamma_{12}\gamma_{12}\gamma_{13}\gamma_{15}\gamma_{18}\gamma_{13}\gamma_{18}\gamma_{15}\gamma_{10}$	$\mathrm{Z}_2=\langle -(t+2)/(7t+1)\rangle$
D_{35}	$\{\infty,0,1,-1,2,-2,-3,-4,7,9\}$	$\gamma_{10}\gamma_{13}\gamma_{10}\gamma_{13}\gamma_{27}\gamma_{30}\gamma_{30}\gamma_{27}\gamma_{29}\gamma_{29}$	$\mathrm{Z}_2=\langle -2/t\rangle$
D_{36}	$\{\infty,0,1,-1,2,-2,-3,5,6,-7\}$	$\gamma_{14}\gamma_{17}\gamma_{14}\gamma_{18}\gamma_{18}\gamma_{17}\gamma_{31}\gamma_{20}\gamma_{20}\gamma_{31}$	$\mathrm{Z}_2=\langle (1-t)/(1+t)\rangle$
D_{37}	$\{\infty,0,1,-1,2,-2,-3,5,-6,7\}$	$\gamma_{15}\gamma_{16}\gamma_{19}\gamma_{16}\gamma_{31}\gamma_{29}\gamma_{19}\gamma_{31}\gamma_{29}\gamma_{15}$	$\mathrm{Z}_2=\langle (2-t)/(1+8t)\rangle$
D_{38}	$\{\infty,0,1,-1,2,-2,-3,5,-7,8\}$	$\gamma_{17}\gamma_{18}\gamma_{17}\gamma_{18}\gamma_{22}\gamma_{23}\gamma_{22}\gamma_{23}\gamma_{20}\gamma_{20}$	$\mathrm{Z}_2=\langle 2/t\rangle$
D_{39}	$\{\infty,0,1,-1,2,-2,-3,5,8,-9\}$	$\gamma_{18}\gamma_{18}\gamma_{21}\gamma_{30}\gamma_{24}\gamma_{24}\gamma_{24}\gamma_{30}\gamma_{30}\gamma_{18}$	$\mathrm{Z}_3=\langle (t+3)/(2t+3)\rangle$
D_{40}	$\{\infty,0,1,-1,2,-2,6,-4,5,9\}$	$\gamma_{22}\gamma_{23}\gamma_{26}\gamma_{23}\gamma_{26}\gamma_{23}\gamma_{26}\gamma_{22}\gamma_{26}\gamma_{23}$	$\mathrm{V}_4=\langle (t+2)/(4t-1),-(t+4)/(9t+1)\rangle$
D_{41}	$\{\infty,0,1,-1,2,-2,6,-4,-6,9\}$	$\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{23}$	$\mathrm{D}_5=\langle (2+t)/(9-t),(t-6)/(t-1)\rangle$

D_{42}	$\{\infty,0,1,-1,2,-2,6,4,5,-5\}$	$\gamma_{24}\gamma_{25}\gamma_{27}\gamma_{24}\gamma_{27}\gamma_{31}\gamma_{24}\gamma_{27}\gamma_{31}\gamma_{31}$	$Z_2=\langle 4/(2-t)\rangle$
D_{43}	$\{\infty,0,1,-1,2,-2,6,4,5,-8\}$	$\gamma_{24}\gamma_{26}\gamma_{28}\gamma_{30}\gamma_{24}\gamma_{30}\gamma_{30}\gamma_{30}\gamma_{26}\gamma_{28}$	$Z_2=\langle (t-5)/(5t-1)\rangle$
D_{44}	$\{\infty,0,1,-1,2,-3,9,4,-4,5\}$	$\gamma_{31}\gamma_{31}\gamma_{31}\gamma_{31}\gamma_{31}\gamma_{31}\gamma_{31}\gamma_{31}\gamma_{31}\gamma_{31}$	$Z_9=\langle (4+t)/(2-t)\rangle$

5-The partition of $PG(1,19)$

Each decad D_i , and its complement D_i^c i partition $PG(1,19)$.The stabilizer G_{D_i} of the decad D_i also fixes the complement D_i^c. In Table 8, all D_i^c are listed with their types of the nonads. Also the projective equation from each D_j to its equivalent decad D_i^c is given[1].

Table 8: Classification of the complements of the decads in $PG(1,19)$

$D^c{}_i$	Types of nonads	D_j	Projective equation
$D^c{}_1$	$\gamma_{30}\gamma_{30}\gamma_{13}\gamma_{13}\gamma_{27}\gamma_{27}\gamma_{10}\gamma_{10}\gamma_{29}\gamma_{29}$	D_{35}	$(t+6)/(4t-1)$
D_2	$\gamma_6\gamma_6\gamma_4\gamma_{10}\gamma_4\gamma_{11}\gamma_{11}\gamma_{29}\gamma_{10}\gamma_{29}$	D_{22}	$(t+5)/(3t+4)$
D_3	$\gamma_{24}\gamma_{30}\gamma_3\gamma_1\gamma_{28}\gamma_9\gamma_4\gamma_{27}\gamma_{29}\gamma_{10}$	D_3	$(t+7)/(4t-1)$
D_4	$\gamma_5\gamma_1\gamma_{16}\gamma_{12}\gamma_3\gamma_4\gamma_9\gamma_{13}\gamma_{11}\gamma_{27}$	D_4	$(t+7)/(7t-1)$
D_5	$\gamma_{21}\gamma_6\gamma_5\gamma_{24}\gamma_{12}\gamma_6\gamma_1\gamma_{30}\gamma_{10}\gamma_{13}$	D_5	$(t+7)/(4t-1)$
D_6	$\gamma_6\gamma_1\gamma_6\gamma_{30}\gamma_1\gamma_6\gamma_{30}\gamma_{30}\gamma_6\gamma_{30}$	D_6	$(t+7)/(7t-1)$
D_7	$\gamma_{19}\gamma_{31}\gamma_{31}\gamma_{19}\gamma_{15}\gamma_{16}\gamma_{15}\gamma_{16}\gamma_{29}\gamma_{29}$	D_{37}	$(t+7)/(3t-4)$
D_8	$\gamma_2\gamma_{18}\gamma_{17}\gamma_8\gamma_{14}\gamma_4\gamma_{12}\gamma_{11}\gamma_{16}\gamma_{10}$	D_8	$(9-t)/(1+4t)$
D_9	$\gamma_{18}\gamma_{13}\gamma_{12}\gamma_{13}\gamma_{15}\gamma_{18}\gamma_{12}\gamma_{10}\gamma_{15}\gamma_{10}$	D_{34}	$(5-t)/(2+6t)$
D_{10}	$\gamma_5\gamma_{24}\gamma_{22}\gamma_{17}\gamma_2\gamma_{18}\gamma_4\gamma_{10}\gamma_6\gamma_{27}$	D_{10}	$(t-6)/(4t-1)$
D_{11}	$\gamma_3\gamma_9\gamma_{25}\gamma_{17}\gamma_2\gamma_{15}\gamma_{14}\gamma_{11}\gamma_{15}\gamma_{27}$	D_{11}	$(t-1)/(4t-1)$
D_{12}	$\gamma_6\gamma_{23}\gamma_{23}\gamma_{17}\gamma_{17}\gamma_{13}\gamma_8\gamma_6\gamma_{19}\gamma_{13}$	D_{25}	$(7+t)/(2-3t)$
D_{13}	$\gamma_{23}\gamma_2\gamma_{23}\gamma_{25}\gamma_{22}\gamma_{12}\gamma_{17}\gamma_4\gamma_{31}\gamma_{13}$	D_{13}	$(t-1)/(8t-1)$
D_{14}	$\gamma_{26}\gamma_2\gamma_{23}\gamma_9\gamma_{24}\gamma_{13}\gamma_{18}\gamma_6\gamma_{31}\gamma_{30}$	D_{14}	$(t-1)/(8t-1)$
D_{15}	$\gamma_{26}\gamma_{23}\gamma_6\gamma_3\gamma_5\gamma_{18}\gamma_2\gamma_4\gamma_{19}\gamma_{30}$	D_{15}	$(9-t)/(1+4t)$
D_{16}	$\gamma_{18}\gamma_{26}\gamma_{32}\gamma_{10}\gamma_{20}\gamma_{12}\gamma_{25}\gamma_9\gamma_{16}\gamma_{29}$	D_{32}	$(t+9)/(5t+6)$
D_{17}	$\gamma_4\gamma_{20}\gamma_{20}\gamma_{20}\gamma_4\gamma_{15}\gamma_{20}\gamma_4\gamma_{15}\gamma_4$	D_{23}	$(t-8)/(5t-3)$
D_{18}	$\gamma_{30}\gamma_{20}\gamma_{19}\gamma_{20}\gamma_9\gamma_9\gamma_{10}\gamma_{30}\gamma_{19}\gamma_{10}$	D_{31}	$(3-t)/(9+6t)$
D_{19}	$\gamma_{26}\gamma_3\gamma_{19}\gamma_{20}\gamma_{27}\gamma_{14}\gamma_{32}\gamma_{28}\gamma_{31}\gamma_4$	D_{19}	$-(t+5)/(8t+1)$

D_{20}	$\gamma_{23}\gamma_{3}\gamma_{20}\gamma_{20}\gamma_{13}\gamma_{17}\gamma_{26}\gamma_{24}\gamma_{31}\gamma_{6}$	D_{20}	$-(t+5)/(8t+1)$
D_{21}	$\gamma_{5}\gamma_{6}\gamma_{17}\gamma_{8}\gamma_{26}\gamma_{23}\gamma_{12}\gamma_{12}\gamma_{4}\gamma_{9}$	D_{21}	$(9-t)/(1+4t)$
D_{22}	$\gamma_{10}\gamma_{2}\gamma_{3}\gamma_{3}\gamma_{8}\gamma_{2}\gamma_{1}\gamma_{10}\gamma_{8}\gamma_{1}$	D_{2}	$(t-6)/(4t+5)$
D_{23}	$\gamma_{17}\gamma_{3}\gamma_{3}\gamma_{32}\gamma_{17}\gamma_{17}\gamma_{3}\gamma_{32}\gamma_{17}\gamma_{3}$	D_{17}	$(t-9)/(8t+6)$
D_{24}	$\gamma_{5}\gamma_{5}\gamma_{5}\gamma_{5}\gamma_{5}\gamma_{5}\gamma_{5}\gamma_{5}\gamma_{5}\gamma_{5}$	D_{24}	$(4-t)/(3+2t)$
D_{25}	$\gamma_{12}\gamma_{2}\gamma_{20}\gamma_{12}\gamma_{23}\gamma_{2}\gamma_{6}\gamma_{20}\gamma_{23}\gamma_{6}$	D_{12}	$(t-3)/(2t+7)$
D_{26}	$\gamma_{9}\gamma_{22}\gamma_{27}\gamma_{7}\gamma_{15}\gamma_{7}\gamma_{9}\gamma_{32}\gamma_{15}\gamma_{27}$	D_{26}	$(t+5)/(8t-1)$
D_{27}	$\gamma_{10}\gamma_{7}\gamma_{7}\gamma_{7}\gamma_{10}\gamma_{7}\gamma_{10}\gamma_{6}\gamma_{6}\gamma_{10}$	D_{27}	$(t+5)/(2t-6)$
D_{28}	$\gamma_{30}\gamma_{31}\gamma_{26}\gamma_{18}\gamma_{7}\gamma_{7}\gamma_{12}\gamma_{26}\gamma_{19}\gamma_{27}$	D_{28}	$(9-t)/(1+5t)$
D_{29}	$\gamma_{20}\gamma_{7}\gamma_{31}\gamma_{13}\gamma_{7}\gamma_{32}\gamma_{32}\gamma_{20}\gamma_{31}\gamma_{13}$	D_{29}	$(t-8)/(3t-1)$
D_{30}	$\gamma_{8}\gamma_{12}\gamma_{11}\gamma_{12}\gamma_{8}\gamma_{25}\gamma_{8}\gamma_{12}\gamma_{11}\gamma_{11}$	D_{30}	$(t-6)/(4t-1)$
D_{31}	$\gamma_{6}\gamma_{28}\gamma_{3}\gamma_{7}\gamma_{3}\gamma_{6}\gamma_{18}\gamma_{7}\gamma_{18}\gamma_{28}$	D_{18}	$(t-7)/(4t-5)$
D_{32}	$\gamma_{24}\gamma_{22}\gamma_{4}\gamma_{14}\gamma_{7}\gamma_{8}\gamma_{9}\gamma_{7}\gamma_{13}\gamma_{3}$	D_{16}	$-(t+8)/(4t+3)$
D_{33}	$\gamma_{31}\gamma_{9}\gamma_{22}\gamma_{27}\gamma_{28}\gamma_{26}\gamma_{20}\gamma_{32}\gamma_{13}\gamma_{24}$	D_{33}	$(t-7)/(8t-1)$
D_{34}	$\gamma_{21}\gamma_{2}\gamma_{9}\gamma_{2}\gamma_{7}\gamma_{11}\gamma_{17}\gamma_{7}\gamma_{17}\gamma_{9}$	D_{9}	$(7+t)/(9-3t)$
D_{35}	$\gamma_{24}\gamma_{7}\gamma_{2}\gamma_{1}\gamma_{28}\gamma_{29}\gamma_{2}\gamma_{7}\gamma_{24}\gamma_{1}$	D_{1}	$-(t+5)/(3t+4)$
D_{36}	$\gamma_{14}\gamma_{31}\gamma_{20}\gamma_{17}\gamma_{18}\gamma_{14}\gamma_{17}\gamma_{20}\gamma_{18}\gamma_{31}$	D_{36}	$-(t+4)/(2t+1)$
D_{37}	$\gamma_{2}\gamma_{3}\gamma_{33}\gamma_{7}\gamma_{27}\gamma_{2}\gamma_{27}\gamma_{3}\gamma_{7}\gamma_{14}$	D_{7}	$(t+5)/(8t-1)$
D_{38}	$\gamma_{18}\gamma_{20}\gamma_{23}\gamma_{22}\gamma_{17}\gamma_{17}\gamma_{23}\gamma_{20}\gamma_{18}\gamma_{22}$	D_{38}	$-(t+4)/(2t+1)$
D_{39}	$\gamma_{18}\gamma_{30}\gamma_{18}\gamma_{24}\gamma_{21}\gamma_{18}\gamma_{24}\gamma_{30}\gamma_{30}\gamma_{24}$	D_{39}	$(t+8)/(8t-1)$
D_{40}	$\gamma_{23}\gamma_{26}\gamma_{22}\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{26}\gamma_{22}\gamma_{26}\gamma_{26}$	D_{40}	$(t+8)/(8t-1)$
D_{41}	$\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{23}\gamma_{23}$	D_{41}	$(t-8)/(4t+7)$
D_{42}	$\gamma_{31}\gamma_{31}\gamma_{24}\gamma_{24}\gamma_{24}\gamma_{27}\gamma_{31}\gamma_{27}\gamma_{25}\gamma_{27}$	D_{42}	$(t+6)/(2t-1)$
D_{43}	$\gamma_{26}\gamma_{28}\gamma_{30}\gamma_{30}\gamma_{28}\gamma_{30}\gamma_{24}\gamma_{24}\gamma_{26}\gamma_{30}$	D_{43}	$-(t+9)/(2t+1)$
D_{44}	$\gamma_{31}\gamma_{33}\gamma_{31}\gamma_{31}\gamma_{31}\gamma_{31}\gamma_{31}\gamma_{31}\gamma_{31}\gamma_{31}$	D_{44}	$(t-7)/(8t-1)$

Amongst the 44 decads D_i there are sixteen of them which are not equivalent to their complements as shown in Table 8. So, the followings are deduced.

Theorem 5.1[5]: The projective line $PG(1,19)$ has

(i) twenty eight projectively distinct partitions into two equivalent decads;

(ii) sixteen projectively distinct partitions into two inequivalent decads.

They are given in Table9and Table 10with their stabilizer groups in $PGL(2,19)$ and the number of partitions of that type.

Table 9: Partition of $PG(1,19)$ into two equivalent decads

$\{D_i, D_i^c\}$	Stabilizer of the partition	Number
$\{D_3, D_3^c\}$	$Z_2 = \langle (t+7)/(4t-1) \rangle$	3420
$\{D_4, D_4^c\}$	$Z_2 = \langle (t+7)/(7t-1) \rangle$	3420
$\{D_5, D_5^c\}$	$Z_2 = \langle (t+7)/(4t-1) \rangle$	3420
$\{D_6, D_6^c\}$	$D_4 = \langle (t+7)/(7t-1), -(t+1) \rangle$	855
$\{D_8, D_8^c\}$	$Z_2 = \langle (9-t)/(1+4t) \rangle$	3420
$\{D_{10}, D_{10}^c\}$	$Z_2 = \langle (t-6)/(4t-1) \rangle$	3420
$\{D_{11}, D_{11}^c\}$	$Z_2 = \langle (t-1)/(4t-1) \rangle$	3420
$\{D_{13}, D_{13}^c\}$	$Z_2 = \langle (t-1)/(8t-1) \rangle$	3420
$\{D_{14}, D_{14}^c\}$	$Z_2 = \langle (t-1)/(8t-1) \rangle$	3420
$\{D_{15}, D_{15}^c\}$	$Z_2 = \langle (9-t)/(1+4t) \rangle$	3420
$\{D_{19}, D_{19}^c\}$	$Z_2 = \langle -(t+5)/(8t+1) \rangle$	3420
$\{D_{20}, D_{20}^c\}$	$Z_2 = \langle -(t+5)/(8t+1) \rangle$	3420
$\{D_{21}, D_{21}^c\}$	$Z_2 = \langle (9-t)/(1+4t) \rangle$	3420
$\{D_{24}, D_{24}^c\}$	$D_{20} = \langle (4-t)/(3+2t), -2/t \rangle$	171
$\{D_{26}, D_{26}^c\}$	$V_4 = \langle (t+5)/(8t-1), (t-1)/(5t-1) \rangle$	1710
$\{D_{27}, D_{27}^c\}$	$D_4 = \langle (t+5)/(2t-6), 2t/(t-1) \rangle$	855
$\{D_{28}, D_{28}^c\}$	$Z_2 = \langle (9-t)/(1+5t) \rangle$	3420
$\{D_{29}, D_{29}^c\}$	$V_4 = \langle (t-8)/(3t-1), (t+2)/(t-1) \rangle$	1710
$\{D_{30}, D_{30}^c\}$	$S_3 = \langle (t-6)/(4t-1), (t-6)/(6t+6) \rangle$	1140
$\{D_{33}, D_{33}^c\}$	$Z_2 = \langle (t-7)/(8t-1) \rangle$	3420
$\{D_{36}, D_{36}^c\}$	$V_4 = \langle -(t-4)/(2t+1), (1-t)/(1+t) \rangle$	1710
$\{D_{38}, D_{38}^c\}$	$V_4 = \langle -(t-4)/(2t+1), 2/t \rangle$	1710
$\{D_{39}, D_{39}^c\}$	$S_3 = \langle (t+8)/(8t-1), (t+3)/(2t+3) \rangle$	1140
$\{D_{40}, D_{40}^c\}$	$D_4 = \langle (t+5)/(3t+8), (t+2)/(4t-1) \rangle$	855

$\{D_{41}, D_{41}^c\}$	$D_{10} = \langle (t-8)/(4t+7), (t-6)/(t-1) \rangle$	342
$\{D_{42}, D_{42}^c\}$	$S_3 = \langle (t+6)/(2t-1), 4/(2-t) \rangle$	1140
$\{D_{43}, D_{43}^c\}$	$V_4 = \langle -(t+9)/(2t+1), (t-5)/(5t-1) \rangle$	1710
$\{D_{44}, D_{44}^c\}$	$D_9 = \langle (t-7)/(8t-1), (4+t)/(2-t) \rangle$	380

Table10: Partition of $PG(1,19)$ into two inequivalent decads

$\{D_i, D_i^c\}$	Stabilizer of the partition	Number
$\{D_1, D_1^c\}$	$Z_2 = \langle -t \rangle$	3420
$\{D_2, D_2^c\}$	$Z_2 = \langle (t-5)/(t-1) \rangle$	3420
$\{D_7, D_7^c\}$	$Z_2 = \langle 4/t \rangle$	3420
$\{D_9, D_9^c\}$	$Z_2 = \langle -(t+2) \rangle$	3420
$\{D_{12}, D_{12}^c\}$	$Z_2 = \langle (2-t)/(1+5t) \rangle$	3420
$\{D_{16}, D_{16}^c\}$	$I = \langle t \rangle$	6840
$\{D_{17}, D_{17}^c\}$	$V_4 = \langle -(t+2,)(5-t)/(1+t) \rangle$	1710
$\{D_{18}, D_{18}^c\}$	$Z_2 = \langle (t+1)/(9t-1) \rangle$	3420
$\{D_{22}, D_{22}^c\}$	$Z_2 = \langle -(t+1)/(5t+1) \rangle$	3420
$\{D_{23}, D_{23}^c\}$	$V_4 = \langle -(t+2), (1-t)/(1+t) \rangle$	1710
$\{D_{25}, D_{25}^c\}$	$Z_2 = \langle -(t+2) \rangle$	3420
$\{D_{31}, D_{31}^c\}$	$Z_2 = \langle (t-7)/(3t-1) \rangle$	3420
$\{D_{32}, D_{32}^c\}$	$I = \langle t \rangle$	6840
$\{D_{34}, D_{34}^c\}$	$Z_2 = \langle -(t+2)/(7t+1) \rangle$	3420
$\{D_{35}, D_{35}^c\}$	$Z_2 = \langle -2/(t) \rangle$	3420
$\{D_{37}, D_{37}^c\}$	$Z_2 = \langle (2-t)/(1+8t) \rangle$	3420

6-Splitting $PG(1,19)$ into Five Tetrads

As in Theorem 4.1[i], there are four types of tetrads H, E, N_1, N_2 on $PG(1,19)$. The third aim of this research is to find out if $PG(1,19)$ can split into five disjoint harmonic tetrads, five equianharmonic tetrads,and five tetrads of type N_1 or five tetrads of type N_2?

The answer is yes for each type as given below. Here the symbol $CR(a_i)$ refers to the cross-ratio of the set a_i **[5]**.

(i) Harmonic

$t_1 = \{\infty, 0, 1, -1\}$ $CR(t_1) = -1;$

$t_2 = \{2, -2, 3, -5\}$ $CR(t_2) = -1;$

$t_3 = \{-3, 4, -4, -6\}$ $CR(t_3) = 2;$

$t_4 = \{5, 7, 8, -8\}$ $CR(t_4) = 2;$

$t_5 = \{6,-7,9,-9\}$ $CR(t_5) = -9.$

(ii) Equianharmonic

$t_1 = \{\infty,0,1,-7\}$ $CR(t_1) = -7;$

$t_2 = \{-1,2,-2,5\}$ $CR(t_2) = -7;$

$t_3 = \{3-,3,-4,7\}$ $CR(t_3) = 8;$

$t_4 = \{4,8,9,-9\}$ $CR(t_4) = 8;$

$t_5 = \{5,6,-6,-8\}$ $CR(t_5) = -7.$

(iii) Tetrads of type N_1

$t_1 = \{\infty,0,1,-2\}$ $CR(t_1) = -2;$

$t_2 = \{-1,2,3,4\}$ $CR(t_2) = -6;$

$t_3 = \{-3,-4,5,-5\}$ $CR(t_3) = -8;$

$t_4 = \{6,-6,8,-8\}$ $CR(t_4) = 7;$

$t_5 = \{7,-7,9,-9\}$ $CR(t_5) = -8.$

(iii) Tetrads of type N_2

$t_1 = \{\infty,0,1,-3\}$ $CR(t_1) = 3;$

$t_2 = \{-1,2,-2,3\}$ $CR(t_2) = 6;$

$t_3 = \{4,-4,5,-5\}$ $CR(t_3) = 4;$

$t_4 = \{6,-6,8,-9\}$ $CR(t_4) = 6;$

$t_5 = \{7,-7,-8,9\}$ $CR(t_5) = 6.$

7-MDS Codes of Dimension Two

Maximum distance separable codes are at heart of combinatory and finite geometries. In their book [7] Mac Williams and Sloane describe MDS codes as "*one of the most fascinating chapters in all of coding theory*". In the following the relation between finite geometry and coding theory are explained[5, 7].

Let $v_1, v_2, \ldots, v_k$ be the rows of a generator matrix G for a projective $[n,k,d]_q$ -code and for $i = 1,2,\ldots,n$ define vectors u_i of $V(k,q)$ by the rule

$(u_i)_j = (v_j)_i$.

In other words, the j th coordinate of u_i is the i th coordinate of v_j; that is, u_i is column vector of G. For all $a \in V(k,q) \setminus \{0\}$ the vector $\sum_{j=1}^{k} a_j v_j$ has at most $n-d$ zero coordinates and so, for $i = 1,2,\ldots,n$,

$$\sum_{j=1}^{k} a_j (v_j)_i = 0$$

has at most $n-d$ solutions. Hence

$$\sum_{j=1}^{k} a_j (u_i)_j = 0.$$

So, this gives the following fundamental theorem.

Theorem 7.1[2] There exists a projective $[n,k,d]_q$ -code if and only if there exists an $(n,n-d)$ -arc in $PG(k-1,q)$.

From Theorem 7.1, the following is deduced.

Corollary 7.2[5]: If $k = 2$ and $n-d = 1$, then

(1) there is a one-to-one correspondence between n -sets in $PG(1,19)$ and projective $[n,2,n-1]_{19}$ -codes C.

(2) $d(C)$ ofthe code C is equal to $n-k+1$, thus the projective code C is MDS.

In Table 11, the MDS codes corresponding to the n -sets in $PG(1,19)$ and theparameter e of errors corrected are given.

Table 11: MDS code over $PG(1,19)$

n-Set	Tetrad	Pentad	hexad	heptad	octad	nonad	decad
MDS code	$[4,2,3]_{19}$	$[5,2,4]_{19}$	$[6,2,5]_{19}$	$[7,2,6]_{19}$	$[8,2,7]_{19}$	$[9,2,8]_{19}$	$[10,2,9]_{19}$
e	1	1	2	2	3	3	4

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