Iraqi Journal of Science

ISSN: 0067-2904

On Nano Generalized Semi Generalized Closed Sets

Qays Hatem Imran*

Department of Mathematics, College of Education for Pure Science, Al-Muthanna University, Samawah, Iraq

Abstract:

In this paper we introduced a new class of N-CS called Ngsg-CS and study their basic properties in nano topological spaces. We also introduce Ngsg-closure and Ngsg-interior and study some of their fundamental properties.

Keywords: Ngsg-CS, Ngsg-OS, Ngsg-closure and Ngsg-interior.

مجموعات النانو المعممة شبه المعممة المغلقة

قيس حاتم عمران*

قسم الرياضيات، كلية التربية للعلوم الصرفة، جامعة المثنى، السماوة، العراق

الخلاصة:

في هذا البحث قدمنا فئة جديدة من مجموعات النانو المغلقة تسمى بمجموعات النانو المعممة شبه المعممة المغلقة و دراسة خصائصها الأساسية في الفضاءات النانو التبولوجية. قدمنا أيضا انغلاق النانو المعممة شبه المعممة و مجموعة النقاط الداخلية النانو المعممة شبه المعممة و دراسة بعض خصائصها الأساسية.

1. Introduction

M. Lellis Thivagar and Carmel Richard [1] introduced nano topological space (or simply NTS) with respect to a subset X of a universe which is defined in terms of lower and upper approximations of X. He has also defined nano closed sets (briefly N-CS), nano interior and nano closure of a set. In 2014, Ng-CS was introduced by K. Bhuvaneswari and K. Mythili Gnanapriya [2]. K. Bhuvaneswari and A. Ezhilarasi [3] introduced the concept of Nsg-CS and Ngs-CS in NTS. The concept gsg-CS have been introduced and studied by M. Lellis et al [4] in classical topology. The purpose of this paper is to introduce the concept of Ngsg-CS and study their basic properties in NTS. We also introduce Ngsg-closure and Ngsg-interior and obtain some of its properties.

2. Preliminaries

Throughout this paper, $(\mathcal{U},\tau_{\mathcal{R}}(X))$ and $(\mathcal{V},\sigma_{\mathcal{R}}(Y))$ (or simply $\mathcal{U}$ and $\mathcal{V}$) always mean NTS on which no separation axioms are assumed unless otherwise mentioned. For a set $\mathcal{A}$ in a NTS $(\mathcal{U},\tau_{\mathcal{R}}(X))$, N$cl(\mathcal{A})$, N$int(\mathcal{A})$ and $\mathcal{A}^c = \mathcal{U} - \mathcal{A}$ denote the nano closure of $\mathcal{A}$, the nano interior of $\mathcal{A}$ and the nano complement of $\mathcal{A}$ respectively.

Definition 2.1:[5] Let $\mathcal{U}$ be a non-empty finite set of objects called the universe and $\mathcal{R}$ be an equivalence relation on $\mathcal{U}$ named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with in another. The pair $(\mathcal{U},\mathcal{R})$ is called the approximation space.

Remark 2.2:[5] Let $(\mathcal{U},\mathcal{R})$ be an approximation space and $X \subseteq \mathcal{U}$. Then:

i. The lower approximation of X with respect to $\mathcal{R}$ is the set of all objects, which can be for certain classified as X with respect to $\mathcal{R}$ and it is denoted by $L_{\mathcal{R}}(X)$. That is, $L_{\mathcal{R}}(X) = \cup\{\mathcal{R}(x): \mathcal{R}(x) \subseteq X, x \in \mathcal{U}\}$, where $\mathcal{R}(x)$ denotes the equivalence class determined by x.

*Email: qays.imran@mu.edu.iq

ii. The upper approximation of X with respect to $\mathcal{R}$ is the set of all objects, which can be possibly classified as X with respect to $\mathcal{R}$ and it is denoted by $U_{\mathcal{R}}(X)$. That is, $U_{\mathcal{R}}(X) = \bigcup\{\mathcal{R}(x): \mathcal{R}(x) \cap X \neq \phi, x \in \mathcal{U}\}$.

iii. The boundary region of X with respect to $\mathcal{R}$ is the set of all objects, which can be classified neither as X nor as not X with respect to $\mathcal{R}$ and it is denoted by $B_{\mathcal{R}}(X)$. That is, $B_{\mathcal{R}}(X) = U_{\mathcal{R}}(X) - L_{\mathcal{R}}(X)$.

Proposition 2.3:[6] If $(\mathcal{U}, \mathcal{R})$ is an approximation space and $X, Y \subseteq \mathcal{U}$. Then:

i. $L_{\mathcal{R}}(X) \subseteq X \subseteq U_{\mathcal{R}}(X)$.

ii. $L_{\mathcal{R}}(\phi) = U_{\mathcal{R}}(\phi) = \phi$ and $L_{\mathcal{R}}(\mathcal{U}) = U_{\mathcal{R}}(\mathcal{U}) = \mathcal{U}$.

iii. $U_{\mathcal{R}}(X \cup Y) = U_{\mathcal{R}}(X) \cup U_{\mathcal{R}}(Y)$.

iv. $U_{\mathcal{R}}(X \cap Y) \subseteq U_{\mathcal{R}}(X) \cap U_{\mathcal{R}}(Y)$.

v. $L_{\mathcal{R}}(X \cup Y) \supseteq L_{\mathcal{R}}(X) \cup L_{\mathcal{R}}(Y)$.

vi. $L_{\mathcal{R}}(X \cap Y) = L_{\mathcal{R}}(X) \cap L_{\mathcal{R}}(Y)$.

vii. $L_{\mathcal{R}}(X) \subseteq L_{\mathcal{R}}(Y)$ and $U_{\mathcal{R}}(X) \subseteq U_{\mathcal{R}}(Y)$ whenever $X \subseteq Y$.

viii. $U_{\mathcal{R}}(X^c) = (L_{\mathcal{R}}(X))^c$ and $L_{\mathcal{R}}(X^c) = (U_{\mathcal{R}}(X))^c$.

ix. $U_{\mathcal{R}}U_{\mathcal{R}}(X) = L_{\mathcal{R}}U_{\mathcal{R}}(X) = U_{\mathcal{R}}(X)$.

x. $L_{\mathcal{R}}L_{\mathcal{R}}(X) = U_{\mathcal{R}}L_{\mathcal{R}}(X) = L_{\mathcal{R}}(X)$.

Definition 2.4:[1] Let $\mathcal{U}$ be the universe, $\mathcal{R}$ be an equivalence relation on $\mathcal{U}$ and $\tau_{\mathcal{R}}(X) = \{\phi, \mathcal{U}, L_{\mathcal{R}}(X), U_{\mathcal{R}}(X), B_{\mathcal{R}}(X)\}$ where $X \subseteq \mathcal{U}$. Then by proposition (2.3), $\tau_{\mathcal{R}}(X)$ satisfies the following axioms:

i. $\phi, \mathcal{U} \in \tau_{\mathcal{R}}(X)$.

ii. The union of the elements of any subcollection of $\tau_{\mathcal{R}}(X)$ is in $\tau_{\mathcal{R}}(X)$.

iii. The intersection of the elements of any finite subcollection of $\tau_{\mathcal{R}}(X)$ is in $\tau_{\mathcal{R}}(X)$.

That is, $\tau_{\mathcal{R}}(X)$ is a topology on $\mathcal{U}$ called the nano topology on $\mathcal{U}$ with respect to X and the pair $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ is called a nano topological space (or simply NTS). The elements of $\tau_{\mathcal{R}}(X)$ are called as nano open sets (briefly N-OS).

Remark 2.5:[1] Let $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ be a NTS with respect to X where $X \subseteq \mathcal{U}$ and $\mathcal{R}$ be an equivalence relation on $\mathcal{U}$. Then $\mathcal{U}/\mathcal{R}$ denotes the family of equivalence classes of $\mathcal{U}$ by $\mathcal{R}$.

Definition 2.6:[1] A subset $\mathcal{A}$ of a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ is said to be:

i. a nano semi-open set (briefly Ns-OS) if $\mathcal{A} \subseteq \mathrm{N}cl(\mathrm{N}int(\mathcal{A}))$ and a nano semi-closed set (briefly Ns-CS) if $\mathrm{N}int(\mathrm{N}cl(\mathcal{A})) \subseteq \mathcal{A}$. The nano semi-closure of a set $\mathcal{A}$ of a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ is the intersection of all Ns-CS that contain $\mathcal{A}$ and is denoted by $\mathrm{N}scl(\mathcal{A})$.

ii. a nano α-open set (briefly Nα-OS) if $\mathcal{A} \subseteq \mathrm{N}int(\mathrm{N}cl(\mathrm{N}int(\mathcal{A})))$ and a nano α-closed set (briefly Nα-CS) if $\mathrm{N}cl(\mathrm{N}int(\mathrm{N}cl(\mathcal{A}))) \subseteq \mathcal{A}$. The nano α-closure of a set $\mathcal{A}$ of a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ is the intersection of all Nα-CS that contain $\mathcal{A}$ and is denoted by $\mathrm{N}\alpha cl(\mathcal{A})$.

Definition 2.7: A subset $\mathcal{A}$ of a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ is said to be:

i. a nano generalized closed set (briefly Ng-CS) [2] if $\mathrm{N}cl(\mathcal{A}) \subseteq \mathcal{M}$ whenever $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{M}$ is a N-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$. The complement of a N$g$-CS is a N$g$-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

ii. a nano αg-closed set (briefly Nαg-CS) [7] if $\mathrm{N}\alpha cl(\mathcal{A}) \subseteq \mathcal{M}$ whenever $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{M}$ is a N-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$. The complement of a N$\alpha g$-CS is a N$\alpha g$-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

iii. a nano $g\alpha$-closed set (briefly N$g\alpha$-CS) [7] if $\mathrm{N}\alpha cl(\mathcal{A}) \subseteq \mathcal{M}$ whenever $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{M}$ is a Nα-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$. The complement of a N$g\alpha$-CS is a N$g\alpha$-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

iv. a nano sg-closed set (briefly Nsg-CS) [3] if $\mathrm{N}scl(\mathcal{A}) \subseteq \mathcal{M}$ whenever $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{M}$ is a Ns-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$. The complement of a N$sg$-CS is a N$sg$-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

v. a nano gs-closed set (briefly Ngs-CS) [3] if $\mathrm{N}scl(\mathcal{A}) \subseteq \mathcal{M}$ whenever $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{M}$ is a N-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$. The complement of a N$gs$-CS is a N$gs$-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

Proposition 2.8:[1,2] In a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, then the following statements hold and the converse of each statements are not true:

i. Every N-OS (resp. N-CS) is a Nα-OS (resp. Nα-CS).

ii. Every N-OS (resp. N-CS) is a Ng-OS (resp. Ng-CS).

iii. Every Nα-OS (resp. Nα-CS) is a Ns-OS (resp. Ns-CS).

Proposition 2.9:[7] In a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, then the following statements hold and the converse of each statements are not true:

i. Every Ng-OS (resp. Ng-CS) is a Nαg-OS (resp. Nαg-CS).

ii. Every Nα-OS (resp. Nα-CS) is a N$g\alpha$-OS (resp. N$g\alpha$-CS).
iii. Every N$g\alpha$-OS (resp. N$g\alpha$-CS) is a Nαg-OS (resp. Nαg-CS).

Proposition 2.10:[3] In a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, then the following statements hold and the converse of each statements are not true:

i. Every Ng-OS (resp. Ng-CS) is a Ngs-OS (resp. Ngs-CS).
ii. Every Ns-OS (resp. Ns-CS) is a Nsg-OS (resp. Nsg-CS).
iii. Every Nsg-OS (resp. Nsg-CS) is a Ngs-OS (resp. Ngs-CS).
iv. Every N$g\alpha$-OS (resp. N$g\alpha$-CS) is a Ngs-OS (resp. Ngs-CS).

3. Nano Generalized sg-Closed Sets

In this section we introduce and study the nano generalized sg-closed sets and some of its properties.

Definition 3.1: A subset $\mathcal{A}$ of a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ is said to be a nano generalized sg-closed set (briefly Ngsg-CS) if N$cl(\mathcal{A}) \subseteq \mathcal{M}$ whenever $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{M}$ is a Nsg-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$. The family of all N$gsg$-CS of a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ is denoted by Ngsg-C$(\mathcal{U}, X)$.

Proposition 3.2: In a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, the following statements are true:

i. Every N-CS is a Ngsg-CS.
ii. Every Ngsg-CS is a Ng-CS.

Proof: (i) Let $\mathcal{A}$ be a N-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ and let $\mathcal{M}$ be a Nsg-OS in $\mathcal{U}$ such that $\mathcal{A} \subseteq \mathcal{M}$. Then N$cl(\mathcal{A}) = \mathcal{A} \subseteq \mathcal{M}$. Therefore $\mathcal{A}$ is a Ngsg-CS.

(ii) Let $\mathcal{A}$ be a Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ and let $\mathcal{M}$ be a N-OS in $\mathcal{U}$ such that $\mathcal{A} \subseteq \mathcal{M}$. Since every N-OS is a Nsg-OS, we have N$cl(\mathcal{A}) \subseteq \mathcal{M}$. Therefore $\mathcal{A}$ is a Ng-CS.

The converse of the above proposition need not be true which can be seen from the following examples.

Example 3.3: Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/\mathcal{R} = \{\{a\}, \{d\}, \{b, c\}\}$ and $X = \{a, c\}$.
Let $\tau_{\mathcal{R}}(X) = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, \mathcal{U}\}$ be a NTS. Then the set $\{b, c\}$ is a Ngsg-CS but not N-CS.

Example 3.4: Let $\mathcal{U} = \{a, b, c, d, e\}$ with $\mathcal{U}/\mathcal{R} = \{\{d\}, \{a, b\}, \{c, e\}\}$ and $X = \{a, d\}$.
Let $\tau_{\mathcal{R}}(X) = \{\phi, \{d\}, \{a, b\}, \{a, b, d\}, \mathcal{U}\}$ be a NTS. Then the set $\{a, c, d\}$ is a Ng-CS but not Ngsg-CS.

Proposition 3.5: In a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, the following statements are true:

i. Every Ngsg-CS is a Nαg-CS.
ii. Every Ngsg-CS is a N$g\alpha$-CS.
iii. Every Ngsg-CS is a Nsg-CS.
iv. Every Ngsg-CS is a Ngs-CS.

Proof:

i. Let $\mathcal{A}$ be a Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ and let $\mathcal{M}$ be a N-OS in $\mathcal{U}$ such that $\mathcal{A} \subseteq \mathcal{M}$. Since every N-OS is a Nsg-OS, we have N$\alpha cl(\mathcal{A}) \subseteq$ N$cl(\mathcal{A}) \subseteq \mathcal{M}$ implies N$\alpha cl(\mathcal{A}) \subseteq \mathcal{M}$. Therefore $\mathcal{A}$ is a Nαg-CS.

ii. Let $\mathcal{A}$ be a Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ and let $\mathcal{M}$ be a Nα-OS in $\mathcal{U}$ such that $\mathcal{A} \subseteq \mathcal{M}$. Since every N$\alpha$-OS is a N$s$-OS which is a N$sg$-OS, we have N$\alpha cl(\mathcal{A}) \subseteq$ N$cl(\mathcal{A}) \subseteq \mathcal{M}$ implies N$\alpha cl(\mathcal{A}) \subseteq \mathcal{M}$. Therefore $\mathcal{A}$ is a N$g\alpha$-CS.

iii. Let $\mathcal{A}$ be a Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ and let $\mathcal{M}$ be a Ns-OS in $\mathcal{U}$ such that $\mathcal{A} \subseteq \mathcal{M}$. Since every N$s$-OS is a N$sg$-OS, we have N$scl(\mathcal{A}) \subseteq$ N$cl(\mathcal{A}) \subseteq \mathcal{M}$ implies N$scl(\mathcal{A}) \subseteq \mathcal{M}$. Therefore $\mathcal{A}$ is a Nsg-CS.

iv. Let $\mathcal{A}$ be a Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ and let $\mathcal{M}$ be a N-OS in $\mathcal{U}$ such that $\mathcal{A} \subseteq \mathcal{M}$. Since every N-OS is a Nsg-OS, we have N$scl(\mathcal{A}) \subseteq$ N$cl(\mathcal{A}) \subseteq \mathcal{M}$ implies N$scl(\mathcal{A}) \subseteq \mathcal{M}$. Therefore $\mathcal{A}$ is a Ngs-CS.

The converse of the above proposition need not be true as shown in the following examples.

Example 3.6: Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/\mathcal{R} = \{\{a\}, \{c\}, \{b, d\}\}$ and $X = \{a, b\}$.
Let $\tau_{\mathcal{R}}(X) = \{\phi, \{a\}, \{b, d\}, \{a, b, d\}, \mathcal{U}\}$ be a NTS. Then the set $\{a, c\}$ is a N$g\alpha$-CS and hence Nαg-CS but not Ngsg-CS.

Example 3.7: Let $\mathcal{U} = \{p, q, r, s\}$ with $\mathcal{U}/\mathcal{R} = \{\{p\}, \{r\}, \{q, s\}\}$ and $X = \{p, q\}$.
Let $\tau_{\mathcal{R}}(X) = \{\phi, \{p\}, \{q, s\}, \{p, q, s\}, \mathcal{U}\}$ be a NTS. Then the set $\{p\}$ is a Nsg-CS and hence Ngs-CS but not Ngsg-CS.

Remark 3.8: The Ngsg-CS are independent of Nα-CS and Ns-CS.

Definition 3.9: A subset $\mathcal{A}$ of a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ is said to be a nano generalized sg-open set (briefly Ngsg-OS) iff $\mathcal{U} - \mathcal{A}$ is a Ngsg-CS. The family of all Ngsg-OS of a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ is denoted by Ngsg-O$(\mathcal{U}, X)$.

Proposition 3.10: Let $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ be a NTS. If $\mathcal{A}$ is a N-OS, then it is a Ngsg-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

Proof: Let $\mathcal{A}$ be a N-OS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, then $\mathcal{U} - \mathcal{A}$ is a N-CS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$. By proposition (3.2) part (i), $\mathcal{U} - \mathcal{A}$ is a Ngsg-CS. Hence $\mathcal{A}$ is a Ngsg-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

Proposition 3.11: Let $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ be a NTS. If $\mathcal{A}$ is a Ngsg-OS, then it is a Ng-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

Proof: Let $\mathcal{A}$ be a Ngsg-OS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, then $\mathcal{U} - \mathcal{A}$ is a Ngsg-CS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$. By proposition (3.2) part (ii), $\mathcal{U} - \mathcal{A}$ is a Ng-CS. Hence $\mathcal{A}$ is a Ng-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

Proposition 3.12: In a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, the following statements are true:

- **i.** Every Ngsg-OS is a Nαg-OS and N$g\alpha$-OS.
- **ii.** Every Ngsg-OS is a Nsg-OS and Ngs-OS.

Proof: Similar to above proposition.

Theorem 3.13: If $\mathcal{A}$ and $\mathcal{B}$ are Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, then $\mathcal{A} \cup \mathcal{B}$ is a Ngsg-CS.

Proof: Let $\mathcal{A}$ and $\mathcal{B}$ be two Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ and let $\mathcal{M}$ be any Nsg-OS in $\mathcal{U}$ such that $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{B} \subseteq \mathcal{M}$. Then we have $\mathcal{A} \cup \mathcal{B} \subseteq \mathcal{M}$. Since $\mathcal{A}$ and $\mathcal{B}$ are Ngsg-CS in $\mathcal{U}$, N$cl(\mathcal{A}) \subseteq \mathcal{M}$ and N$cl(\mathcal{B}) \subseteq \mathcal{M}$. Now, N$cl(\mathcal{A} \cup \mathcal{B}) =$ N$cl(\mathcal{A}) \cup$N$cl(\mathcal{B}) \subseteq \mathcal{M}$ and so N$cl(\mathcal{A} \cup \mathcal{B}) \subseteq \mathcal{M}$. Hence $\mathcal{A} \cup \mathcal{B}$ is a Ngsg-CS in $\mathcal{U}$.

Theorem 3.14: If a set $\mathcal{A}$ is Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, then N$cl(\mathcal{A}) - \mathcal{A}$ contains no non-empty N-CS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

Proof: Let $\mathcal{A}$ be a Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ and let $\mathcal{F}$ be any N-CS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ such that $\mathcal{F} \subseteq$ N$cl(\mathcal{A}) - \mathcal{A}$. Since $\mathcal{A}$ is a Ngsg-CS, we have N$cl(\mathcal{A}) \subseteq \mathcal{U} - \mathcal{F}$. This implies $\mathcal{F} \subseteq \mathcal{U} -$ N$cl(\mathcal{A})$. Then $\mathcal{F} \subseteq$ N$cl(\mathcal{A}) \cap (\mathcal{U} -$ N$cl(\mathcal{A})) = \phi$. Thus, $\mathcal{F} = \phi$. Hence N$cl(\mathcal{A}) - \mathcal{A}$ contains no non-empty N-CS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

Theorem 3.15: A set $\mathcal{A}$ is Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ iff N$cl(\mathcal{A}) - \mathcal{A}$ contains no non-empty Nsg-CS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

Proof: Let $\mathcal{A}$ be a Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ and let $\mathcal{D}$ be any Nsg-CS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ such that $\mathcal{D} \subseteq$ N$cl(\mathcal{A}) - \mathcal{A}$. Since $\mathcal{A}$ is a Ngsg-CS, we have N$cl(\mathcal{A}) \subseteq \mathcal{U} - \mathcal{D}$. This implies $\mathcal{D} \subseteq \mathcal{U} -$ N$cl(\mathcal{A})$. Then $\mathcal{D} \subseteq$ N$cl(\mathcal{A}) \cap (\mathcal{U} -$ N$cl(\mathcal{A})) = \phi$. Thus, $\mathcal{D}$ is empty.

Conversely, suppose that N$cl(\mathcal{A}) - \mathcal{A}$ contains no non-empty Nsg-CS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$. Let $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{M}$ is Nsg-OS. If N$cl(\mathcal{A}) \subseteq \mathcal{M}$ then N$cl(\mathcal{A}) \cap (\mathcal{U} - \mathcal{M})$ is non-empty. Since N$cl(\mathcal{A})$ is N-CS and $\mathcal{U} - \mathcal{M}$ is Nsg-CS, we have N$cl(\mathcal{A}) \cap (\mathcal{U} - \mathcal{M})$ is non-empty Nsg-CS of N$cl(\mathcal{A}) - \mathcal{A}$ which is a contradiction. Therefore N$cl(\mathcal{A}) \nsubseteq \mathcal{M}$. Hence $\mathcal{A}$ is a Ngsg-CS.

Theorem 3.16: If $\mathcal{A}$ is a Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ and $\mathcal{A} \subseteq \mathcal{B} \subseteq$ N$cl(\mathcal{A})$, then $\mathcal{B}$ is a Ngsg-CS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

Proof: Suppose that $\mathcal{A}$ is a Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$. Let $\mathcal{M}$ be a Nsg-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ such that $\mathcal{B} \subseteq \mathcal{M}$. Then $\mathcal{A} \subseteq \mathcal{M}$. Since $\mathcal{A}$ is a Ngsg-CS, it follows that N$cl(\mathcal{A}) \subseteq \mathcal{M}$. Now, $\mathcal{B} \subseteq$ N$cl(\mathcal{A})$ implies N$cl(\mathcal{B}) \subseteq$ N$cl($N$cl(\mathcal{A})) =$ N$cl(\mathcal{A})$. Thus, N$cl(\mathcal{B}) \subseteq \mathcal{M}$. Hence $\mathcal{B}$ is a Ngsg-CS.

Theorem 3.17: Let $\mathcal{A} \subseteq \mathcal{V} \subseteq \mathcal{U}$ and if $\mathcal{A}$ is a Ngsg-CS in $\mathcal{U}$ then $\mathcal{A}$ is a Ngsg-CS relative to $\mathcal{V}$.

Proof: $\mathcal{A} \subseteq \mathcal{V} \cap \mathcal{M}$ where $\mathcal{M}$ is a Nsg-OS in $\mathcal{U}$. Then $\mathcal{A} \subseteq \mathcal{M}$ and hence N$cl(\mathcal{A}) \subseteq \mathcal{M}$. This implies that $\mathcal{V} \cap$N$cl(\mathcal{A}) \subseteq \mathcal{V} \cap \mathcal{M}$. Thus $\mathcal{A}$ is a Ngsg-CS relative to $\mathcal{V}$.

Proposition 3.18: If $\mathcal{A}$ is a Nsg-OS and a Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, then $\mathcal{A}$ is a N-CS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

Proof: Suppose that $\mathcal{A}$ is a Nsg-OS and a Ngsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, then N$cl(\mathcal{A}) \subseteq \mathcal{A}$ and since $\mathcal{A} \subseteq$ N$cl(\mathcal{A})$. Thus, N$cl(\mathcal{A}) = \mathcal{A}$. Hence $\mathcal{A}$ is a N-CS.

Theorem 3.19: For each $x \in \mathcal{U}$ either $\{x\}$ is a Nsg-CS or $\mathcal{U} - \{x\}$ is a Ngsg-CS in $\mathcal{U}$.

Proof: If $\{x\}$ is not a Nsg-CS in $\mathcal{U}$ then $\mathcal{U} - \{x\}$ is not a Nsg-OS and the only Nsg-OS containing $\mathcal{U} - \{x\}$ is the space $\mathcal{U}$ itself. Therefore N$cl(\mathcal{U} - \{x\}) \subseteq \mathcal{U}$ and so $\mathcal{U} - \{x\}$ is a Ngsg-CS in $\mathcal{U}$.

Theorem 3.20: If $\mathcal{A}$ and $\mathcal{B}$ are Ngsg-OS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, then $\mathcal{A} \cap \mathcal{B}$ is a Ngsg-OS.

Proof: Let $\mathcal{A}$ and $\mathcal{B}$ be Ngsg-OS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$. Then $\mathcal{U} - \mathcal{A}$ and $\mathcal{U} - \mathcal{B}$ are Ngsg-CS. By theorem (3.13), $(\mathcal{U} - \mathcal{A}) \cup (\mathcal{U} - \mathcal{B})$ is a Ngsg-CS. Since $(\mathcal{U} - \mathcal{A}) \cup (\mathcal{U} - \mathcal{B}) = \mathcal{U} - (\mathcal{A} \cap \mathcal{B})$. Hence $\mathcal{A} \cap \mathcal{B}$ is a Ngsg-OS.

Theorem 3.21: A set $\mathcal{A}$ is Ngsg-OS iff $\mathcal{C} \subseteq$ N$int(\mathcal{A})$ where $\mathcal{C}$ is a Ngsg-CS and $\mathcal{C} \subseteq \mathcal{A}$.

Proof: Suppose that $\mathcal{C} \subseteq \mathrm{N}int(\mathcal{A})$ where $\mathcal{C}$ is a $\mathrm{N}gsg$-CS and $\mathcal{C} \subseteq \mathcal{A}$. Then $\mathcal{U} - \mathcal{A} \subseteq \mathcal{U} - \mathcal{C}$ and $\mathcal{U} - \mathcal{C}$ is a $\mathrm{N}sg$-OS by proposition (3.12) part (ii). Now, $\mathrm{N}cl(\mathcal{U} - \mathcal{A}) = \mathcal{U} - \mathrm{N}int(\mathcal{A}) \subseteq \mathcal{U} - \mathcal{C}$. Then $\mathcal{U} - \mathcal{A}$ is a $\mathrm{N}gsg$-CS. Hence $\mathcal{A}$ is a $\mathrm{N}gsg$-OS.

Conversely, let $\mathcal{A}$ be a $\mathrm{N}gsg$-OS and $\mathcal{C}$ be a $\mathrm{N}gsg$-CS and $\mathcal{C} \subseteq \mathcal{A}$. Then $\mathcal{U} - \mathcal{A} \subseteq \mathcal{U} - \mathcal{C}$. Since $\mathcal{U} - \mathcal{A}$ is a $\mathrm{N}gsg$-CS and $\mathcal{U} - \mathcal{C}$ is a $\mathrm{N}sg$-OS, we have $\mathrm{N}cl(\mathcal{U} - \mathcal{A}) \subseteq \mathcal{U} - \mathcal{C}$. Then $\mathcal{C} \subseteq \mathrm{N}int(\mathcal{A})$.

Theorem 3.22: If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{U}$ where $\mathcal{A}$ is a $\mathrm{N}gsg$-OS relative to $\mathcal{B}$ and $\mathcal{B}$ is a $\mathrm{N}gsg$-OS in $\mathcal{U}$, then $\mathcal{A}$ is a $\mathrm{N}gsg$-OS in $\mathcal{U}$.

Proof: Let $\mathcal{F}$ be a $\mathrm{N}sg$-CS in $\mathcal{U}$ and suppose that $\mathcal{F} \subseteq \mathcal{A}$. Then $\mathcal{F} = \mathcal{F} \cap \mathcal{B}$ is a $\mathrm{N}sg$-CS in $\mathcal{B}$. But $\mathcal{A}$ is a $\mathrm{N}gsg$-OS relative to $\mathcal{B}$. Therefore $\mathcal{F} \subseteq \mathrm{N}int_{\mathcal{B}}(\mathcal{A})$. Since $\mathrm{N}int_{\mathcal{B}}(\mathcal{A})$ is a N-OS relative to $\mathcal{B}$. We have $\mathcal{F} \subseteq \mathcal{M} \cap \mathcal{B} \subseteq \mathcal{A}$, for some N-OS $\mathcal{M}$ in $\mathcal{U}$. Since $\mathcal{B}$ is a $\mathrm{N}gsg$-OS in $\mathcal{U}$, we have $\mathcal{F} \subseteq \mathrm{N}int(\mathcal{B}) \subseteq \mathcal{B}$. Therefore $\mathcal{F} \subseteq \mathrm{N}int(\mathcal{B}) \cap \mathcal{M} \subseteq \mathcal{B} \cap \mathcal{M} \subseteq \mathcal{A}$. It follows that $\mathcal{F} \subseteq \mathrm{N}int(\mathcal{A})$. Thus $\mathcal{A}$ is a $\mathrm{N}gsg$-OS in $\mathcal{U}$.

Theorem 3.23: If $\mathcal{A}$ is a $\mathrm{N}gsg$-OS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ and $\mathrm{N}int(\mathcal{A}) \subseteq \mathcal{B} \subseteq \mathcal{A}$, then $\mathcal{B}$ is a $\mathrm{N}gsg$-OS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

Proof: Suppose that $\mathcal{A}$ is a $\mathrm{N}gsg$-OS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ and $\mathrm{N}int(\mathcal{A}) \subseteq \mathcal{B} \subseteq \mathcal{A}$. Then $\mathcal{U} - \mathcal{A}$ is a $\mathrm{N}gsg$-CS and $\mathcal{U} - \mathcal{A} \subseteq \mathcal{U} - \mathcal{B} \subseteq \mathrm{N}cl(\mathcal{U} - \mathcal{A})$. Then $\mathcal{U} - \mathcal{B}$ is a $\mathrm{N}gsg$-CS by theorem (3.16). Hence, $\mathcal{B}$ is a $\mathrm{N}gsg$-OS.

Theorem 3.24: For a subset $\mathcal{A}$ of a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, the following statements are equivalent:

- **i.** $\mathcal{A}$ is a $\mathrm{N}gsg$-CS.
- **ii.** $\mathrm{N}cl(\mathcal{A}) - \mathcal{A}$ contains no non-empty $\mathrm{N}sg$-CS.
- **iii.** $\mathrm{N}cl(\mathcal{A}) - \mathcal{A}$ is a $\mathrm{N}gsg$-OS.

Proof: Follows from theorem (3.15) and theorem (3.17).

Remark 3.25: The following diagram shows the relation between the different types of N-CS:

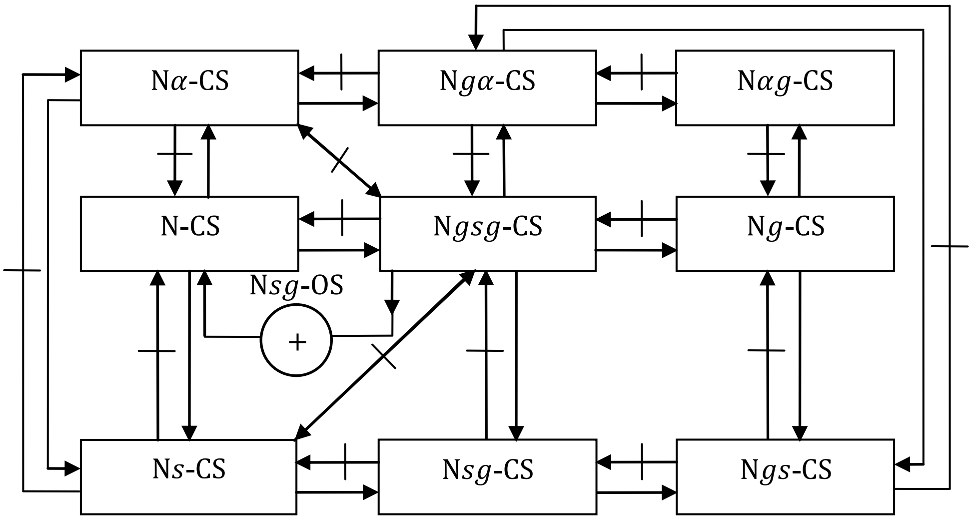

Diagram (3.1)

4. Nano gsg-Closure and Nano gsg-Interior

We introduce nano gsg-closure and nano gsg-interior and obtain some of its properties in this section.

Definition 4.1: The intersection of all $\mathrm{N}gsg$-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ containing $\mathcal{A}$ is called nano gsg-closure of $\mathcal{A}$ and is denoted by $\mathrm{N}gsg\text{-}cl(\mathcal{A})$, $\mathrm{N}gsg\text{-}cl(\mathcal{A}) = \cap\{\mathcal{B}: \mathcal{A} \subseteq \mathcal{B}, \mathcal{B}$ is a $\mathrm{N}gsg$-CS$\}$.

Definition 4.2: The union of all $\mathrm{N}gsg$-OS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ contained in $\mathcal{A}$ is called nano gsg-interior of $\mathcal{A}$ and is denoted by $\mathrm{N}gsg\text{-}int(\mathcal{A})$, $\mathrm{N}gsg\text{-}int(\mathcal{A}) = \cup\{\mathcal{B}: \mathcal{A} \supseteq \mathcal{B}, \mathcal{B}$ is a $\mathrm{N}gsg$-OS$\}$.

Proposition 4.3: Let $\mathcal{A}$ be any set in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$. Then the following properties hold:

- **i.** $\mathrm{N}gsg\text{-}int(\mathcal{A}) = \mathcal{A}$ iff $\mathcal{A}$ is a $\mathrm{N}gsg$-OS.
- **ii.** $\mathrm{N}gsg\text{-}cl(\mathcal{A}) = \mathcal{A}$ iff $\mathcal{A}$ is a $\mathrm{N}gsg$-CS.
- **iii.** $\mathrm{N}gsg\text{-}int(\mathcal{A})$ is the largest $\mathrm{N}gsg$-OS contained in $\mathcal{A}$.
- **iv.** $\mathrm{N}gsg\text{-}cl(\mathcal{A})$ is the smallest $\mathrm{N}gsg$-CS containing $\mathcal{A}$.

Proof: (i), (ii), (iii) and (iv) are obvious.

Proposition 4.4: Let $\mathcal{A}$ be any set in a NTS $(\mathcal{U},\tau_{\mathcal{R}}(X))$. Then the following properties hold:

- **i.** $\text{N}gsg\text{-}int(\mathcal{U}-\mathcal{A})=\mathcal{U}-(\text{N}gsg\text{-}cl(\mathcal{A}))$,
- **ii.** $\text{N}gsg\text{-}cl(\mathcal{U}-\mathcal{A})=\mathcal{U}-(\text{N}gsg\text{-}int(\mathcal{A}))$.

Proof: (i) By definition, $\text{N}gsg\text{-}cl(\mathcal{A})=\bigcap\{\mathcal{B}:\mathcal{A}\subseteq\mathcal{B},\mathcal{B}$ is a $\text{N}gsg$-CS$\}$

$$\begin{aligned}\mathcal{U}-(\text{N}gsg\text{-}cl(\mathcal{A}))&=\mathcal{U}-\bigcap\{\mathcal{B}:\mathcal{A}\subseteq\mathcal{B},\mathcal{B}\text{ is a N}gsg\text{-CS}\}\\&=\bigcup\{\mathcal{U}-\mathcal{B}:\mathcal{A}\subseteq\mathcal{B},\mathcal{B}\text{ is a N}gsg\text{-CS}\}\\&=\bigcup\{\mathcal{M}:\mathcal{U}-\mathcal{A}\supseteq\mathcal{M},\mathcal{M}\text{ is a N}gsg\text{-OS}\}\\&=\text{N}gsg\text{-}int(\mathcal{U}-\mathcal{A}).\end{aligned}$$

(ii) The proof is similar to (i).

Theorem 4.5: Let $\mathcal{A}$ and $\mathcal{B}$ be two sets in a NTS $(\mathcal{U},\tau_{\mathcal{R}}(X))$. Then the following properties hold:

- **i.** $\text{N}gsg\text{-}cl(\phi)=\phi$, $\text{N}gsg\text{-}cl(\mathcal{U})=\mathcal{U}$.
- **ii.** $\mathcal{A}\subseteq\text{N}gsg\text{-}cl(\mathcal{A})$.
- **iii.** $\mathcal{A}\subseteq\mathcal{B}\Longrightarrow\text{N}gsg\text{-}cl(\mathcal{A})\subseteq\text{N}gsg\text{-}cl(\mathcal{B})$.
- **iv.** $\text{N}gsg\text{-}cl(\mathcal{A}\cap\mathcal{B})\subseteq\text{N}gsg\text{-}cl(\mathcal{A})\cap\text{N}gsg\text{-}cl(\mathcal{B})$.
- **v.** $\text{N}gsg\text{-}cl(\mathcal{A}\cup\mathcal{B})=\text{N}gsg\text{-}cl(\mathcal{A})\cup\text{N}gsg\text{-}cl(\mathcal{B})$.
- **vi.** $\text{N}gsg\text{-}cl(\text{N}gsg\text{-}cl(\mathcal{A}))=\text{N}gsg\text{-}cl(\mathcal{A})$.

Proof: (i) and (ii) are obvious.

(iii) By part (ii), $\mathcal{B}\subseteq\text{N}gsg\text{-}cl(\mathcal{B})$. Since $\mathcal{A}\subseteq\mathcal{B}$, we have $\mathcal{A}\subseteq\text{N}gsg\text{-}cl(\mathcal{B})$. But $\text{N}gsg\text{-}cl(\mathcal{B})$ is a $\text{N}gsg$-CS. Thus $\text{N}gsg\text{-}cl(\mathcal{B})$ is a $\text{N}gsg$-CS containing $\mathcal{A}$. Since $\text{N}gsg\text{-}cl(\mathcal{A})$ is the smallest $\text{N}gsg$-CS containing $\mathcal{A}$, we have $\text{N}gsg\text{-}cl(\mathcal{A})\subseteq\text{N}gsg\text{-}cl(\mathcal{B})$. Hence, $\mathcal{A}\subseteq\mathcal{B}\Longrightarrow\text{N}gsg\text{-}cl(\mathcal{A})\subseteq\text{N}gsg\text{-}cl(\mathcal{B})$.

(iv) We know that $\mathcal{A}\cap\mathcal{B}\subseteq\mathcal{A}$ and $\mathcal{A}\cap\mathcal{B}\subseteq\mathcal{B}$. Therefore, by part (iii), $\text{N}gsg\text{-}cl(\mathcal{A}\cap\mathcal{B})\subseteq\text{N}gsg\text{-}cl(\mathcal{A})$ and $\text{N}gsg\text{-}cl(\mathcal{A}\cap\mathcal{B})\subseteq\text{N}gsg\text{-}cl(\mathcal{B})$. Hence $\text{N}gsg\text{-}cl(\mathcal{A}\cap\mathcal{B})\subseteq\text{N}gsg\text{-}cl(\mathcal{A})\cap\text{N}gsg\text{-}cl(\mathcal{B})$.

(v) Since $\mathcal{A}\subseteq\mathcal{A}\cup\mathcal{B}$ and $\mathcal{B}\subseteq\mathcal{A}\cup\mathcal{B}$, it follows from part (iii) that $\text{N}gsg\text{-}cl(\mathcal{A})\subseteq\text{N}gsg\text{-}cl(\mathcal{A}\cup\mathcal{B})$ and $\text{N}gsg\text{-}cl(\mathcal{B})\subseteq\text{N}gsg\text{-}cl(\mathcal{A}\cup\mathcal{B})$. Hence $\text{N}gsg\text{-}cl(\mathcal{A})\cup\text{N}gsg\text{-}cl(\mathcal{B})\subseteq\text{N}gsg\text{-}cl(\mathcal{A}\cup\mathcal{B})$ **(1)**

Since $\text{N}gsg\text{-}cl(\mathcal{A})$ and $\text{N}gsg\text{-}cl(\mathcal{B})$ are $\text{N}gsg$-CS, $\text{N}gsg\text{-}cl(\mathcal{A})\cup\text{N}gsg\text{-}cl(\mathcal{B})$ is also $\text{N}gsg$-CS by theorem (3.13). Also $\mathcal{A}\subseteq\text{N}gsg\text{-}cl(\mathcal{A})$ and $\mathcal{B}\subseteq\text{N}gsg\text{-}cl(\mathcal{B})$ implies that $\mathcal{A}\cup\mathcal{B}\subseteq\text{N}gsg\text{-}cl(\mathcal{A})\cup\text{N}gsg\text{-}cl(\mathcal{B})$. Thus $\text{N}gsg\text{-}cl(\mathcal{A})\cup\text{N}gsg\text{-}cl(\mathcal{B})$ is a $\text{N}gsg$-CS containing $\mathcal{A}\cup\mathcal{B}$. Since $\text{N}gsg\text{-}cl(\mathcal{A}\cup\mathcal{B})$ is the smallest $\text{N}gsg$-CS containing $\mathcal{A}\cup\mathcal{B}$, we have $\text{N}gsg\text{-}cl(\mathcal{A}\cup\mathcal{B})\subseteq\text{N}gsg\text{-}cl(\mathcal{A})\cup\text{N}gsg\text{-}cl(\mathcal{B})$ **(2)**

From (1) and (2), we have $\text{N}gsg\text{-}cl(\mathcal{A}\cup\mathcal{B})=\text{N}gsg\text{-}cl(\mathcal{A})\cup\text{N}gsg\text{-}cl(\mathcal{B})$.

(vi) Since $\text{N}gsg\text{-}cl(\mathcal{A})$ is a $\text{N}gsg$-CS, we have by proposition (4.3) part (ii), $\text{N}gsg\text{-}cl(\text{N}gsg\text{-}cl(\mathcal{A}))=\text{N}gsg\text{-}cl(\mathcal{A})$.

Theorem 4.6: Let $\mathcal{A}$ and $\mathcal{B}$ be two sets in a NTS $(\mathcal{U},\tau_{\mathcal{R}}(X))$. Then the following properties hold:

- **i.** $\text{N}gsg\text{-}int(\phi)=\phi$, $\text{N}gsg\text{-}int(\mathcal{U})=\mathcal{U}$.
- **ii.** $\text{N}gsg\text{-}int(\mathcal{A})\subseteq\mathcal{A}$.
- **iii.** $\mathcal{A}\subseteq\mathcal{B}\Longrightarrow\text{N}gsg\text{-}int(\mathcal{A})\subseteq\text{N}gsg\text{-}int(\mathcal{B})$.
- **iv.** $\text{N}gsg\text{-}int(\mathcal{A}\cap\mathcal{B})=\text{N}gsg\text{-}int(\mathcal{A})\cap\text{N}gsg\text{-}int(\mathcal{B})$.
- **v.** $\text{N}gsg\text{-}int(\mathcal{A}\cup\mathcal{B})\supseteq\text{N}gsg\text{-}int(\mathcal{A})\cup\text{N}gsg\text{-}int(\mathcal{B})$.
- **vi.** $\text{N}gsg\text{-}int(\text{N}gsg\text{-}int(\mathcal{A}))=\text{N}gsg\text{-}int(\mathcal{A})$.

Proof: (i), (ii), (iii), (iv), (v) and (vi) are obvious.

Definition 4.7: A NTS $(\mathcal{U},\tau_{\mathcal{R}}(X))$ is said to be a nano $T_{\frac{1}{2}}$-space (briefly $\text{N}T_{\frac{1}{2}}$-space) if every $\text{N}g$-CS in it is a N-CS.

Definition 4.8: A NTS $(\mathcal{U},\tau_{\mathcal{R}}(X))$ is said to be a nano T_{gsg}-space (briefly $\text{N}T_{gsg}$-space) if every $\text{N}gsg$-CS in it is a N-CS.

Proposition 4.9: Every $\text{N}T_{\frac{1}{2}}$-space is a $\text{N}T_{gsg}$-space.

Proof: Let $(\mathcal{U},\tau_{\mathcal{R}}(X))$ be a $\text{N}T_{\frac{1}{2}}$-space and let $\mathcal{A}$ be a $\text{N}gsg$-CS in $\mathcal{U}$. Then $\mathcal{A}$ is a $\text{N}g$-CS, by proposition (3.2) part (ii). Since $(\mathcal{U},\tau_{\mathcal{R}}(X))$ is a $\text{N}T_{\frac{1}{2}}$-space, then $\mathcal{A}$ is a N-CS in $\mathcal{U}$. Hence $(\mathcal{U},\tau_{\mathcal{R}}(X))$ is a $\text{N}T_{gsg}$-space.

The following example shows that the converse of the above proposition not be true.

Example 4.10: Let $\mathcal{U}=\{x,y,z\}$ with $\mathcal{U}/\mathcal{R}=\{\{x\},\{y,z\}\}$ and $X=\{x,z\}$.

Let $\tau_{\mathcal{R}}(X)=\{\phi,\{x\},\{y,z\},\mathcal{U}\}$ be a NTS. Then $(\mathcal{U},\tau_{\mathcal{R}}(X))$ is a $\text{N}T_{gsg}$-space but not $\text{N}T_{\frac{1}{2}}$-space.

Theorem 4.11: For a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$, the following statements are equivalent:

- **i.** $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ is a NT_{gsg}-space.
- **ii.** Every singleton of a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ is either Nsg-CS or N-OS.

Proof: (i) ⟹(ii) Assume that for some $x \in \mathcal{U}$ the set $\{x\}$ is not a Nsg-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$. Then the only N$sg$-OS containing $\mathcal{U} - \{x\}$ is the space $\mathcal{U}$ itself and $\mathcal{U} - \{x\}$ is a Ngsg-CS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$. By assumption $\mathcal{U} - \{x\}$ is a N-CS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ or equivalently $\{x\}$ is a N-OS.

(ii) ⟹(i) Let $\mathcal{A}$ be a Ngsg-CS in $(\mathcal{U}, \tau_{\mathcal{R}}(X))$ and let $x \in$ N$cl(\mathcal{A})$. By assumption $\{x\}$ is either Nsg-CS or N-OS.

Case (1). Suppose $\{x\}$ is a Nsg-CS. If $x \notin \mathcal{A}$ then N$cl(\mathcal{A}) - \mathcal{A}$ contains a non-empty Nsg-CS $\{x\}$ which is a contradiction to theorem (3.17). Therefore $x \in \mathcal{A}$.

Case (2). Suppose $\{x\}$ is a N-OS. Since $x \in$ N$cl(\mathcal{A})$, $\{x\} \cap \mathcal{A} \neq \phi$ and therefore N$cl(\mathcal{A}) \subseteq \mathcal{A}$ or equivalently $\mathcal{A}$ is a N-CS in a NTS $(\mathcal{U}, \tau_{\mathcal{R}}(X))$.

5. Conclusion

The class of Ngsg-CS defined using Nsg-CS forms a nano topology and lies between the class of N-CS and the class of Ng-CS. The Ngsg-CS can be used to derive a new decomposition of nano continuity and new nano separation axioms.

References

1. Lellis Thivagar, M. and Carmel Richard. **2013**. On nano forms of weakly open sets, *International Journal of Mathematics and Statistics Invention*, 1(1), pp:31-37.
2. Bhuvaneswari, K. and Mythili Gnanapriya, K. **2014**. Nano generalized closed sets in nano topological spaces, *International Journal of Scientific and Research Publication*, 4(5), pp:1-3.
3. Bhuvaneswari, K. and Ezhilarasi, A. **2014**. On nano semi-generalized and nano generalized-semi closed sets in nano topological spaces, *International Journal of Mathematics and Computer Applications Research*, 4(3), pp:117-124.
4. Lellis Thivagar, M. Nirmala Rebecca Paul and Saeid Jafari.**2011**. On new class of generalized closed sets, *Annals of the University of Craiova, Mathematics and Computer Science Series*, 38(3), pp:84-93.
5. Pawlak, Z. **1982**. Rough Sets, *International Journal of Information and Computer Sciences*, 11, pp:341-356.
6. Reilly, L. and Vamanamurthy.**1985**. On α-sets in topological spaces, *Tamkang Journal Math.*, 16, pp:7-11.
7. Thanga Nachiyar, R. and Bhuvaneswari, K. **2014**. On nano generalized α-closed sets and nano α-generalized closed sets in nano topological spaces, *International Journal of Engineering Trends and Technology*, 13(6), pp:257-260.