Estimate the parameters of the Generalized Goel-okumoto model using the Maximum likelihood and the shrinkage methods

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Abstract:

By this research we will deal with the Generalized Goel- okumoto model parameters, since this model contains three parameters (α, β, γ), that represents the time-rate function of the heterogeneous Poisson's processes, the parameters of this model will be estimated in two ways of estimating , the Maximum likelihood method as well as Shrinkage method , and to look for the best method the simulation method has used by selecting four different sample sizes (30, 60, 90, 120) in order to show the effect of the change in the volumes of different samples on the parameters consider estimated as well as Three initial values were imposed for each of the model parameters used in this research, and to do so the mean squares error was used, with the results showing that the shrinkage method is the best method of estimation.

Keywords: Generalized Goel-okumoto model, Maximum likelihood method, Shrinkage method

تقدير معلمات انموذج Generalized Goel-okumoto باستخدام طريقتي الإمكان الأعظم والتقلص

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المستخلص:

في هذا البحث سنتناول تقدير معلمات انموذج Generalized Goel-okumoto اذ يحتوي هذا الانموذج على ثلاثة معلمات (α, β, γ) الذي يمثل دالة المعدل الزمني لعمليات بواسون غير المتجانسة، سوف يتم تقدير معلمات هذا الانموذج بطريقتين لتقدير وهما طريقة الإمكان الأعظم وكذلك طريقة التقلص وللبحث عن الطريقة الأفضل تم استخدام أسلوب المحاكاة وباختيار أربعة حجوم عينات مختلفة (٣٠، ٦٠، ٩٠، ١٢٠) وذلك لبيان تأثير التغير الحاصل في حجوم العينات المختلفة على المعلمات التي سيتم تقديرها وكذلك تم فرض ثلاثة قيم أولية لكل معلمة من معلمات الانموذج المستخدم في هذا البحث، ولإجراء ذلك تم استخدام معيار المقارنة متوسط مربعات الخطأ حيث أوضحت النتائج ان طريقة التقلص هي الطريقة الأفضل في التقدير .

الكلمات المفتاحية: انموذج Generalized Goel-okumoto، طريقة الإمكان الأعظم، طريقة التقلص.

1. Introduction:

Non homogeneous Poisson processes are one of the important topics which have a role in technological and scientific progress and development. Non homogeneous Poisson process is a general case of Poisson homogeneous processes where as the Non Homogeneous Poisson Process, if the rate of occurrence of events changes by time (T), Either the time rate of event φ (t) does not change by changing time (i.e. the φ (t) is fixed by time), then the Poisson process is called the (homogeneous Poisson Process).

The counting process is called {W (t, t>0)} the Non homogeneous Poisson process (NHPP) with a density function of φ (t), t>0 if the following conditions are met [8]:

(i) W (0) = 0

(ii) The process {W(t) , t>0 } has independent and unstable increments.

(iii) $p\{W(t+q)-W(t)>2\}=0(q)$.

(iv) $P\{W(t+q)-W(t)=1\}=\lambda(t)+0(q)$.

Where: (0) *q* is quantity approaching zero at the time period *q*.

And so, the Poisson process {W(t), t>0} follows the distribution of Poisson with a probability mass function: -

$$P[M(b)-M(a)=n]=\frac{[\mu(t)^n]e^{-\mu(t)}}{n!} \qquad ;\ n=1,2,\ldots$$

Where: µ(t) represents the mean-value function and Non homogeneous Poisson process parameters, and it's derivable as follow:

$$\lambda(t)=\frac{\partial\mu(t)}{dt}$$

Where: λ(t) represents the rate of occurrence and is variable by time change, and the relationship between the Mean-value function and time rate function is:

$$\mu(t)=E[W(t)]=\int_0^t \lambda(u)du, \qquad t\geq o$$

2. Research Objective:

Aimed the research to estimate the parameters of the Generalized Goel-Okumoto model using the Maximum likelihood method, which is one of the models of the Non homogeneous Poisson process and the Shrinkage method, then to compare the two methods of estimation used in the research and determine the best way to estimate the parameters of the model, and this is done by using the scale of the mean squares error (MSE) in order to reach the best way to estimate the parameters of the models used in the Research.

3. Research Model:

There are many researchers who have used or suggested several functions of the Non homogeneous Poisson process (as a time rate of occurrence the events that based on time).

In our research, we will examine one of the models of the Non homogeneous Poisson process, the Generalized Goel -Okumoto model.

In this research, we will examine one of the models of the Non homogeneous Poisson process, the Generalized Goel-Okumoto model:
In 1979, the researchers Goel and Okumoto suggested the following average function [4, 5]:

$$\mu(t) = \alpha[1 - \exp(-\beta t)] \qquad \ldots\ldots(1)$$

In 1983, the researcher Goel decided to generalize the model to three parameters where the density function of the model is:

$$\lambda(t) = \alpha\beta\,\gamma t^{\gamma-1} e^{-\beta t^{\gamma}} \qquad \ldots\ldots(2)$$

As for Mean value function is:

$$\mu(t) = \alpha[1 - exp(-\beta t^{\gamma})] \qquad \ldots\ldots(3)$$

The density function behavior of this model is constant as it is always in a low state.

4. Estimation Methods:

4-1. Maximum Likelihood Estimation Method: Maximum Likelihood Estimation Method is one of the most commonly used methods in estimating the parameters of the Non homogeneous Poisson processes models because of their characteristics that distinguish them from the other methods, including Minimum Variance Unbiased estimators and stability (Invariant Property), and the capabilities of this method can be defined as the values of parameters that make the Maximum likelihood function of observations at its maximize .

let (t_1, t_2, ..., t_n) elements of a random sample of size (n) derived from a population with a known probability density function, the Joint-distribution function of the observations are as follows [2]:

$$f_n(t_1, t_2, \ldots, t_n) = \prod_{i=1}^{n} \lambda(t_i) e^{[-\int_0^t \lambda(tu)du]} \qquad \ldots(4)$$

where the likelihood function for the used model is as follows:

$$f_n = (\alpha\beta\gamma)^n \left(\prod_{i=1}^{n} t_i^{\gamma-1}\right) \exp\left(-\beta\sum_{i=1}^{n} t_i^{\gamma}\right) \exp\left[-\alpha\left(1-e^{-\beta t_n^{\gamma}}\right)\right] \quad ...(5)$$

when taking the natural logarithm for the likelihood function in formula (5) we get:

$$lnf_n = nln\alpha + nln\beta + nln\gamma + \sum_{i=1}^{n} lnt_i^{\gamma-1} - \beta\sum_{i=1}^{n} t_i^{\gamma} - \alpha\left(1-\exp\left(-\beta t_n^{\gamma}\right)\right) \quad ...(6)$$

To estimate the α parameter, we derive equation (6) for α and equal it to zero, so we get:

$$\hat{\alpha} = \frac{n}{1-\exp\left(-\hat{\beta} t_n^{\hat{\gamma}}\right)} \quad ...(7)$$

To estimate the β parameter, we derive equation (6) for β and equal it to zero, so we get:

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n} t_i^{\hat{\gamma}} + \hat{\alpha}\exp\left(-\hat{\beta} t_n^{\hat{\gamma}}\right)(t_n^{\hat{\gamma}})} \quad ...(8)$$

To estimate the γ parameter, we derive equation (6) for γ and equal it to zero, so we get:

$$\hat{\gamma} = \frac{n}{\hat{\beta}\sum_{i=1}^{n} t_i^{\hat{\gamma}}(lnt_i) - \sum_{i=1}^{n} lnt_i^{\hat{\gamma}}(\ln(lnt_i)) + \hat{\alpha}(\exp\left(-\hat{\beta} t_n^{\hat{\gamma}}\right))(\hat{\beta} t_n^{\hat{\gamma}})(\ln(lnt_n))} \quad ...(9)$$

By our observation of the equations above we find that, these equations cannot be solved in the usual methods, the reason for this is the high degree of nonlinear in them, so we will use one of numerical methods in solving nonlinear equations and one of the most important and most widely used methods is (Newton-Raphson).

4-2. shrinkage method: Shrinkage function means the amount of the researcher's confidence in the previous information and what is available from them and writes the formula of the amount of contraction as follows [1, 3, 6]:

$$\hat{\delta}_{sh} = z\hat{\delta} + (1-z)\delta_0 \quad ...(10)$$

where:

$\hat{\delta}$: is an unbiased initial amount.

δ_0 advance information about the parameter.

z: is the amount of shrinkage and takes the value between (0 1,) in this research will be assumed a value for this amount which is (0.5).

For imposing initial values of the amount $\hat{\delta}$ we will use estimations that have already been estimated in the Maximum likelihood method as follows:

$$\hat{\delta}_{sh} = z\hat{\delta}_{MLE} + (1-z)\delta_0 \quad \dots (11)$$

To estimate the parameters of the model used for this method, we will follow the following:

$$\hat{\alpha}_{sh} = z\hat{\alpha}_{MLE} + (1-z)\alpha_0 \quad \dots (12)$$

$$\hat{\beta}_{sh} = z\hat{\beta}_{MLE} + (1-z)\beta_0 \quad \dots (13)$$

$$\hat{\gamma}_{sh} = z\hat{\gamma}_{MLE} + (1-z)\gamma_0 \quad \dots (14)$$

5. Comparison standard :

The mean squares error were relied upon as a measure of comparison between the methods of estimation as this measure indicates the accuracy of the estimate and its decrease in value indicates the quality and accuracy of the capabilities and is calculated as follows[1] :

$$\text{MSE}(\hat{\theta}) = \frac{\sum_{i=1}^{R}(\hat{\theta}-\theta)^2}{R} \quad \dots\dots\dots (15)$$

Where:

R, represents the number of times the experiment repeats.

6. Simulation:

Simulation has a major and important role in addressing many complex problems and dilemmas in various applications, including statistical applications, which led to the adoption of this method by many researchers in many studies dealing with the behavior of certain statistical models or distributions.

Simulation is defined as a mathematical method of solving complex problems that arise during the preview, where a sample of the theoretical community is designing to represent the phenomenon rather than the real society.

The simulation method is based on generating random numbers that simulate the random process under study to generate certain data. In addition, any simulation experiment is only a certain type of inspection, as this sample is calculated from the virtual community represented by the phenomenon studied and then the appropriate statistical methods are applied to reach the required results for the purpose of comparison and analysis.

7. Description the phases of simulation experiment:

Phase-1: - This phase includes:

- ❖ Choosing sizes of the samples, where we have selected four sizes of the samples which are (30, 60, 90, 120) in order to show the effect of the change in the sizes of the different samples on the parameters to show the effect of the change in sample sizes on the estimation of the parameters of the research model.
- ❖ Choice the default values, if we impose three values for each of the model parameters that have been used in this research, which are as follows:

Table (1): default values

	1	2	3
α	0.3	0.5	0.7
β	0.5	1.5	2
γ	1	1.5	3.5

Phase-2: At this stage, random data is generated subject to the Non homogeneous Poisson processes, is done by rejection and acceptance method, where it is one of the most commonly used simulation methods in the generation of random variables, where the program MATLAB2014A was used.

Algorithm for generating Non homogeneous Poisson processes is as follows [7]:

Step-1: to put,

Ko =0 and K* =0

Step-2: generate a random variable that follows the exponential distribution E On average $\bar{\lambda}$

Step-3:

K* = K* + E

Step-4: generate a random variable that follows uniform distribution.

$U \sim U\ (0,1)$

Step-5: If($U > \lambda(K^*)/\bar{\lambda}$)go back to step two Otherwise, we're going to make:

$K_i = K^*$

Phase-3: At this phase, the results obtained to estimate the time-rate function of the research model and to deduce the best and most efficient method of estimation are presented and analyzed.

The results of the parameters values and the average values of the error boxes will be displayed as follows:

Table (2): Shows estimated values for parameter estimates α , β , γ for repeated experience 500 times

N=30	α =0.3, β =0.5, γ =1			α =0.5 , β =1.5 , γ =1.5			α =0.7 , β =2, γ =3.5		
GGO	α	β	γ	α	β	γ	α	β	γ
MLE	0.122	0.963	0.345	0.972	2.345	1.927	0.967	2.894	4.984
shrinkage	0.295	0.484	0.985	0.504	1.445	1.456	0.692	1.923	3.482

N=60	α =0.3, β =0.5, γ =1			α =0.5 , β =1.5 , γ =1.5			α =0.7 , β =2 , γ =3.5		
GGO	α	β	γ	α	β	γ	α	β	γ
MLE	0.774	0.773	0.434	0.956	2.106	1.856	0.877	2.786	4.343
shrinkage	0.305	0.498	0.991	0.492	1.502	1.491	0.699	1.932	3.503

N=90	α =0.3, β =0.5, γ =1			α =0.5 , β =1.5 , γ =1.5			α =0.7 , β =2 , γ =3.5		
GGO	α	β	γ	α	β	γ	α	β	γ
MLE	0.685	0.763	0.534	0.913	1.941	1.863	0.875	2.392	4.182
shrinkage	0.973	0.497	0.991	0.501	1.492	1.483	0.690	1.952	3.492

N=120	α =0.3, β =0.5, γ =1			α =0.5 , β =1.5 , γ =1.5			α =0.7 , β =2 , γ =3.5		
GGO	α	β	γ	α	β	γ	α	β	γ
MLE	0.588	0.683	0.723	0.802	1.872	1.772	0.792	2.202	3.932
shrinkage	0.992	0.501	0.993	0.499	1.501	1.490	0.703	1.982	3.499

Table (3): Shows the mean squares error for parameter estimates

N=30	α =0.3, β =0.5, γ =1			α =0.5 , β =1.5 , γ =1.5			α =0.7 , β =2, γ =3.5		
GGO	α	β	γ	α	β	γ	α	β	γ
MLE	0.8154	0.9307	2.5675	0.9009	0.9794	2.9919	1.4176	1.3001	3.623
shrinkage	0.04315	0.06387	0.0045	0.0501	0.0303	0.0522	0.0233	0.0974	0.0247

N=60	α =0.3, β =0.5, γ =1			α =0.5 , β =1.5 , γ =1.5			α =0.7 , β =2 , γ =3.5		
GGO	α	β	γ	α	β	γ	α	β	γ
MLE	0.7694	0.8757	3.5638	0.7478	0.5467	2.9628	0.9473	1.2984	1.6573
shrinkage	0.0397	0.0407	0.0023	0.0415	0.0106	0.05746	0.0672	0.0885	0.0157

N=90	α =0.3, β =0.5, γ =1			α =0.5 , β =1.5 , γ =1.5			α =0.7 , β =2 , γ =3.5		
GGO	α	β	γ	α	β	γ	α	β	γ
MLE	0.5685	0.7344	3.2354	0.6344	0.3269	2.8743	0.8997	1.1823	1.4873
shrinkage	0.0328	0.0353	0.0016	0.0372	0.0161	0.0172	0.0587	0.07823	0.0853

N=120	α =0.3, β =0.5, γ =1			α =0.5 , β =1.5 , γ =1.5			α =0.7 , β =2 , γ =3.5		
GGO	α	β	γ	α	β	γ	α	β	γ
MLE	0.5198	0.5464	3.04638	0.5729	0.2856	2.3926	0.7295	0.9564	1.2743
shrinkage	0.02678	0.02875	0.0032	0.0289	0.0115	0.0754	0.0463	0.0643	0.0943

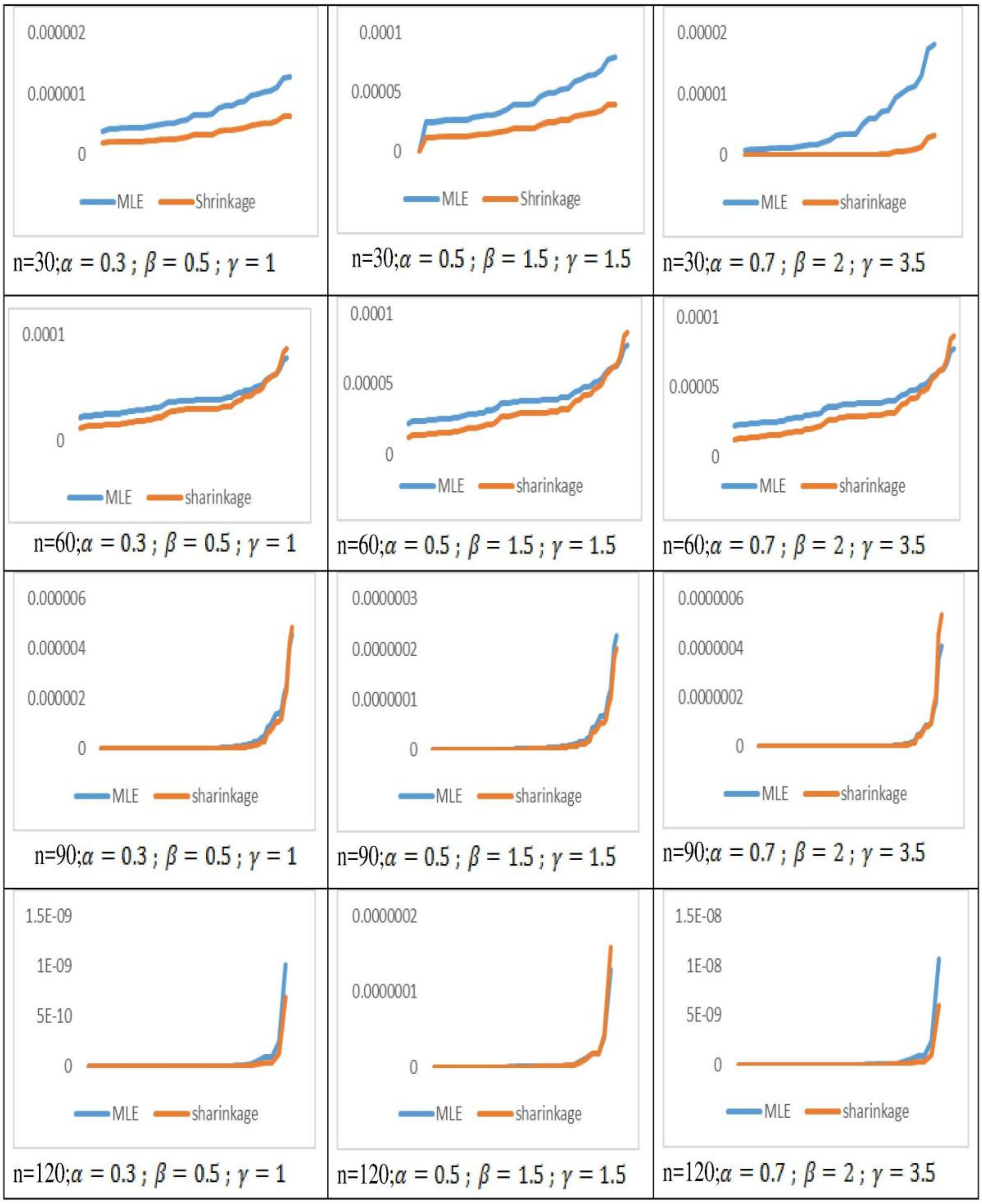

Figure (1): Rate of occurrence function curve for Generalized Goel-okumoto model

8. Conclusions and Recommendations:

8.1: Conclusions:

- ❖ From table (2) we note that, there is a convergence between the default values and the estimated values for all sample sizes.

- ❖ From table (3) we note from the results of the simulation that, the Shrinkage method is better than Maximum likelihood method for all sample sizes as well as for all default values.
- ❖ From Figure (1) we note that the shrinkage method gives stationary in results than the maximum likelihood method

8.2: Recommendations:

- ❖ The researcher recommends using Shrinkage method in estimating the parameters of the models of the Non homogeneous Poisson process because of the high efficiency and flexibility of the estimate.
- ❖ The researcher recommends the use of other methods to estimate the parameters of the Rate of occurrence of events of the Non homogeneous Poisson process such as moment method.

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