The Basic and Extended Identities for Certain q −Polynomials

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Abstract.

In this paper, we introduce the generating function, Mehler's formula, Roger's formula, the linearization formula and the inverse relation of the linearization formula of the polynomials $G_n(x;q)$, which is defined by L.Carlitz [3]. Also we introduce an extension of the generating function and extension of the Roger's formula, the extended generating function of $G_n(x;q)$ involves a ${}_2\phi_1$ sum and the extended Rogers formula involves a ${}_3\phi_2$ sum.

All identities will be derived depending on the roles of the exponential operator $E(\theta)$ after representing the polynomials $G_n(x;q)$ by this operator.

الملخص :

سنقدم في هذا البحث الدالة المولدة وصيغة ملر وصيغة روجر والصيغة الخطية ومعكوس الصيغة الخطية لمتعددات الحدود $G_n(x;q)$ (**المعرفة في المصدر** [3]). كذلك سنقدم توسيعاً **للدالة المولدة و**توسيعاً لصيغة روجر لمتعددات الحدود $G_n(x;q)$, حيث سيحتوي توسيع الدالة **المولدة على المجموع** ${}_2\phi_1$, أما توسيع صيغة روجر فسيحتوي المجموع ${}_3\phi_2$.

كل المتطابقات المقدمة في هذا البحث ستشتق بالأعتماد على قوانين المؤثر الأسي $E(\theta)$ **بعد** أستخدام هذا المؤثر في تمثيل متعددات الحدود $G_n(x;q)$.

Keywords: The exponential operator; generating function; Mehler's formula; Rogers formula; linearization formula, extended generating function; extended Rogers formula.

1. Introduction and Notation

Using of operators approach to some basic hypergeometric series given in the work of Goldman and Rota [13, 14], Andrews [1] and Roman [15]. Chen and Liu [7] developed a method of deriving hypergeometric identities by parameter augmentation, this method has more realizations as in [6, 8, 9, 10, 17, 18]. In this paper, we derive some new identities of the polynomials $G_n(x;q)$ and give an operator proof for these identities.

Let us review some common notation and terminology for basic hypergeometric series in [11]. Throughout this paper, we assume that $|q| < 1$ and $q \neq 0$, the q −shifted factorial is defined for any real or complex variable a by:

$$(a;q)_0 = 1, \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n \in Z^+.$$

The following notation refers to the multiple q −shifted factorials:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \dots (a_m;q)_n,$$
$$(a_1, a_2, \dots, a_m; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \dots (a_m;q)_\infty.$$

The q −binomial coefficients, or the Gaussian polynomials, are given by:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

The basic hypergeometric series ${}_{r+1}\phi_r$ are defined by:

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, b_2, \dots, b_r; q)_n} x^n,$$

where a_i, b_j, q and x may be real or complex [16].

The Cauchy identity is defined as:

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1. \qquad \dots (1.1)$$

Putting $a = 0$, (1.1) becomes Euler's identity:

$$\sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty}, \quad |x| < 1, \qquad \dots (1.2)$$

and its inverse relation:

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} x^k}{(q;q)_k} = (x;q)_\infty, \qquad \dots (1.3)$$

The q −difference operator $D_q$and the$q$ −shift operator $\eta$are given by:

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a} \, and \, \eta\{f(a)\} = f(aq).$$

In 1998, Chen and Liu [6] constructed operator $\theta = \eta^{-1} D_q$. They introduced the exponential operator as:

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{\binom{n}{2}}}{(q;q)_n}, \qquad \dots (1.4)$$

with the Leibniz formula for θ:

$$\theta^n\{f(a)g(a)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \theta^k\{f(a)\} \theta^{n-k}\{g(aq^{-k})\},$$

and gave the following operator identities [6]:

Proposition 1.1.

$$E(b\theta)\{(at;q)_\infty\}$$
$$= (at,bt;q)_\infty \qquad \text{... (1.5)}$$

$$E(b\theta)\{(as,at;q)_\infty\}$$
$$= \frac{(as,at,bs,bt;q)_\infty}{(abst/q\,;q)_\infty} \qquad \text{... (1.6)}$$

$$where|abst/q| < 1.$$

Also in 2006, Zhang and Liu [18] derived two operator identities as:

<u>Proposition 1.2.</u>

$$E(b\theta)\{a^n(as;q)_\infty\}$$
$$= a^n(as,bs;q)_\infty\, {}_2\phi_1\left(\begin{matrix} q^{-n}, & q/as \\ & 0 \end{matrix};q,bs\right), \qquad \text{... (1.7)}$$

$$where|bs| < 1.$$

$$E(b\theta)\{a^n(as,at;q)_\infty\}$$
$$= a^n\frac{(as,at,bs,bt;q)_\infty}{(abst/q)_\infty}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, q/as\,, q/at \\ 0, q^2/abst \end{matrix};q,q\right), \qquad \text{... (1.8)}$$

$$where|abst/q| < 1.$$

In 1958, Carlitz defined a sequence of polynomials(see [3,4,5,12]) as:

$$G_n(x;q)$$
$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)}x^k, \qquad \text{... (1.9)}$$

and gave its bilinear generating function [4] in the form:

$$\sum_{k=0}^{\infty} (-1)^n q^{n(n+1)/2} G_n(x;q)G_n(y;q)\frac{z^n}{(q;q)_n} = \frac{(qz,qxz,qyz,qxyz;q)_\infty}{(qxyz^2;q)_\infty},$$

also he showed that the polynomials (1.9) satisfy the following three-term recurrence relation:

$$G_{n+1}(x;q) = (x+1)G_n(x;q) + (q^{-n}-1)xG_{n-1}(x;q)\,.$$

In 2009, Cao [5] represented the polynomials $G_n(x;q)$ by the exponential operator $E(\theta)$ as:

$$E(\theta)\{x^n\} = G_n(x;q)\,, \qquad \text{... (1.10)}$$

and used this representation to solve some identities of $G_n(x;q)$.

In this paper, we use Cao representation (1.10) to introduce the basic and extended identities for $G_n(x;q)$, where in Section 2 we derive the generating function and Mehler's formula, in Section 3 the Rogers formula for the polynomials $G_n(x;q)$ will be derived with two of its applications: the linearization formula and the inverse linearization formula, then in Section 4 we introduce two extended identities for $G_n(x;q)$ which are the extended generating function and the extended Rogers formula.

2. The Generating Function and Mehler's Formula for $G_n(x;q)$

In this section, we derive the generating function and Mehler's formula for $G_n(x;q)$ polynomials depending on the exponential operator representation (1.10) and the identities (1.5,1.6) of the exponential operator.

Theorem 2.1(**The generating function for $G_n(x;q)$).** *We have*:

$$\sum_{n=0}^{\infty} G_n(x;q)\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n} = (t,xt;q)_\infty. \qquad \dots (2.1)$$

Proof.

$$\sum_{n=0}^{\infty} G_n(x;q)\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} E(\theta)\{x^n\}\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n} \qquad [from\ (1.10)]$$

$$= E(\theta)\left\{\sum_{n=0}^{\infty}\frac{(-1)^n (xt)^n q^{\binom{n}{2}}}{(q;q)_n}\right\}$$

$$= E(\theta)\{(xt;q)_\infty\} \qquad [from\ (1.3)]$$

$$= (t,xt;q)_\infty. \qquad [from\ (1.5)]$$

Now we derive Mehler's formula by using identity (1.6) of exponential operator, where we can represent the polynomials $G_n(x;q)$ by this operator as $E(\theta)\{x^n\}$or the polynomials $G_n(y;q)$ as $E(\theta)\{y^n\}$or using the two representations together to get the same result.

Theorem 2.2(**Mehler's formula for $\boldsymbol{G_n(x;q)}$).** *We have*:

$$\sum_{n=0}^{\infty} G_n(x;q)G_n(y;q)\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= \frac{(t,xt,yt,xyt;q)_\infty}{(xyt^2/q\,;q)_\infty}, \qquad \ldots (2.2)$$

$$where |xyt^2/q| < 1.$$

Proof.

$$\sum_{n=0}^{\infty} G_n(x;q)G_n(y;q)\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} E(\theta)\{x^n\}G_n(y;q)\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n} \qquad [from\ (1.10)]$$

$$= E_x(\theta)\left\{\sum_{n=0}^{\infty} G_n(y;q)\frac{(-1)^n (xt)^n q^{\binom{n}{2}}}{(q;q)_n}\right\}$$

$$= E_x(\theta)\{(xt,xyt;q)_\infty\} \qquad [from\ (2.1)]$$

$$= \frac{(t,xt,yt,xyt;q)_\infty}{(xyt^2/q\,;q)_\infty}. \qquad [from\ (1.6)]$$

3. The Roger's Formula and It's Applications

In this section, we introduce two forms of the Roger’s formula depending on identity (1.6) of the exponential operator $E(\theta)$, then as an application of the Roger’s formula, we give the linearization formula and the inverse linearization formula for $G_n(x;q)$ polynomials.

Theorem 3.1(**The Roger's formula for $\boldsymbol{G_n(x;q)}$).** *We have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} G_{n+m}(x;q)\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m} q^{\binom{n}{2}+\binom{m}{2}}$$

$$= \frac{(t,s,xt,xs;q)_\infty}{(xts/q;q)_\infty}, \qquad \ldots (3.1)$$

$$where|xts/q| < 1.$$

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} G_{n+m}(x;q)\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m}q^{\binom{n}{2}+\binom{m}{2}}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E(\theta)\{x^{n+m}\}\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m}q^{\binom{n}{2}+\binom{m}{2}} \quad [from\ (1.10)]$$

$$= E(\theta)\left\{\sum_{n=0}^{\infty}\frac{(-1)^n (xt)^n q^{\binom{n}{2}}}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(-1)^m (xs)^m}{(q;q)_m}q^{\binom{m}{2}}\right\}$$

$$= E(\theta)\{(xt,xs;q)_\infty\} \qquad [from\ (1.3)]$$

$$= \frac{(t,s,xt,xs;q)_\infty}{(xts/q;q)_\infty}. \quad [from\ (1.6)]$$

In the R.H.S. of (3.1), the terms $(t,xt;q)_\infty$and $(s,xs;q)_\infty$can be written as generating functions for the polynomials $G_n(x;q))$ and $G_m(x;q)$, respectively to get the following identity:

$$\frac{(t,s,xt,xs;q)_\infty}{(xts/q;q)_\infty}$$

$$= \frac{1}{(xts/q;q)_\infty}\sum_{n=0}^{\infty} G_n(x;q)\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}\sum_{n=0}^{\infty} G_m(x;q)\frac{(-1)^m s^m q^{\binom{m}{2}}}{(q;q)_m}$$

$$= \frac{1}{(xts/q;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} G_n(x;q)G_m(x;q)\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m}q^{\binom{n}{2}+\binom{m}{2}}$$

Therefore, we get another Roger's formula as:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} G_{n+m}(x;q)\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m}q^{\binom{n}{2}+\binom{m}{2}}$$

$$= \frac{1}{(xts/q;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} G_n(x;q) G_m(x;q) \frac{(-1)^n t^n}{(q;q)_n} \frac{(-1)^m s^m}{(q;q)_m} q^{\binom{n}{2}+\binom{m}{2}} \quad \dots (3.2)$$

Now we derive the linearization formula (3.3) and the inverse linearization formula (3.4) as an application of the Roger's formula as follows:

Corollary 3.1.1. *For $n, m \in N$, we have*

$$G_n(x;q) G_m(x;q)$$
$$= \sum_{k=0}^{\min\{n,m\}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} (q;q)_k (-x)^k q^{k(3k-2m-2n-1)/2} G_{n+m-2k}(x;q). \quad \dots (3.3)$$

Proof. From (3.2) we have:

$$(xts/q;q)_\infty \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} G_{n+m}(x;q) \frac{(-1)^n t^n}{(q;q)_n} \frac{(-1)^m s^m}{(q;q)_m} q^{\binom{n}{2}+\binom{m}{2}}$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} G_n(x;q) G_m(x;q) \frac{(-1)^n t^n}{(q;q)_n} \frac{(-1)^m s^m}{(q;q)_m} q^{\binom{n}{2}+\binom{m}{2}}.$$

Verify the L.H.S. by using Euler's identity (1.3):

$$\sum_{k=0}^{\infty} \frac{(-xts/q)^k q^{\binom{k}{2}}}{(q;q)_k} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} G_{n+m}(x;q) \frac{(-1)^n t^n}{(q;q)_n} \frac{(-1)^m s^m}{(q;q)_m} q^{\binom{n}{2}+\binom{m}{2}}$$
$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-x/q)^k q^{\binom{k}{2}}}{(q;q)_k} G_{n+m}(x;q) \frac{(-1)^n t^{n+k}}{(q;q)_n} \frac{(-1)^m s^{m+k}}{(q;q)_m} q^{\binom{n}{2}+\binom{m}{2}}.$$

Set $n \to n-k$, $m \to m-k$, the L.H.S. equals:

$$\sum_{n=k}^{\infty} \sum_{m=k}^{\infty} \sum_{k=0}^{\infty} \frac{(-x/q)^k q^{\binom{k}{2}}}{(q;q)_k} G_{n+m-2k}(x;q) \frac{(-1)^{n-k} t^n}{(q;q)_{n-k}} \frac{(-1)^{m-k} s^m}{(q;q)_{m-k}} q^{\binom{n-k}{2}+\binom{m-k}{2}}$$
$$\sum_{n=0}^{\infty} \sum_{m=o}^{\infty} \sum_{k=0}^{\min\{n,m\}} \frac{(-x/q)^k q^{\binom{k}{2}}}{(q;q)_k} G_{n+m-2k}(x;q) \frac{(-1)^n t^n}{(q;q)_{n-k}} \frac{(-1)^m s^m}{(q;q)_{m-k}} q^{\binom{n-k}{2}+\binom{m-k}{2}}.$$

By comparing the coefficients of $t^n s^m$ with the R.H.S. get:

$$\frac{G_n(x;q) G_m(x;q)}{(q;q)_n (q;q)_m} q^{\binom{n}{2}+\binom{m}{2}} = \sum_{k=0}^{\min\{n,m\}} \frac{(-x/q)^k}{(q;q)_k} G_{n+m-2k}(x;q) \frac{q^{\binom{n-k}{2}+\binom{m-k}{2}+\binom{k}{2}}}{(q;q)_{n-k} (q;q)_{m-k}}.$$

Then

$$G_n(x;q)G_m(x;q) =$$

$$\sum_{k=0}^{\min\{n,m\}} \frac{(q;q)_n(q;q)_m}{(q;q)_k(q;q)_{n-k}(q;q)_{m-k}}(-x)^k G_{n+m-2k}(x;q)q^{\binom{n-k}{2}+\binom{m-k}{2}+\binom{k}{2}-\binom{n}{2}-\binom{m}{2}-k}$$

$$= \sum_{k=0}^{\min\{n,m\}} \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix}(q;q)_k(-x)^k G_{n+m-2k}(x;q)q^{\binom{n-k}{2}+\binom{m-k}{2}+\binom{k}{2}-\binom{n}{2}-\binom{m}{2}-k}.$$

By simplifying the term up to q, the desired identity follows.

Here we give the second application of the Roger's formula, it is the inverse relation of the linearization formula for $G_n(x;q)$ polynomials.

Corollary 3.1.2. *For* $n, m \in N$, *we have*

$$G_{n+m}(x;q) = \sum_{k=0}^{\min\{n,m\}} \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix}(q;q)_k x^k q^{k(k-m-n)} G_{n-k}(x;q)G_{m-k}(x;q) \quad \dots (3.4)$$

$$where |xts/q| < 1.$$

Proof. In (3.2), expand $1/(xts/q\,;q)_\infty$ by the Euler's identity (1.2), the R.H.S. can be rewritten as:

$$\sum_{k=0}^{\infty}\frac{(xts/q)^k}{(q;q)_k}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}G_n(x;q)G_m(x;q)\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m}q^{\binom{n}{2}+\binom{m}{2}}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(x/q)^k}{(q;q)_k}G_n(x;q)G_m(x;q)\frac{(-1)^n t^{n+k}}{(q;q)_n}\frac{(-1)^m s^{m+k}}{(q;q)_m}\ q^{\binom{n}{2}+\binom{m}{2}}.$$

Set $n \to n-k$, $m \to m-k$, the R.H.S. equals:

$$\sum_{n=k}^{\infty}\sum_{m=k}^{\infty}\sum_{k=0}^{\infty}\frac{(x/q)^k}{(q;q)_k}G_{n-k}(x;q)G_{m-k}(x;q)\frac{(-1)^{n-k}t^n}{(q;q)_{n-k}}\frac{(-1)^{m-k}s^m}{(q;q)_{m-k}}\ q^{\binom{n-k}{2}+\binom{m-k}{2}}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\min\{n,m\}}\frac{(x/q)^k}{(q;q)_k}G_{n-k}(x;q)G_{m-k}(x;q)\frac{(-1)^n t^n}{(q;q)_{n-k}}\frac{(-1)^m s^m}{(q;q)_{m-k}}\ q^{\binom{n-k}{2}+\binom{m-k}{2}}$$

By comparing the coefficients of $t^n s^m$ with the L.H.S. of (3.2), get:

$$\frac{G_{n+m}(x;q)}{(q;q)_n(q;q)_m} q^{\binom{n}{2}+\binom{m}{2}}$$

$$= \sum_{k=0}^{\min\{n,m\}} (x/q)^k G_{n-k}(x;q) G_{m-k}(x;q) \frac{q^{\binom{n-k}{2}+\binom{m-k}{2}}}{(q;q)_{n-k}(q;q)_{m-k}(q;q)_k}.$$

Therefore

$$G_{n+m}(x;q) = \sum_{k=0}^{\min\{n,m\}} \frac{(q;q)_n(q;q)_m}{(q;q)_k(q;q)_{n-k}(q;q)_{m-k}} x^k G_{n-k}(x;q) G_{m-k}(x;q) q^{\binom{n-k}{2}+\binom{m-k}{2}-\binom{n}{2}-\binom{m}{2}-k}$$

Hence

$$G_{n+m}(x;q) = \sum_{k=0}^{\min\{n,m\}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} (q;q)_k x^k q^{k(k-m-n)} G_{n-k}(x;q) G_{m-k}(x;q).$$

The proof is completed.

4. The Extended Identities for $G_n(x;q)$ Polynomials

In this section, we introduce two extended identities. The first is an extension of the generating function which is deriving by using the identity (1.7) of the exponential operator

involving a ${}_2\phi_1$ sum. The second is an extension of the Roger's formula which is deriving by using the identity (1.8) of the exponential operator involving a ${}_3\phi_2$ sum.

<u>Theorem 4.1</u>(Extended generating function). *We have*:

$$\sum_{n=0}^{\infty} G_{n+k}(x;q) \frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= x^k (t, xt; q)_\infty \, {}_2\phi_1 \left(\begin{matrix} q^{-k}, & q/xt \\ & 0 \end{matrix} ; q, t \right), \qquad \ldots (4.1)$$

$$where |t| < 1.$$

Proof.

$$\sum_{n=0}^{\infty} G_{n+k}(x;q) \frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} E(\theta)\{x^{n+k}\} \frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= E(\theta)\left\{x^k \sum_{n=0}^{\infty} \frac{(-1)^n (xt)^n q^{\binom{n}{2}}}{(q;q)_n}\right\}$$

$$= E(\theta)\{x^k (xt;q)_\infty\}$$

$$= x^k\,(t,xt;q)_\infty\, {}_2\phi_1\left(\begin{matrix} q^{-k}, & q/xt \\ & 0 \end{matrix} ; q, t\right).$$

<u>Theorem 4.2</u>(Extended Roger's formula). *We have*:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} G_{n+m+k}(x;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}(-1)^{n+m} q^{\binom{n}{2}+\binom{m}{2}}$$

$$= \frac{x^k (t,s,xt,xs;q)_\infty}{(xts/q;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} q^{-k}, & q/xt, & q/xs \\ & q^2/xts, & 0 \end{matrix} ; q, q\right), \qquad \text{... (4.2)}$$

$$\text{where} |xts/q| < 1.$$

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} G_{n+m+k}(x;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}(-1)^{n+m} q^{\binom{n}{2}+\binom{m}{2}}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E(\theta)\{x^{n+m+k}\}\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}\frac{(-1)^m s^m q^{\binom{m}{2}}}{(q;q)_m}$$

$$= E(\theta)\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n (xt)^n q^{\binom{n}{2}}}{(q;q)_n}\frac{(-1)^m (xs)^m}{(q;q)_m} q^{\binom{m}{2}}\right\}$$

$$= E(\theta)\{x^k (xt,xs;q)_\infty\}$$

$$= \frac{x^k (t,s,xt,xs;q)_\infty}{(xts/q;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} q^{-k}, & q/xt, & q/xs \\ & q^2/xts, & 0 \end{matrix} ; q, q\right).$$

Cao [5] gave the extension of Mehler’s formula for the polynomials $G_n(x;q)$ as:

$$\sum_{n=0}^{\infty} G_{n+k}(x;q)G_n(y;q)\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= \frac{(t,xt,yt,xyt;q)_\infty}{(xyt^2/q;q)_\infty} \times \frac{(q/t;q)_k}{(ty/q)^k (q^2/xyt^2;q)_k}\, {}_2\phi_1\left(\begin{matrix} q^{-k}, & q/xt \\ & 0 \end{matrix} ; q, t\right),$$

where he used identity (1.8) and the following ${}_3\phi_2$ transformation [11]:

$${}_3\phi_2\left(\begin{matrix} q^{-n}, & a, & c \\ & b, & d \end{matrix} ; q, q\right) = \frac{a^n (d/a;q)_n}{(d;q)_n}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, & a, & b/c \\ & b, & a\,q^{1-n}/d \end{matrix} ; q, cq/d\right).$$

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