Fractal Dimension Based on Pixel Covering Method

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الخلاصة

البعد الكسوري هو احد الخصائص الاساسية التي تستخدم في تمييزالصور حيث يعتبرالمفهوم الاساسي في هندسة الكسوريات والذي يستعمل لقياس نسبة التعقيد الهندسي للمجموعة الكسورية. التعريف الاساس للبعد الكسوري في هندسة الكسوريات هو من خلال البعد الهاوسدورفي والذي يصعب حسابه في أكثر الحالات.هناك العديد من الاتجاهات لحساب البعد الكسوري، قسم منها يعد غير كفوء حيث يعطي نتائج غير مقبولة عند استخدامه لتمييز الصفات المحلية للصورة. وللتغلب على هذا النوع من المشاكل قمنا في هذا البحث باقتراح خوارزميات جديدة تستخدم في تخمين البعد الكسوري والتي تعتبر خوارزميات موسعة للخوارزميتين التقليدية في حساب البعد الكسوري وهما (البعد التكعيبي وخوارزمية زمن الهروب) بالاعتماد على طريقة التغطية النقطية. الطرق المقترحة تستعمل كخاصية مهمة في العديد من التطبيقات في الطب،الهندسة والعلوم حيث انها تساعد على تحديد الخواص المحلية للصورة خلافا عن الطرق السابقة المصممة لايجاد البعد الكسوري للصورة بشكلها الكامل. الحسابات العددية تشير الى كفاءة هذة الطريقة مقارنة مع قسم من الطرق السابقة الواسعة الاستخدام.

ABSTRACT

Fractal dimension is an important feature of images, which is considered as a basic concept in fractal geometry used to measure the geometrical complexity of fractal set. In fractal geometry theory, the fundamental definition of fractal dimension have been based on Hausdorff dimension that is not easy to be estimated in most cases. There are many approaches to estimate the fractal dimension of an object, they compute inefficiently and the present of the local features of image invalidly. This paper addresses this problem by presenting a new estimated algorithm based on pixel covering method. The proposed approach will serve as an important characteristic for several applications in medical, engineering, and sciences, it helps to determine the local structure feature of image upon other conventional approaches used to determine the fractal dimension for the whole image. Experimental investigations indicate the efficiency of this approach compared with a well known widely used approaches such as; the box counting dimension, and the escape time dimension.

Keywords: Fractal Dimension (FD), Attractor, Box Counting Dimension (BCD), Escape Time Dimension (ETD), Pixel Covering Method (PCM).

1.Introduction

The description of irregular and random phenomenon in nature is performed through fractal that was established by B.B. Mandelbrot [1]. Fractal theory is a new system describe self-similarity, it has been studied by many researchers and successfully applied in many fields. Self-similarity could be also regarded as a measure of geometrical complexity of an object under discussion. Mandelbrot was the first that handled the irregularity of surfaces in an image through introducing the concept of FD, and described an approach to calculate it that is when he tried to estimate the length of the coastline. Although he did not give a precise definition of fractal, one can understand why, depends on the object studied, for this reason he give several non equivalent definitions of FD, that each problem should entail an appropriate notation of dimension [2].

Mandelbrot singled out “Hausdorff dimension”, because most of the works of this subject was studied by Besicovitch, another type of dimension is belonging to the Bouligand-Minkowski dimension (Minkowski fractal dimension method) [3]. Some of them are equivalent but others are not. The most celebrated one is the Hausdorff (or Hausdorff–Besicovitch) dimension [4]. The most popular one is the box-counting dimension [4] a very similar method to Mandelbrot approach that is given by the Box-Counting theorem. There exist several equivalent definitions termed as a box-counting dimension. A capacity dimension (due to the definition given by Kolmogorov) and the Minkowski–Bouligand dimension are among them [4,5,6]. Although in many cases the Hausdorff dimension equals the box-counting dimension, in general the Hausdorff is used only in theoretical settings and is too subtle for practitioners [1,6]. Popular methods for estimating FD are also correlation dimensions [7] (Grassberger–Procaccia, Takens estimators) and information (or entropy) dimensions [8]. Bernsley [5] introduced the fractal interpolation method which applies iterated function system (IFS) to produce a fractal with known FD through N+1 points. These methods generate graphs which is attractor of the IFS of N contractive affine transformations. In the thesis of A. J. Mohammed [9], a new method to find the dimension of some fractals based on escape time principle was proposed using the method of spreading of the points inside a specific window with I=[-1,1]. In this paper a new estimated approach based on pixel covering method (PCM) is proposed. The proposed approach will serve as an important characteristic for several applications in image classification, object modeling, texture analysis and many other application in medical, engineering and sciences. It helps to determine the local structure feature of image upon other conventional approaches used to determine the FD for the whole image. Many other researchers proposed new approaches that used to improve the efficiency of FD estimation [8,10,11,12].

The material of this paper is arranged into six sections. Section 2 deals with the theoretical background of FD with some known types of fractal dimensions. Box-counting dimension with the proposed Box counting algorithm based on PCM is described in section 3. The ETD with the proposed escape time algorithm is presented in section 4. Section 5 is devoted to present the algorithm implementation with the numerical experiments. Finally, some conclusions are summarized in section 6.

2. Theoretical Background

Theory of fractal sets is a modern domain of research; whereas, the complexity of fractal set can be reflected using fractal dimension. This section presents an overview of major concepts and results that help to understand the FD and their counting methods, a more detailed review of the topics are as in [1,5,13,14].

Let (X,d) be a metric space, $Y \subseteq X$. Then Y is called totally bounded (precompact) if for each $\varepsilon > 0$, there exists a finite set of points $\{x_j\}_{j\in J_m} = \{x_1, x_2, ..., x_m\}$ such that $\bigcup_{j\in J_m} \overline{B}(x_j, \in) \supseteq Y$ where $\overline{B}(x_j, \varepsilon) = \{x \in X : d(x, x_j) \le \varepsilon\}$. When Y is totally bounded. Let $N_\varepsilon(Y) = min\{|J| : \bigcup_{j\in J} \overline{B}(x_j, \varepsilon) \supseteq Y\} = min\{m : \bigcup_{j\in J_m} \overline{B}(x_j, \varepsilon) \supseteq Y\}$.

Let C(Y) be the capacity (length, area, volume) of the subset Y of X. Therefore, $C(Y) = N_\varepsilon(Y)\varepsilon^{d(\varepsilon)}$. Then $d(\varepsilon)$ is a function of ε, where $d(\varepsilon) = \frac{ln(N_\varepsilon(Y))}{ln(1/\varepsilon)} - \frac{ln(C(Y))}{ln(1/\varepsilon)}$. If the $\lim_{\varepsilon\to 0} d(\varepsilon)$ exists, and equal to a real number d, then d is called the capacity dimension of Y. If the capacity dimension d not integer, then Y is called a fractal, and d is called a fractal dimension.

When $X = R^m$, and $Y \subseteq X = R^m$. Let $0 < r < 1$, and $r^n < \varepsilon \le r^{n+1}$ for some positive integer n. Then $d = \lim_{\varepsilon\to 0} d(\varepsilon) = \lim_{n\to\infty} d(r^n)$.

Therefore $d = \lim_{\varepsilon\to 0} \frac{ln(N_\varepsilon(Y))}{ln(1/\varepsilon)} = \lim_{n\to\infty} \frac{ln(N_n(Y))}{n\, ln(1/r)}$. Then d is called box dimension, and the way to calculate d is called box counting dimension.

Let $X = R^m$, and $d(x,y) = (\sum_{j\in J_m} (x_j - y_j)^2)^{1/2}$ is a metric mapping defined on R^m. Then $(X,d) = (R^m, d)$ is a metric space. The mapping $f : X \to X$ is called contraction mapping when $d(f(x), f(y)) \le sd(x,y)\ \forall\ x,y \in X, s \in [0,1)$, which is a contractivity of f and similarity mapping when $d(f(x), f(y)) = sd(x,y)$ for all $x,y \in X, s \ge 0$, when $s = 1$, f is called isometry.

Let (R^m, d) be a complete metric space. Then $H(R^m)$ denote the set of all non-empty compact subsets of R^m and D is the Hausdorff metric on $H(R^m)$ defined by

$$D(A,B) = max\{\max_{b\in B}.\min_{a\in A} d(a,b), \max_{a\in A}.\min_{b\in B} d(a,b)\}$$ for all $A,B \in H(R^m)$.

Then $(H(R^m), D)$ is a metric space.

Let $\{ f_j \}_{j\in J_N}$ be the set of contraction mappings on R^m, with contractivity s_j $for\ j \in J_N$. Then $F(A) = \bigcup_{j\in J_N} f_j(A)$, for all $A \in H(R^m)$ is a contraction mapping on the complete metric space $(H(R^m),D))$ with contractivity $s = max\{ s_j : j \in J_N \}$. By the contraction mapping theorem, there exists a fixed point $A \in H(R^m)$ such that $F(A) = A$, and $\lim_{n\to\infty} F^n(B) = A$ for all $B \in H(R^m)$. These fixed point A is called the attractor set. Let $B = I^m = [0,1]^m \in H(R^m)$, $F(I^m) = \bigcup_{j\in J_m} f_j(I^m) \subseteq I^m$ and

$I^m \supseteq F(I^m) \supseteq F^2(I^m) \supseteq \cdots \supseteq A$. Then $A = \bigcap_{n\in N} F^n(I^m) = \lim_{n\to\infty} F^n(I^m)$.

When f_j is a family of affine transformation $f_j(x) = s_j R_j(x) + b_j$, where R_j is an isometry and b_j is a transformation on I^m for $j \in J_N$, and for all $x \in I^m$. Then $A = f(A)$ is called self similarity in I^m, and d=dim A is the solution of $\sum_{j\in J_N} s_j^d = 1$, is called similarity dimension of A it is calculated by $y = \sum_{j\in J_N} s_j^d$ when y=1.

Let (X,d) be a metric space, and $f : X \to X$ be a contraction mapping, then (X,f) is called dynamical system, when $f^\circ(x) = x, f^{n+m}(x) = f^n(f^m(x)), \forall x \in X$ and n,m are positive integers. Let $x_{n+1} = f(x_n)$ for all positive integer n.

Then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x)$. Let $Y = \{ x \in X : f(x) = x \} \subseteq$ X. It is not easy to find Y, but $Y = \{ x \in X : \exists x_\circ \in X\ such\ that \lim_{n\to\infty} f^n(x_\circ) = x \}$, when $X = I^m$ and $f = (f_1, f_2,..., f_m)$ where $f_j(x) = x_j$ for all $j \in J_m$. Then $x = (x_1, x_2,..., x_m)$ implies $f(x) = x \in Y$. To find this x, let $x_n = (x_1^n, x_2^n,..., x_m^n)$ and $x_j^{n+1} = f_j(x_n)$. Therefore $\lim_{n\to\infty} f_j(x_n) = x_j = \lim_{n\to\infty}(f^n(x_\circ)$. This will form the escape time method.

3. Box Counting Dimension

Since the fractal dimension may be found by using the box counting dimension which is used on totally bounded sets by means of PCM for 2D monochrome images, which means that it is useful when we deals with fractals with dimension between 1 and 2, since we are aiming to find the dimension for the 3-dimentional object, this requires these images to be binarized in order to be estimated using pixel covering method; this may cause loss of information, which is not acceptable in many image processing applications, in addition, there is an underlying uncertainty accompanying of any estimation of FD according to the truncated error accrued as a result of iterative process and to using a sample of points for any object. To resolve this restriction the box counting method needs some modification to be general and suitable for all application based on 3- dimensional objects. Thus, a new approach is introduced in this work.[15]

Also, For computer applications, the data is usually discretized, the PCM is proposed to estimate the FD of fractal binarized images where points are represented by 1, while the background is represented by 0. The image Y is divided into squares with width ε, where $N_\varepsilon(Y)$ represents the minimum number of sets with radius less than or equals ε that covers Y. Hence, a group of data (-log δ_i, log $N_{\delta i}(Y)$) is obtained and the FD is estimated by changing the value of δ, it is the slop of the line derived from these data using the least squares linear regression. This is possible when the contraction mapping for the fractal Y is known, if not, so, it is not easy to calculate the box dimensions of a locally bounded subset $Y \subseteq I^m$, where $\delta(r^n) = \lim_{n\to\infty} \frac{ln(N_{(\frac{1}{2})^n}(Y)}{n\,ln 2}$. Hence, we proposed a new estimation method to find δ, which is presented as follows:

3.1 The proposed algorithm to estimate FD

For a sufficiently large integer $n \in N$, let $S_n=\{0,1,2,\ldots,2^n-1\}$,

and $p_n = \{ \frac{0}{2^n}, \frac{1}{2^n}, \ldots, \frac{2^n-1}{2^n} \}$

Where $p_n \subseteq I = [0,1]$ with $|s_n|=|p_n|=2^n$.

Let $x^j = \frac{1}{2}(1 + \frac{(-1)^{j_1}}{2} + \frac{(-1)^{j_2}}{2^2} + \cdots + \frac{(-1)^{j_n}}{2^n})$, where $j_i \in \{0,1\}$.

Then $j=(j_1,j_2,\ldots j_n)=j_1+2j_2+\ldots+2^n j_n$, is a permutation of S_n

and $x^j = (x_1^{j_1}, x_2^{j_2}, \cdots, x_m^{j_m}) \in I^m$ is a permutation of p_n.

Then $\hat{I}^m = \{ x^j = (x_1^{j_1}, x_2^{j_2}, \cdots, x_m^{j_m}) : j_i \text{ is a permution of } p_n \}$, then we have $|\hat{I}^m| = 2^{nm}$ pixels, and these pixels will form partition of I^m as a small box region, where their vertices are these pixels.

Let $\hat{Y}\hat{Y} = \{ x^j \in Y \}$ be the set of pixels in $Y \in H(I^m)$, since $N_{(1/2)^{n+1}}(Y) = min\{ m : \bigcup_{j \in J_m} \hat{B}(x_j^j, (\frac{1}{2})^{n+1} \supseteq Y \} = N_{(1/2)^{n+1}}(\hat{Y})$.

Therefore, $\hat{\delta} = \frac{N_{(1/2)^{n+1}}(Y)}{n\,ln 2} = \frac{N_{(1/2)^{n+1}}(\hat{Y})}{n\,ln 2}$ is an approximate box dimension of $Y \in H(I^m)$.

The number of pixels in $\hat{Y}\,\hat{Y}$ can be calculated through the scanning way to all of these pixels in a given space, and as follows:
For each $x^j \in \hat{I}^m$, factorize $x^j = (x_1^{j_1}, x_2^{j_2}, \cdots, x_m^{j_m})$ into m collections of subsets, each has 2^n pixels that represent the value of $x=(x_1,x_2,\ldots,x_m)$. This can be performed using the recursive sequence of points $t_{i-1}=2^n t_i+k_i$, and $x_i=(k_i/2^n)\in[0,1]$ that scans all $\hat{I}^m$ pixels. Hence, it is easy to count the number of pixels $N_n(\hat{Y})$ in $\hat{Y}$. Then $\hat{\delta}=\frac{N_n(\hat{Y})}{n\,ln\,2}$ that represents the box dimension of the set $Y\in H(I^m)$, as in the following algorithm.

PCMD1 Algorithm

Input $m,n\in N$, where n is a large positive integer
$p=2^n$, $q=p^m-1$
Input $t_0\in Z_q$
Input Y
Factor $(t_0)=(k_1,k_2,\ldots k_m)\in Z_q^m$
  For $t_0=0 : 2^{nm}-1$
    For i=1: m
      $t_{i-1}=t_i p+k_i$
      $x_i=k_i/p$
    End i
      If $x=(x_1,x_2,\ldots,x_m)\in Y$, then $No=No+1$
  End t_0
Output $(x_1,x_2,\ldots,x_m)\in I^m$
Output $\hat{\delta}=\frac{N_0}{n\,ln\,2}$

Then $\hat{\delta}\,\hat{\delta}$ is a box dimension of Y, and if Y is a fractal, then $\hat{\delta}$ is called box dimension of the fractal Y.

Example 1

Let m=2, n=6, let Y is the given set as follows, where Dim(Y)=0.6813.
Y={(0.15625,0),(0.03125,0.03125),(0.125,0.5),(0,0.75),(0.5,0.03125), (0.28125,0.0635),(0.25,0.25),(0.96875,0.0625), (0.78125,0.09375),(0.5,0.125)(0.375,0.25),(0.25,0.3125), (0.25,0.4375),(0.375,0.5635), (0.875,0.625),(0.125,0.953125), (0.859357,0.156625) }.

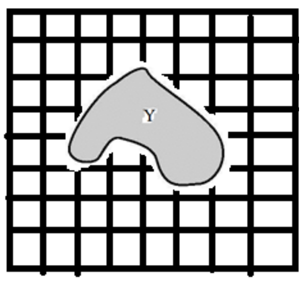


Figure-1:A given set Y in I^2

For n=2 , $\hat{\delta}=\frac{ln\,2}{2\,ln\,2}=0.5$ For n=3, $\hat{\delta}=\frac{ln\,6}{3\,ln\,2}$ =0.8617

For n=4, $\hat{\delta}=\frac{ln\,9}{4\,ln\,2}$ =0.7925   For n=5, $\hat{\delta}=\frac{ln\,15}{5\,ln\,2}$ = 0.7814

Then for large n, $\hat{\delta}=0.6813$.

4. Escape Time Dimension (ETD)

In this section, a new ETD is proposed, it is considered as a general algorithm whih is applicable for fractals generated using “Escape Time Algorithm”, [5]. These fractals are generated by repeatedly applying a transformation to a given point in the plane, using an initial point; the resulted series of the transformed points is called the orbit of this point. The orbit is called diverges when its points grow further apart without bounds. In this case, a fractal can be defined as; “the set of points whose orbit does not diverge”. With the existence of the same restriction mentioned in section 3, the proposed algorithm is considered useful, and also it is more efficient than the one proposed by [9], as we will show in the next section.

4.1Fractal generated by ETA

Let (X,d) be a metric space, for $f:X\to X$, $f^{0}(x)=x$, and $f^{n}(x)=f(f^{n-1}(x))$, for all $x\in X$. The sequence $\{x_n\}_{n=0}^{\infty}$ in X is generated as; $x_1=f(x_0)$, $x_2=f(x_1),\ldots,x_n=f(x_{n-1})$, where $x_n=f^{n}(x_0)$, for all $n\in N$, and $x_0\in X$. The convergence of the sequence $\{x_n\}$ in X is called *the attractor* of f in X, where $\{x=\lim_{n\to\infty} x_n\in X\}=Y=\{x\in X : f(x)=x\}\subseteq X$.

4.2 The Proposed Escape Time Dimension (ETD)

Let (R^m, d') is a complete metric space where d' is the usual metric and $Y\in H(R^m)$ where Y is closed and totally bounded (i.e there exist $M>0$, such that $D(x,y)\le M$, for all $x,y\in Y$). Now if $M>1$, then the dimension of the attractor Y can be calculated by $\delta=\lim_{\varepsilon\to\infty}\delta(\varepsilon)=\lim_{\varepsilon\to 0}\frac{ln(N_\varepsilon(Y))}{ln(1/\varepsilon)}$, where $Y\subseteq I^m=[0,1]^m\subseteq R^m$, and $d(x,y)=(1/M)$ $d'(x,y)\le 1$.

Hence δ is the ETD of the attractor in the set Y and it can be calculated using the same proposed algorithm, and as follows.

PCMD2 Algorithm

Input $m,n\in N$, where n is a large positive integer
$p=2^n$, $q=p^m$-1
Input $t_0\in Z_q$, $S\in N$
Factor $(t_0)=(k_1,k_2,\ldots k_m)\in Z_q^{\,m}$
  For $t_0=0 : 2^{nm}-1$
    For $I=1: m$
      $t_{i-1}=t_i p+k_i$
      $x_i=k_i/p$
    End I
     For $j=0: S$
      If $f^{j}(x)\in I^m$

End j

If $j>S$, then $x=(x_1,x_2,\ldots,x_m)\in Y$, and $No=No+1$

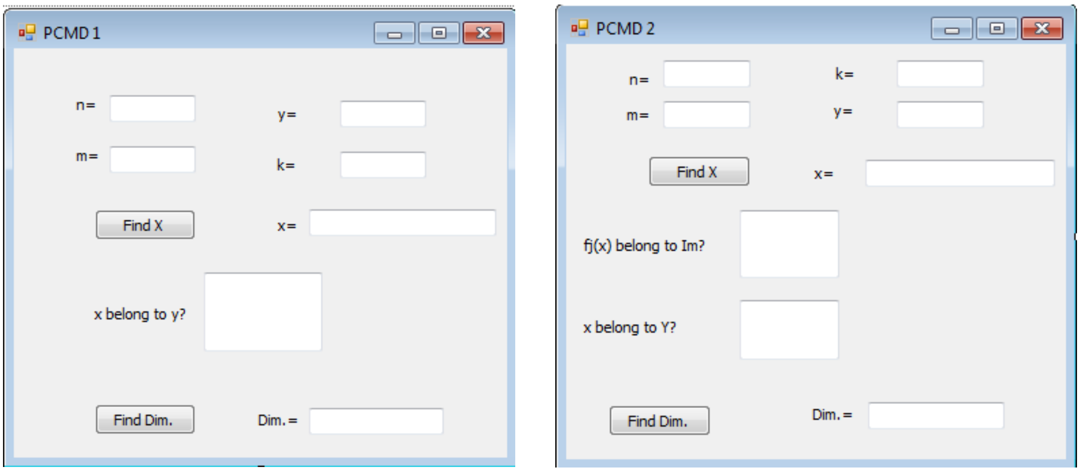
End t_0

Output Y, $\hat{\delta}=\dfrac{N_0}{n\ln 2}$

Then $\hat{\delta}\,\hat{\delta}$ is the ETD of Y, and if Y is a fractal, then $\hat{\delta}$ is called ETD of the fractal Y.

5. Experimental results and Implementation

The PCMD1 and PCMD2 algorithms with their graphic user interface (Figure 2) are carried out using Visual Basic. The results have been obtained by using a computer with the specifications 2.0 GHz Intel COR i5 CPU and 2 GB RAM.



A. General BCD using PCM   B. General ETD using PCM

Figure-2: User interface to calculate FD using PCMD1 and PCMD2

Table-1:Fractal Dimension comparison using the classical and proposed methods for some known fractal sets.

Iterated Function f	**BCD**	**ETD**	**PCMD1**	**PCMD2**
Sierpinski triangle	1.58496	1.58	1.58012	-
Filled Julia sets c=(0.2,0.7)	-	1.520263	-	1.51973
Cantor set	0.63093	0.63	0.629456	-
$Y=f(z)=z^2-1.25$	-	1.2845	-	1.27952

6.CONCLUSIONS

The fractal dimension is the basic concepts of fractal geometry and serves as an important feature of image. In many applications, the data sets do not strictly follow the definition of fractal, but only follow a certain range of scales. With the unavailability of a contraction mapping for many fractal images, the conventional methods that based on the using the coefficients in these mapping became useless, which motivating the researchers to always search for new methods to overcome these restrictions. In this paper, two new algorithms for the estimation of the FD are proposed. They are considered as a generalization for the well known and widely used algorithms; "*the box counting dimension*" and "the *escape time dimension*". It seems that from the experimental results, the proposed methods could be particularly useful to be applicable for various real world applications that based on gray scale and 3- dimensional levels in comparison with the corresponding classical methods used to estimate fractal dimension.

REFERENCES

1. Mandelbrot, B.B., "*The Fractal Geometry of Nature*", W.H. Freeman, NewYork, 1983.
2. Feng, Z.and Zhou, H., "*Computing method of fractal dimension of image and its application*". Journal of Jiangsu University of Science and Technology, No. 6, 2001, pp. 59-66.
3. Shuai, L.a, Xiangjiu, C.and Zhengxuan, W., "*Improvement of Escape Time Algorithm by No-Escape-Point*". Journal of Computers, Vol 6, No 8 (2011), pp. 1648-1653.
4. Falconer, K., "*Fractal Geometry Mathematical Foundations and Application*". John Wiley & Sons Ltd, 2nd edition, 2003.
5. Barnsley, M.F., "*Fractals everywhere*", 2ed. Academic Press Professional, Inc., San Diego, CA, USA, 1993.
6. Cherbit (Eds.), G., "*Fractals. Non-integral Dimensions and Applications*", Wiley, New York, 1990.
7. Grassberger, P.,"An Optimized Box-assisted al-gorithm for Fractal Dimensions" Phys. Lett. A 148 (1990),pp63-68.
8. Skubalska-Rafajłowicz, E., "*A new method of estimation of the box-counting dimension of multivariate objects using space-filling curves*". Nonlinear Analysis 63 (2005) e1281 – e1287.
9. Mohammed, A.J., "*On dimensions of some fractals*". Ph.D. thesis, Al- Mustansiriah University, 2005.
10. BÃlachowski ,A.and Ruebenbauer, K., "*Roughness Method to Estimate Fractal Dimension*". ACTA PHYSICA POLONICA A, Vol. 115 (2009).
11. André Backes, R.and Bruno Odemir, M., "*Fractal and Multi-Scale Fractal Dimension analysis: a comparative, study of Bouligand-Minkowski method*". CoRR abs/1201.3153: (2012).
12. Cholhui, Y., "*Box-Counting dimension of a kind of fractal interpolation surface on rectangular grids*". Romanian Journal of Mathematics and Computer science, 2(2)(2012), pp. 61-69

13. Peinke, J., Parisi,J., Rossler, O., E. and Stoop, R., “*Encounter with Chaos*”. Springer, New York, 1992.
14. Peitgen, H., Jürgens, H.and Saupe, D. ,“*Chaos and Fractals: New Frontiers of Science*”. 2ed, Springer, 2004.
15. Du, J. Gao, Y. Peng Qu, X. Xu, Zheng, Y., “*Multi- Multi-Feature Edge Extraction for Gray-Scale Images with Local Fuzzy Fractal Dimension*”. Seventh International Conference on Fuzzy Systems and Knowledge Discovery (FSKD 2010), 2010.