Using Semi-Analytic Technique for Solving Lane Emden Equations

Luma. N. M. Tawfiq , Reem. W. Hussein*
Department of Mathematics, College of Education Ibn Al-Haitham, University of Baghdad, Baghdad, Iraq
*ralazawy@yahoo.com

Abstract

This paper propose the semi - analytic technique using two point osculatory interpolation to construct polynomial solution for solving some well-known classes of Lane-Emden type equations which are linear ordinary differential equations, and disusse the behavior of the solution in the neighborhood of the singular points along with its numerical approximation. Many examples are presented to demonstrate the applicability and efficiency of the methods. Finally , we discuss behavior of the solution in the neighborhood of the singularity point which appears to perform satisfactorily for singular problems.

Keywords: ODE keys, Singular Initial Value Problems

استخدام التقنية شبه التحليلية في حل معادلات لان أمدن

لمى ناجي محمد توفيق ,ريم وليد حسين*
قسم الرياضيات, كلية التربية ابن الهيثم ,جامعة بغداد ,بغداد ,العراق

الخلاصة

الهدف من هذا البحث هو لعرض دراسة تحليلية لمعادلات لان أمدن التفاضلية الاعتيادية وبأنواع مختلفة حيث أننا نقترح التقنية شبه التحليلية باستخدام الاندراج التماسي ذو النقطتين للحصول على الحل كمتعددة حدود, كذلك ناقشنا عدد من الأمثلة لتوضيح الدقة , الكفاءة وسهولة أداء الطريقة المقترحة و أخيرا ناقشنا سلوك الحل في جوار النقاط الشاذة و إيجاد الحل التقريبي لها و اقترحنا صيغة جديدة مطورة لتخمين الخطأ تساعد في تقليل الحسابات العملية وإظهار النتائج بشكل مرضي فيما يخص المسائل الشاذة .

1. Introduction

In the study of nonlinear phenomena in physics, engineering and other sciences, many mathematical models lead to singular initial value problems (SIVP) associated with nonlinear second order ordinary differential equations (ODE) .

Lane-Emden type equations are linear ordinary differential equations on semi-infinite domain. They are categorized as singular initial value problems. These equations describe the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. The polytropic theory of stars essentially follows out of thermodynamic considerations, that deals with the issue of energy transport, through the transfer of material between different levels of the star. These equations are one of the basic equations in the theory of stellar structure and has been the focus of many studies [1,2].

The general form of the Lane-Emden equation :

$$x\, y'' + (A)\, y' = f(x, y) \quad , \quad 0 < x < 1 \qquad (1)$$

On some interval of the real line with some initial conditions.

2. Solution of Second Order Lane – Emden Equations

In this section we suggest semi analytic technique to solve second order Lane – Emden Equations as following, we consider:

$x\, y'' + f(x, y, y') = 0$ ..... (2a)

$y(0) = A\ ,\ y'(0) = B$ ….. (2b)

where f is in general nonlinear function of their arguments.

The simple idea behind the use of two-point polynomials is to replace y(x) in problem (2), or an alternative formulation of it, by osculatory interpolating polynomials of order (2n+1) a P_{2n+1} which enables any unknown initial values or derivatives of y(x) to be computed .

Therefore , the first step is to construct the P_{2n+1} , to do this we need the Taylor coefficients of y at x = 0 :

$$Y(x) = a_0 + a_1 x + \sum_{i=2}^{\infty} a_i x^i \quad \ldots\ldots \quad (3)$$

where $y(0)= a_0$,$y'(0)= a_1$,$y''(0) / 2! = a_2$,…, $y^{(i)}(0) / i! = a_i$, i= 3, 4,…

then inserts the series form (3) into (2a) and put x=0 and equate coefficients of powers of x to obtain a_2 . Also we need Taylor coefficient of y(x) about x = 1 :

$$y = b_0 + b_1 (x-1) + \sum_{i=2}^{\infty} b_i (x-1)^i \quad \ldots(4)$$

where $y(1) = b_0$, $y'(1) = b_1$, $y''(1) / 2! = b_2$,…, $y^{(i)}(1) / i! = b_i$, i = 3,4,…

then inserts the series form (4) into (2a) and put x = 1 and equate coefficients of powers of (x-1) to obtain b_2 ,then derive equation (2a) with respect to x to obtain new form of equation say (5) then, inserts the series form (3) into (5) and put x = 0 and equate coefficients of powers of x to obtain a_3 again inserts the series form (4) into (5) and put x = 1 and equate coefficients of powers of (x-1) to obtain b_3 , now iterate the above process many times to obtain a_4 , b_4 ,then a_5 ,b_5 and so on, that is ,we can get a_i and b_i for all $i \geq 2$, the resulting equations can be solved using MATLAB to obtain a_i and b_i for all $i \geq 2$, the notation implies that the coefficients depend only on the indicated unknowns a_0 , a_1, b_0 , b_1, and we get a_0 , a_1, by the initial condition .Now, we can construct a $P_{2n+1}(x)$ from these coefficients (a_is and b_is) by the following :

$$P_{2n+1} = \sum_{i=0}^{n} \{ a_i Q_i(x) + (-1)^i b_i Q_i(1-x) \} \ldots \quad (6)$$

where $(x^j / j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s = Q_j(x) / j!$

we see that (6) have only two unknowns b_0 and b_1 to find this, we integrate equation (2a) on [0, x] to obtain :

$$x\, y'(x) - \int_0^x y'(x)\, dx + \int_0^x f(s, y, y')\, ds = 0 \quad \ldots (7a)$$

and again integrate equation (7a) on [0, x] to obtain :

$$x\, y(x) - 2\int_0^x y(x)\, dx + a_0 x + \int_0^x (1-s) f(s,y,y')\, ds = 0 \quad \ldots(7b)$$

Putting x = 1 in (7), then gives :

$$b_1 - b_0 + a_0 + \int_0^1 f(s, y, y')\, ds = 0 \quad \ldots (8a)$$

and

$$b_0 - 2\int_0^1 y(x)\, dx + a_0 + \int_0^1 (1-s) f(s, y, y')\, ds = 0 \quad \ldots (8b)$$

Use P_{2n+1} as a replacement of y(x) in (8) and substitute the boundary conditions (2b) in (8) then, we have only two unknown coefficients b_1, b_0 and two equations (8) so, we can find b_1, b_0 for any n by solving this system of algebraic equations using MATLAB, so insert b_0 and b_1 into (6) , thus (6) represents the solution of (2) .

Extensive computations have shown that this generally provides a more accurate polynomial representation for a given n .

3. Examples

In this section, many examples of different forms of Lane -Emden will be given to illustrate the efficiency, accuracy , implementation and utility of the suggested method. The bvp4c solver of MATLAB has been modified accordingly so that it can solve some class of SIVP as effectively as it previously solved nonsingular IVP.

Example 1

Consider the following Lane – Emden equation :

$y''(x) + (2/x)\, y'(x) + y(x) = x^5 + 30\, x^3$, $0 \leq x \leq 1$ .With IC : $y(0) = 0$, $y'(0) = 0$.

The exact solution is : $y(x) = x^5$

It is clear that x = 0, is regular singular point and it is singularity of first kind . Now, we solve this example using semi - analytic technique , From equations (6) we have : $P_5(x) = x^5$

For more details ,Table 1 give the results for different nodes in the domain, for n = 2, i.e. P_5 and errors obtained by comparing it with the exact solution. Figure 1 gives the accuracy of the suggested method.

Ertürk [3] solved this example using differential transformation method and also, give the exact solution .

Table 1- The comparison between exact & P_5 of example 1

b_0	1		
b_1	5		
x_i	Exact solution y(x)	P_5	Error $\lvert y(x) - P_5\rvert$
0	0	0	0
0.1	0.00001	0.00001	0
0.2	0.00032	0.00032	0
0.3	0.00243	0.00243	0
0.4	0.01024	0.01024	0
0.5	0.03125	0.03125	0
0.6	0.07776	0.07776	0
0.7	0.16807	0.16807	0
0.8	0.32768	0.32768	0
0.9	0.59049	0.59049	0
1	1	1	0

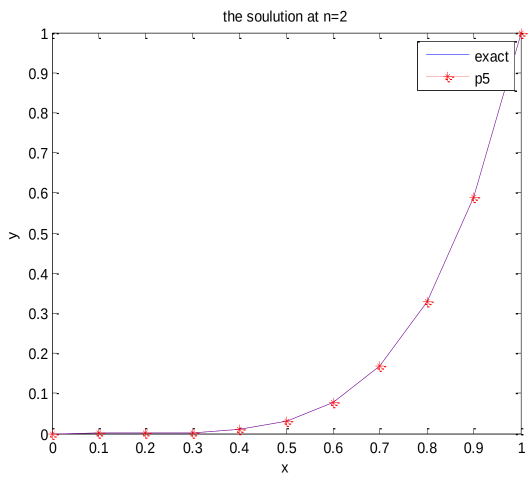

Figure 1- Comparison between the exact & semi-analytic solution P_5 of example1

Example 2

Consider the following Lane–Emden equation

$y'' + (2/x)\, y' + 2(2x^2 + 3)\, y = 0 , 0 \le x \le 1$

With IC : $y(0) = 1$, $y'(0) = 0$.

It is clear that x = 0 , is regular singular point and it is singularity of first kind and the exact solution is $y(x) = \exp(x^2)$. Now, we solve this example using semi-analytic technique ,From equation (6) we have : if n = 12 ,we have P_{25} as follows :

P_{25} = $0.000000015534x^{25} + 0.0000001418229 x^{24} + 0.00000065861 x^{23} - 0.0000018673 x^{22} + 0.000003752997083632007x^{21} - 0.0000051019199x^{20} + 0.000005686369x^{19} - 0.0000017042416x^{18} + 0.0000025706 x^{17} + 0.00002374116 x^{16} + 0.0000002969187 x^{15} + 0.000198362071 x^{14} + 0.00000000397 x^{13} + 0.0013888888889 x^{12} + 0.0083333333333 x^{10} + 0.0416666666667x^8 + 0.16666666667 x^6 + 0.5 x^4 + x^2 + 1$

Now, increase n, to get higher accuracy , let n = 13 , i.e. ,

$P_{27} = 0.00000000131953x^{27} - 0.0000000131885x^{26} + 0.000000066909x^{25} - 0.00000020992416x^{24} + 0.00000046803777x^{23} - 0.000000730448348x^{22} + 0.000000914015x^{21} - 0.0000005615242x^{20} + 0.0000005793768x^{19} + 0.00000245680143x^{18} + 0.00000011164989x^{17} + 0.0000247730217x^{16} + 0.000000004484619x^{15} + 0.0001984123722789x^{14} + 0.001388888889x^{12} + 0.00833333333x^{10} + 0.041666666667x^8 + 0.166666667x^6 + 0.5x^4 + x^2 + 1$

For more details ,Table 2a gives the results of different nodes in the domain, for n=10, 11, 12. Also**,** Figure 2 illustrate suggested method for n=12. Hojjati [4] solved this equation by second derivative multistep method (SDMMs) and the results given in Table 2b . Ramos [5] solved this equation using linearization method and the absolute error given in the Figure 3 Hasan [6] solved this equation using Taylor series and gave the following result :

$$y(x) = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \ldots$$

Also, Mohyud-Din [7] solved this equation using He’s polynomials and gave the result in the following series :

$$y(x) = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \frac{1}{5!}x^{10} + \cdots$$

And , Batiha [8] solved this equation by variational iteration method (VIM) and the results given in the Table 3 .Yaghoobi [9] solved this equation by differential transformation method (DTM) and the result given in the following series :

$$y(x) = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \frac{1}{5!}x^{10} + \cdots$$

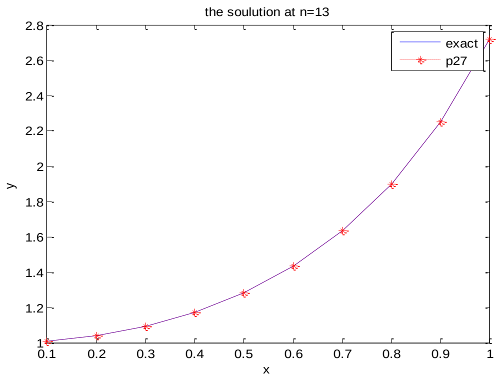

Figure 2- **I**llustrate suggested method for n= 13,i.e.,P_{27} of example 2 .

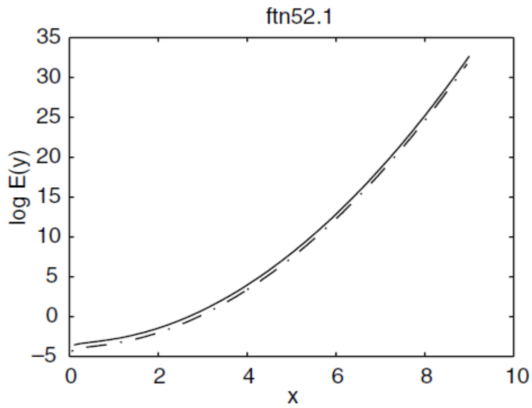

Figure 3-The absolute error of method given in[5]

Table 2a- The results of the suggested method for n= 10,11,12 of example2

b_0	2.718281828459046			
b_1	5.436563656918091			
x_i	P_{27}	Exact solution y(x)	Error of VIM (with 5iterate)	Error $\|y(x)-P_{27}\|$
0.1	1.010050167084168	1.010050167084168	7.60000E-18	2.503132204460492e-016
0.2	1.040810774192390	1.040810774192388	3.23412E-14	1.00257763568394e-015
0.3	1.094174283705210	1.094174283705210	4.47700E-12	2.503132204460492e-016
0.4	1.173510870991810	1.173510870991810	1.54546E-10	2.503132204460492e-016

Table 2b-The results of the suggested method for n= 12& result in[4] for example2

b_0		2.718281828459122	2.718281828459043	2.718281828459045
b_1		5.436563656918276	5.436563656918087	5.436563656918090
x_i	Exact solution y(x)	P_{21}	P_{23}	P_{25}
0.25	1.06449445891786	1.064494458917865	1.064494458917859	1.064494458917859
0.5	1.28402541668774	1.284025416687875	1.284025416687738	1.284025416687742
0.75	1.7550546569603	1.755054656960343	1.755054656960297	1.755054656960299
1	2.71828182845904	2.718281828459122	2.718281828459043	2.718281828459045

Table 3- Absolute error between the VIM (with5-iterate) [8] and the exact solution

x_i	SDMMs $y_1(x)$	Error $\|y(x) - y_1(x)\|$	Error $\| y(x) - p_{25} \|$
0.25	1.06449445891768E+0	1.77 E+13	e-016666.6613381477509e
0.5	1.28402541668753E+0	2.14E+13	1.554312234475219e-015
0.75	1.7550546569600E+0	2.93E+13	1.332267629550188e-015
1	2.71828182845859E+0	4.54E+13	4.884981308350689 1e-015
SSE		3.6932e-025	2.849760020110905e029

4.Behavior of the solution in the eighborhood of the singularity x= 0

Our main concern in this section will be the study of the behavior of the solution in the neighborhood of the singular point x=0.
Consider the following SIVP :

$y''(x) + ((N-1)/x)\, y'(x) = f(y)$, $N \geq 1$, $0 < x < 1$ ……… (9)

$y(0) = y_0$, $\lim_{x\to 0+} x\, y'(x) = 0$ .. (10)

where f(y) is continuous function .

As the same manner in [10], let us look for a solution of this problem in the form :

$y(x) = y_0 - C\, x^k (1 + o(1))$ …. (11)

$y'(x) = - C\, k\, x^{k-1}(1 + o(1))$

$y''(x) = - C\, k\, (k-1)\, x^{k-2}(1 + o(1))$

where C is a positive constant and k > 1. If we substitute (11) in (9) we obtain :

$C = (1/k)\, (f(y_0)\ /N)^{k-1}$ …. (12)

In order to improve representation (11) we perform the variable substitution :

$y(x) = y_0 - C\, x^k (1 + g(x))$ …..(13)

we easily obtain the following result which is similar to the results in [10].

Theorem 1

For each $y_0 > 0$, problem (9), (10) has, in the neighborhood of x = 0, a unique solution that can be represented by :

$y(x, y_0) = y_0 - C\, x^k (1 + g\, x^k + o(x^k))$,

where k, C and g are given by (12) and (13), respectively.

We see that these results are in good agreement with the ones obtained by the method in [10], they are also consistent with the results presented in [11]. In order to estimate the convergence order of the suggested method at x = 0, we have carried out several experiments with different values of n and used the formula:

$c_{y0} = -\log_2 (|y_0^{n3} - y_0^{n2}| / |y_0^{n2} - y_0^{n1}|)$ … (14)

where y_0^{ni} is the approximate value of y_0 obtained with n_i ,n_i = 1,2, 3, 4,…

References

1. Motsa, S. S. and Sibanda, P. **2011**.A new algorithm for solving singular IVPs of Lane-Emden type , Latest Trends on Applied Mathematics, pp: 176-180.
2. Howell K.B. **2009**. Ordinary Differential Equations" , USA , Spring.
3. Ertürk, V. S. **2007**. Differntial Transformation Method for Solving Differential Equations of Lane - Emden Typy, Mathematical and Computational Applications, Vol. **12**, No. 3, pp. 135-139.
4. Hojjati, G.and Parand, K. **2011**, An efficient computational algorithm for solving the nonlinear Lane-Emden type equations , International Journal of engineering and natural science ,vol. **5**, No.3 , pp: 106-111.
5. Ramos, J. I. **2005**, Linearization techniques for singular initial value problems of ordinary differential equations , Applied Mathematics and Computation , vol. **161** , pp: 525-542 .
6. Hasan ,Y. Q. and Zhu, L. M. **2007**, Solving Singular Initial Value Problems in the Second-Order Ordinary Differential Equations, Journal of Applied Sciences , Vol. **7**, No. 17, pp: 2505-2508 .
7. Mohyud-Din, S. T. **2009**, M.A. Noor and K. I. Noor, Solving Second-order Singular Problems Using He's Polynomials , World Applied Sciences Journal, Vol. **6** , No. 6, pp: 769-775 .
8. Batiha, B. **2009**, Numerical Solution of a Class of Singular Second-Order IVPs by Variational Iteration Method , Int. Journal of Math. Analysis, Vol. **3**, No. 40, pp: 1953 – 1968.
9. Yaghoobi, H., Fereidoon, A . , Khoshnevis P.and Ganji D. D. **2010**, Solution of the Lane- Emden Type Equations Arising in Astrophysics Using Differential Transformation Method, World Applied Sciences Journal , Vol. **9**, No. 10, pp: 1164-1170 .
10. Morgado, L.and Lima, P. **2009**, Numerical methods for a singular boundary value problem with application to a heat conduction model in the human head, Proceedings of the International Conference on Computational and Mathematical Methods in Science and Engineering, CMMSE .
11. Abukhaled, M., Khuri, S.A. and Sayfy A., **2011**,A Numerical Approach for Solving a Class of Singular Boundary Value Problems Arising in Physiology , International Journal of Numerical Analysis and Modeling ,Vol.**8**, No.2, PP:353–363.