On Centrally Semiprime Rings and Centrally Semiprime Near-Rings with Derivations

Adil Kadir Jabbar ***and*** ***Abdulrahman Hamed Majeed***
College of Science - University of Sulaimani
College of Science - University of Baghdad

Abstract

In this paper, two new algebraic structures are introduced which we call a centrally semiprime ring and a centrally semiprime right near-ring, and we look for those conditions which make centrally semiprime rings as commutative rings, so that several results are proved, also we extend some properties of semiprime rings and semiprime right near-rings to centrally semiprime rings and centrally semiprime right near-rings.

Introduction

Let R be a ring . A non-empty subset S of R is said to be a multiplicative system in R if $0\notin S$ and $a,b\in S$ implies $ab\in S$ (Larsen & McCarthy,1971). Let S be a multiplicative system in R such that $[S,R]=\{0\}$, where $[S,R]=\{[s,r]:s\in S, r\in R\}$ and $[s,r]$ is the commutator defined by $sr-rs$. Define a relation $(\sim)$ on $R\times S$ as follows : If $(a,s),(b,t)\in R\times S$, then $(a,s)\sim(b,t)$ if and only if there exists $x\in S$ such that $x(at-bs)=0$. Since $[S,R]=\{0\}$, it can be shown that $(\sim)$ is an equivalence relation on $R\times S$. Let us denote the equivalence class of (a,s) in $R\times S$ by a_S, and the set of all equivalence classes determined under this equivalence relation by R_S, that is, let $a_S=\{(b,t)\in R\times S:(a,s)\sim(b,t)\}$ and $R_S=\{a_s:(a,s)\in R\times S\}$. (the equivalence class a_S is also denoted by $\frac{a}{s}$ (Larsen & McCarthy, 1971) or by $s^{-1}a$ (Ranicki, 2006), and R_S is also denoted by $S^{-1}R$ (Larsen & McCarthy, 1971 ; Ranicki, 2006). We define addition $(+)$ and multiplication $(.)$ on R_S as follows: $a_S+b_t=(at+bs)_{st}$ and $a_S.b_t=(ab)_{st}$, for all $a_S,b_t\in R_S$. It can be shown that these two operations are well-defined and that $(R_S,+,.)$ forms a ring which is known as the localization of R at S ,(Fahr, 2002).

Next we mention to some basic definitions:
Let R be a ring. Then R is called semiprime, if for $a \in R$, $aRa = \{0\}$ implies $a=0$ (Ashraf, 2005), where $aRa = \{ara : r \in R\}$. An additive mapping $D : R \to R$ is called a derivation on R if $D(ab)=D(a)b+aD(b)$, for all $a,b \in R$ (Jung & Park, 2006) and a derivation $D : R \to R$ is called a centrally zero derivation if $D(S) = \{0\}$ for each multiplicative system S in R with $[S,R] = \{0\}$. By $Z(R)$ is meant the center of R. We say R satisfies the central commutativity property (CCP) if R_S is commutative for each multiplicative system S in R with $[S,R] = \{0\}$. If n is a positive integer then R is called an $n-$torsion free, if for $x \in R$, $nx=0$ implies $x=0$ (Vukman, 2001). An additive subgroup U of R is called a Lie ideal of R if $[U,R] \subseteq U$ (Ali & Kumar, 2007) and a Lie ideal U of R is called a centrally closed Lie ideal if $US \subseteq U$ for each multiplicative system S in R with $[S,R]=\{0\}$ (Jabbar, 2007). (Note that by US we mean the set $US = \{us : u \in U, s \in S\}$). By a right near-ring is meant a non-empty set N with two operations addition (+) and multiplication (.) such that the following conditions satisfied, (Pilz, 1983 ; Kandasamy, 2002).
(i): $(N,+,.)$ is a group (not necessarily abelian). **(ii):** $(N,.)$ is a semigroup and **(iii):** For all $a,b,c \in N$, we have $(a+b).c = a.c + b.c$.

It is necessary to mention that, in a right near-ring N, we have $0a = 0$, for all $a \in N$ while the identity $a0 = 0$, for $a \in N$ need not satisfied in general, and the following example shows this fact, which can be found in (Jabbar, 2007).

Consider the usual addition (+) of integers and define the multiplication (.) on Z as follows: $a.b = a$, for all $a,b \in Z$, it can be shown that $(Z,+,.)$ is a right near-ring. Clearly $0.a = 0$, for all $a \in R$, while $1.0 = 1 \neq 0$. A right near-ring N is called zero-symmetric, if $a0 = 0$, for all $a \in N$ (Pilz, 1983 ; kandasamy, 2002).

Now we mention to the following remarks which can be found in (Jabbar, 2006).

Let R be a ring and S a multiplicative system in R such that $[S,R]=\{0\}$.If $s \in S$, then s_S is the identity element of R_S and 0_S is the zero of R_S, also it is easy to check that $s_S = t_t$, and $0_s = 0_t$, for all $s,t \in S$. If $a,b \in R$, then we have $a_S + b_S = (a+b)_S$, and this result can be generalized to any n elements of R. If $a_S \in R_S$, for $a \in R$ and $s \in S$

then $(-a)_S$ is the additive inverse of a_S in R_S, that is $-a_S=(-a)_S$, and if $a_S=0$ in R_S, then there exists $t\in S$ such that $ta=0$.

Next we mention to the following results , the proofs of which can found in the indicated references and we will use their statements in driving our main results.

Lemma 1: (Jabbar, 2006)

Let R be a ring and S a multiplicative system in R such that $[S,R]=\{0\}$.

If $D:R\to R$ is a centrally-zero derivation on R, then $D_*:R_S\to R_S$ defined by

$D_*(r_S)=(D(r))_S$, for all $r_S\in R_S$, is a derivation on R_S.(Note that D_* is called the induced derivation by D).

Lemma 2: (Jabbar, 2007)

If a ring R satisfies (CCP) and $Z(R)$ contains no proper zero divisors then R is commutative.

Lemma 3: (Q. Deng and M. Ashraf, 1996)

Let R be a semiprime ring. If R admits a mapping F and a derivation D such that $[F(x),D(y)]=[x,y]$, for all $x,y\in R$, then R is commutative.

Lemma 4: (Hongan, 1997)

Let R be a 2-torsion free semiprime ring and I a nonzero ideal of R. If $D:R\to R$ is a derivation such that $D([x,y])+[x,y]\in Z(R)$ or $D([x,y])-[x,y]\in Z(R)$, for all $x,y\in I$, then R is commutative.

Lemma 5: (Jabbar, 2007)

If R is an $n-$torsion free ring and S is a multiplicative system in R such that $[S,R]=\{0\}$, then R_S is also $n-$torsion free.

Lemma 6: (Jabbar, 2007)

Let R be a ring and S a multiplicative system in R with $[S,R]=\{0\}$.If U is a centrally closed Lie ideal of R, then U_S is a Lie ideal of R_S.

Lemma 7: (Herstien, 1969)

Let R be a 2-torsion free semiprime ring. If U is a commutative Lie ideal of R, then $U\subseteq Z(R)$.

Lemma 8: (Jabbar, 2007)

If N is a zero-symmetric right near-ring and S is a multiplicative system in N with $[S,N]=\{0\}$, then N_S is also a zero-symmetric right near-ring.

Lemma 9: (Jabbar, 2007)

Let N be a zero-symmetric right near-ring and S is a multiplicative system in N with $[S,N]=\{0\}$. If $D:N\to N$ is a centrally zero derivation on N, then $D_*:N_S\to N_S$, defined by $D_*(a_S)=(D(a))_S$, for all $a_S\in N_S$, is a derivation on N_S.(is called the induced derivation by D).

Lemma 10: (Argac, 1997)

Let N be a zero-symmetric semiprime right near-ring and $D:N\to N$ is a derivation on N. If $A\subseteq N$ such that $0\in A$ and $AN\subseteq A$, and D acts as a homomorphism on A or as anti-homomorphism on A, then $D(A)=\{0\}$.

Lemma 11: (Jabbar, 2007)

If R is a ring (resp. a right near-ring), in which $Z(R)$ contains no proper zero divisors of R, then $Z(R)-\{0\}$ is a multiplicative system in R and $[Z(R)-\{0\},R]=\{0\}$.

Lemma 12: (Jabbar, 2007)

If R is a ring and S is a multiplicative system in R with $[S,R]=\{0\}$, then $(Z(R))_S\subseteq Z(R_S)$.

Throughout this paper all rings under consideration are nonzero unless otherwise stated.

The Main Results:

Before giving the main results of the paper we need to introduce some new definitions and giving some examples.

Introduced Definitions

1: Let R be a ring. We say R is a centrally semiprime ring if R_S is a semiprime ring for each multiplicative systems S in R with $[S,R]=\{0\}$.

2: Let N be a zero-symmetric right near-ring. We say N is a centrally semiprime right near-ring if N_S is a semiprime right near-ring for each multiplicative system S in N with $[S,N]=\{0\}$.

Next we give two examples, one for a centrally semiprime ring which is not semiprime and the other, for a centrally semiprime right near-ring which is not semiprime.

Examples :

(1): Let $R=\{\bar{0},\bar{2},\bar{4},\bar{6}\}$.Then $(R,+_8,\cdot_8)$ is a ring. It is clear that R is not semiprime since $\bar{4}R\bar{4}=\{\bar{0}\}$, but $\bar{4}\neq\bar{0}$. On the other hand if R is not centrally semiprime then there exists a multiplicative system S in R with $[S,R]=\{\bar{0}\}$ for which R_S is not semiprime. But R has no multiplicative system since the only subsets of R which does not contain $\bar{0}$ are $\{\bar{2}\},\{\bar{4}\},\{\bar{6}\},\{\bar{2},\bar{4}\},\{\bar{2},\bar{6}\},\{\bar{4},\bar{6}\}$ and $\{\bar{2},\bar{4},\bar{6}\}$, and by simple computations we can see that none of these sets is a multiplicative system in R, and so R is a centrally semiprime ring but not semiprime.

(2): Let $R=\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10},\overline{12},\overline{14}\}$, then clearly $(R,+_{16},\cdot_{16})$ is a right near-ring. Since $\bar{4}R\bar{4}=\{\bar{0}\}$, but $\bar{4}\neq\bar{0}$, so R is not semiprime. On the other hand it can be shown that R has no any multiplicative system, thus R is a centrally semiprime right near-ring but not semiprime.

Now we give two corollaries, which are especial cases of **Lemma 3** and **Lemma 4**. In fact, if we take $F=D$ in **Lemma 3,** then one can get:

Corollary 13:

Let R be a semiprime ring. If R admits a derivation D such that $[D(x),D(y)]=[x,y]$, for all $x,y\in R$, then R is commutative.

Also**,** by taking $I=R$ in **Lemma 4,** we can get:

Corollary 14:

Let R be a 2-torsion free semiprime ring. If $D:R\to R$ is a derivation such that
$D([x,y])+[x,y]\in Z(R)$ or $D([x,y])-[x,y]\in Z(R)$, for all $x,y\in R$, then R is commutative.

Now it is the time to prove the first result of this paper.

Lemma 15:

If R is a ring which has no proper zero divisors then it is centrally semiprime.

Proof :

We will show R is centrally semiprime. If $R=\{0\}$, then it has no any multiplicative system and thus R is centrally semiprime. So let $R\neq\{0\}$. Let S be any multiplicative system in R such that $[S,R]=\{0\}$. To show R_S is a semiprime ring. Let for $a_s\in R_S$ we have $a_S R_S a_s=\{0\}$, where $a\in R$ and $s\in S$. Since $R\neq\{0\}$, so there exists $0\neq r\in R$. Then $r_S\in R_S$. Hence $a_S r_S a_s\in a_S R_S a_s$ which gives $a_S r_S a_s=0$ or $(ara)_{sss}=0$, and hence there exists $t\in S$ such that $t(ara)=0$ or $tara=0$, but R has no proper zero divisors so $t=0$ or $a=0$ or $r=0$. Since $0\notin S$ and $t\in S$ so $t\neq 0$ and $r\neq 0$ (r is chosen non-zero in R) thus we get $a=0$, and then $a_S=0_S=0$. Hence R_S is a semiprime ring, which means that R is a centrally semiprime ring ♦ .

Next we give some conditions which make centrally semiprime rings as commutative rings.

Theorem 16:

Let R be a centrally semiprime ring in which $Z(R)$ contains no proper zero divisors of R. If $D:R\to R$ is a centrally-zero derivation on R such that $[D(x),D(y)]=[x,y]$, for all $x,y\in R$, then R is commutative.

Proof:

First to show that R satisfies (CCP). If R does not satisfy (CCP), then there exists a multiplicative system S' in R with $[S',R]=\{0\}$ for which $R_{S'}$ is not commutative.Let $D_*:R_{S'}\to R_{S'}$ be the induced derivation of **Lemma 1,** on $R_{S'}$. Now if $a_S,b_t\in R_{S'}$ are any elements then we have $[D_*(a_S),D_*(b_t)]=[(D(a))_S,(D(b))_t]=$
$(D(a))_S(D(b))_t-(D(b))_t(D(a))_S=(D(a)D(b))_{st}-(D(b)D(a))_{ts}=$
$(D(a)D(b)-D(b)D(a))_{st}=([D(a),D(b)])_{st}=([a,b])_{st}=(ab-ba)_{st}=$
$(ab)_{st}-(ba)_{ts}=a_Sb_t-b_ta_S=[a_S,b_t]$. (Note that, since $[S,R]=\{0\}$ so $st=ts$). Now $R_{S'}$ is a semiprime ring and $D_*:R_{S'}\to R_{S'}$ is a derivation on $R_{S'}$ such that $[D_*(a_S),D_*(b_t)]=[a_S,b_t]$, for all $a_S,b_t\in R_{S'}$.Hence by **Corollary 13,** $R_{S'}$ is commutative, which is a contradiction and thus R must satisfy (CCP), and since $Z(R)$ contains no proper zero divisors of R, so by **Lemma 2,** we get R is commutative ♦ .

As a corollary to **Theorem 16,** we give:

Corollary 17:

If R is a ring which has no proper zero divisors and $D:R\to R$ is a centrally zero derivation on R such that $[D(x),D(y)]=[x,y]$, for all $x,y\in R$, then R is commutative.

Proof:

Since R has no proper zero divisors so by **Lemma 15,** we get R is a centrally semiprime ring and as R has no proper zero divisors, so $Z(R)$ contains no proper zero divisors of R and thus by **Theorem 16,** we get that R is commutative ♦ .

Theorem 18:

Let R be a 2-torsion free centrally semiprime ring in which $Z(R)$ contains no proper zero divisors of R and $D: R \rightarrow R$ be a centrally zero derivation on R such that $D([x,y])+[x,y]\in Z(R)$, for all $x,y\in R$ or $D([x,y])-[x,y]\in Z(R)$, for all $x,y\in R$, then R is commutative.

Proof:

We will show that R satisfies (CCP). If R does not satisfy (CCP) then there exists a multiplicative system S in R with $[S,R]=\{0\}$ for which R_S is not commutative. Since R is 2-torsion free, so by **Lemma 5** , we have R_S is also a 2-torsion free ring, and since R is centrally semiprime so R_S is semiprime. Now let $D_*: R_S \rightarrow R_S$ be the induced derivation of **Lemma 1,** on R_S, where $D_*(r_s)=(D(r))_s$, for all $r_s \in R_S$. We take the first case when $D([x,y])+[x,y]\in Z(R)$, for all $x,y\in R$. Let $a_s, b_t \in R_S$, where $a,b\in R$ and $s,t\in S$,and now

$$D_*([a_s,b_t])+[a_s,b_t]=D_*(a_s b_t - b_t a_s)+(a_s b_t - b_t a_s)=$$
$$D_*((ab)_{st}-(ba)_{ts})+((ab)_{st}-(ba)_{ts})=D_*((ab-ba)_{st})+(ab-ba)_{st}=$$
$$(D(ab-ba))_{st}+([a,b])_{st}=(D([a,b]))_{st}+([a,b])_{st}=$$
$$(D([a,b])+[a,b])_{st}\in (Z(R))_S \subseteq Z(R_S),$$ (see **Lemma 12**).

Hence R_S is a 2-torsion free semiprime ring and $D_*: R_S \rightarrow R_S$ is a derivation on R_S such that $D_*([a_s,b_t])+[a_s,b_t]\in Z(R_S)$, for all $a_s, b_t \in R_S$ so by **Corollary 14,** we get R_S is commutative, which is a contradiction and hence R must satisfy (CCP). Since $Z(R)$ contains no proper zero divisors of R, so by **Lemma 2,** we get R is commutative. If we take the case when $D([x,y])-[x,y]\in Z(R)$, for all $x,y\in R$, then by using the same technique as in the first case the result is obtained ♦ .

As a corollary to **Theorem 18,** we give:

Corollary 19:

Let R be a 2-torsion free ring which has no proper zero divisors and $D: R \rightarrow R$ is a centrally zero derivation on R. If either $D([x,y])+[x,y]\in Z(R)$, for all $x,y\in R$ or $D([x,y])-[x,y]\in Z(R)$, for all $x,y\in R$, then R is commutative.

Proof:

Since R has no proper zero divisors so by **Lemma 15,** we get R is centrally semiprime and since R has no proper zero divisors, so $Z(R)$

contains no proper zero divisors of R, and thus by **Theorem 18**, we get R is commutative ♦ .

Finally, we prove two properties one for centrally semiprime rings and the other for zero-symmetric centrally semiprime right near-rings.

Theorem 20:

Let R be a 2-torsion free centrally semiprime ring in which $Z(R)$ has no proper zero divisors and U is a commutative centrally closed Lie ideal of R, then $U \subseteq Z(R)$.

Proof:

By **Lemma 11,** we have $S = Z(R) - \{0\}$ is a multiplicative system in R with $[S,R]=\{0\}$. By **Lemma 5,** we have R_S is a 2-torsion free semiprime ring, and by **Lemma 6,** we get U_S is a Lie ideal of R_S. Since U is commutative, it can be shown that U_S is also commutative. Hence by **Lemma 7,** we get $U_S \subseteq Z(R_S)$. To show $U \subseteq Z(R)$. Let $u \in U$ and $r \in R$ be any elements. Then for a fixed $s \in S$ we have $u_S \in U_S$ and $r_S \in R_S$. Now $([u,r])_{SS} = [u_S, r_S] = 0$, and thus there exists $t \in S$ such that $t[u,r] = 0$. Since $Z(R)$ contains no proper zero divisors of R and $0 \neq t \in S \subseteq Z(R)$ so we get $[u,r]=0$. Hence $[U,R]=\{0\}$, that means $U \subseteq Z(R)$ ♦ .

Now we give a property of centrally semiprime right near-rings.

Theorem 21:

Let N be a zero-symmetric centrally semiprime right near-ring in which $Z(N)$ contains no proper zero divisors of N and $D: N \to N$ be a centrally-zero derivation of N and let A be a subset of N such that $0 \in A$ and $AN \subseteq A$, if D acts as a homomorphism on A or as anti-homomorphism on A, then $D(A)=\{0\}$.

Proof:

By **Lemma 11,** we have $S = Z(N) - \{0\}$ is a multiplicative system in N with $[S,N]=\{0\}$. Then from **Lemma 8,** we get that N_S is a zero-symmetric right near-ring, and it is also semiprime (since N is centrally semiprime). Now since D is a cenyrally-zero derivation on N so by **Lemma 9,** we get that $D_* : N_S \to N_S$, which is defined by $D_*(a_S) = (D(a))_S$, for all $a_S \in N_S$, is a derivation on N_S. Next, since $0 \in A$ so $0_S \in A_S$, for all $s \in S$, that means the zero of N_S is in A_S. Also since $A \subseteq N$ and $AN \subseteq A$ so $A_S \subseteq N_S$ and then easily can be shown that $A_S N_S \subseteq (AN)_S \subseteq A_S$. To show D_* acts as a homomorphism on A_S or as anti-homomorphism on A_S. Now if D acts as a homomorphism on A, then for $a_s, b_t \in A_S$, we have

$D_*(a_s b_t) = D_*((ab)_{st}) = (D(ab))_{st} = (D(a)D(b))_{st} = (D(a))_s (D(b))_t =$
$D_*(a_s) D_*(b_t)$, that means D_* acts as a homomorphism on A_S and if D acts as anti-homomorphism on A then by the same technique we can show D_* acts as anti-homomorphism on A_S.Hence we have N_S is a zero-symmetric semiprime right near-ring , D_* is a derivation on N_S, A_S is a subset of N_S such that A_S contains the zero of N_S with $A_S N_S \subseteq A_S$ and D_* acts as a homomorphism on A_S or as anti-homomorphism on A_S. Thus by **Lemma 10,** we get $D_*(A_S)=0$. To show that $D(A)=\{0\}$. Let $a \in A$. Since $S \neq \phi$ so there exists $s \in S$. Hence $a_s \in A_S$ so that $(D(a))_s = D_*(a_s) = 0$. Hence there exists $t \in S$ such that $tD(a)=0$, where $t \in S = Z(N) - \{0\}$. Now if $D(a) \neq 0$ then t is a proper zero divisor, that means $Z(N)$ contains a proper zero divisor of N which is a contradiction and hence $D(a)=0$, and this result is true for all $a \in A$ and hence $D(A)=\{0\}$ ♦ .

As a corollary to **Theorem 21,** we give:

Corollary 22:

Let N be a zero-symmetric centrally semiprime right near-ring in which $Z(N)$ has no proper zero divisors and $D: N \to N$ is a centrally zero derivation on N. If D acts as a homomorphism on N or acts as anti-homomorphism on N then $D=0$.

Proof:

Put $A=N$ in **Theorem 21,** the result will follows ♦ .

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حول الحلقات الشبه الاولية مركزيا و الحلقات- المقتربة الشبه الاولية مركزيا مع الاشتقاقات

عادل قادرجبار و عبدالرحمن حميد مجيد
كلية العلوم ــ جامعة السليمانية
كلية العلوم ــ جامعة بغداد

الخلاصة

في هذا البحث تم تعريف بنيتان جبريتان جديدتان وسميناهما حلقة شبه اولية مركزيا وحلقة- مقتربة شبه اولية مركزيا. لقد تم تحديد بعض الشروط والتي تجعل من الحلقات الشبه الاولية مركزيا حلقات تبادلية وتم ايضا برهان العديد من النتائج .كذلك تم نقل بعض خواص الحلقات الشبه الاولية والحلقات المقتربة والشبه الاولية الى الحلقات الشبه الاولية مركزيا والحلقات المقتربةاليمنى والشبه الاولية مركزيا.