تصنيف خط الإسقاط على حقل كالوز من الرتبة 23

د. نجم عبدالزهره مخرب السراجي
قسم الرياضيات– كلية العلوم– الجامعة المستنصرية–بغداد

الخلاصة

في هذا البحث،المجموعات من النوع k في خط الإسقاط $PG(1,23)$ والتي هي مجموعة غير مرتبة من k نقاط مختلفة عندما $k = 12$ هو تم تصنيفه. التجزئات إلى $PG(1,23)$ إلى مجموعتان من اثنا عشر و كذلك إلى ستة مجاميع غير منفصلة من أربع نقاط تم إيجادها. الأدوات الرئيسية للحساب هي لغة البرمجة الرياضياتية باستخدام برمجة خوارزمية المجموعات. المعادلة الاسقاطية للمجموعات من النوع k و الزمر المثبتة لها تم تشكيلها.

Classification of The Projective Line Over Galois Field of Order 23

Dr. Najm A.M. Al-seraji
Department of Mathematics, College of Science, Al-Mustansiriya University, Baghdad, najem_abdul@yahoo.com

ABSTRACT

In this paper, the k-sets in projective line $PG(1,23)$ which are unordered set of k distinct points up to $k = 12$ are classified. Partitions of $PG(1,23)$ into two 12-sets and into six disjoint 4-sets are found. The main computing tool is the mathematical programming language Groups Algorithms of Programming GAP. The Projective equation of k-sets and the stabilizer groups of them are constructed.

Key words: Projective line, The orbits, stabilizer groups, Partitions on projective line.

1. INTRODUCTION

The main object of this paper is k-sets. In $PG(1,q)$, the projective line over the finite field $\mathbb{F}_q$ of order $q = p^h$ elements, here $q = p = 23$, $h = 1$. The area of research is consider links between the subjects of Projective Geometry, Vector Spaces, Coding Theory, Group Theory, such that

- Projective Geometry, an n-set in $PG(1,q)$, that is , a set of unordered distinct of n points;
- Vector Spaces, a set of n vectors in a vector space of two dimensions $V(2,q)$ with any two linearly independent;
- Coding Theory, a maximum distance separable code of length n, dimension 2 and hence minimum distance $d = n - 1$, that is, an $[n, 2, n-1]$ code.

The three above notions are equivalent for $n \geq 2$, also the concepts of subject of Group Theory are applied to calculate the transformations between the k-sets, the orbits and the stabilizer group. Also, GAP- Groups- Algorithms, Programming- a system for computational Discrete Algebra has been used.

Associated to any topic in mathematics is its history. k-sets in $PG(1,q)$ for $q = 2,3,4,5,7,8,9,11,13,16,17,19$ have been classified; for more details see [2,3,5,6,7,8]. We are looking at the line of order twenty three, as it is the next in the sequence.

We answer the question: How many projectively inequivalent k-sets in $PG(1,q)$ are there and what is the stabilizer group of each one?

The 24 points of $PG(1{,}23)$ are $P(x_0, x_1)$, $x_i \in \mathbb{F}_{23}$. So

$PG(1{,}23) = \{U_0 = P(1{,}0)\} \cup \{P(x,1) | x \in \mathbb{F}_{23}\}$. Each point $P(x_0, x_1)$ with $x_1 \neq 0$ is determined by the non-homogeneous coordinate x_0/x_1; the coordinate for U_0 is ∞. Then, with $\mathbb{F}_{23} \cup \{\infty\}$, each point of $PG(1{,}23)$ is represented by a single element of $\mathbb{F}_{23} \cup \{\infty\}$. Thus

$$PG(1{,}23) = \{P(t,1) | t \in \mathbb{F}_{23} \cup \{\infty\}\};$$

Here, $P(\infty,1)=P(1,0)$. A projectivity $\xi=M(T)$ of $PG(1,23)$ is given by $Y=XT$, where $X=(x_0,x_1)$, $Y=(y_0,y_1)$ and

$$T=\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $s=y_0/y_1$ and $t=x_0/x_1$; then $s=(at+c)/(bt+d)$. If $Q_i=P_i\xi$ for $i=2,3,4$ and P_i and Q_i have the respective coordinate t_i and s_i, then ξ is given by

$$\frac{(s-s_3)(s_2-s_4)}{(s-s_4)(s_2-s_3)}=\frac{(t-t_3)(t_2-t_4)}{(t-t_4)(t_2-t_3)}$$

The following definitions are interesting to area of research:

<u>Definition(1.1)</u>[5]: A *finite field* is a field with only a finite number of elements, the characteristic of a finite field K is the least positive integer p, and hence a prime, such that $pz=\underbrace{z+\cdots+z}_{p}=0$ For all $z\in K$.

<u>Definition(1.2)[5]</u>: The set denoted by $\mathbb{F}_p$, with P prime, consists of the residue classes of the integers modulo P under the natural addition and multiplication.

<u>Definition(1.3)[1]</u>: Let S and S^* be two spaces of $PG(n,K)$, a *projectivity* $\beta: S\to S^*$ is a bijection given by a matrix T, necessarily non-singular, where $P(X^*)=P(X)\beta$ if $tX^*=XT$, with $t\in K-\{0\}$. Write $\beta=M(T)$; then $\beta=M(\lambda T)$ for any λ in $K-\{0\}$. The group of projectivities of $PG(n,K)$ is denoted by $PGL(n+1,K)$.

<u>Definition(1.4)[1]</u>: A group G acts on a set Λ if there is a map $\Lambda\times G\to\Lambda$ such that given g, g′ elements in G and 1 its identity, then

1. $x1 = x$,
2. $(x\mathrm{g})\mathrm{g}' = x(\mathrm{g}\mathrm{g}')$ for any x in Λ.

Definition(1.5)[3]: The orbit of x in Λ under the action of G is the set

$$x\mathrm{G} = \{x\mathrm{g} | \mathrm{g} \in \mathrm{G}\}.$$

Definition(1.6)[3]: The stabilizer of x in Λ under the action of G is the group

$$\mathrm{G}_x = \{\mathrm{g} \in G | x\mathrm{g} = x\}.$$

Definition(1.7)[3]: Let the group G act on the set Λ.

1. If $\mathrm{y} = x\mathrm{g}$, for $x, \mathrm{y} \in \Lambda$, then
   - $\mathrm{yG} = x\mathrm{G}$;
   - $\mathrm{G_y} = \mathrm{g}^{-1}\mathrm{G}_x\mathrm{g}$.
2. $|\mathrm{G}_x| = |G|/|xG|$.

Theorem(1.8)[2]: There is a unique projectivity of $PG(1, q)$ transforming any three points to any three other points.

Theorem(1.9)[7]: There exists a projective $[n, k, d]_q$ −code if and only if there exists an $(n; n - d)$-arc in $PG(k - 1, q)$.

Definition(10)[4]: An $[n, k, d]_q$ code C is a subspace of $V(n, q) = (\mathbb{F}_q)^n$, where the dimension of C is dim $C = k$, and the minimum distance is $d(C) = d = \min d(x, y)$.

Definition(11)[4]: For any $[n, k, d]_q$ code we have $d \leq n - k + 1$.

2. The main results

In the following sections, the k-sets in $PG(1,23)$, $k = 3, \cdots, 12$; are classified by giving the projectively inequivalent k-sets with their stabilizer groups.

2.1 The algorithm for classification of the k -sets in $PG(1, q)$

On $PG(1,23)$, a k-set can be constructed by adding to any $(k-1)$- set one point from the other $q-k+2$ points. According to the Fundamental Theorem of Projective Geometry, Theorem (1.8), any three distinct points on a line are projectively equivalent; so choose a fixed 3-set A. A 4-set is formed by adding to A one point from the other $q-2$ points on $PG(1, q)$; that is, from $PG(1, q) - A = A^c$. A 5-set is formed by adding to any 4-set B one point from the other $q-3$ points on $PG(1, q)$.The stabilizer group G_B fixes B and splits the other $q-3$ points into a number of orbits; so, different 5-sets are formed by adding one point from each different orbit. The procedure can be extended to construct $6,7,8,9, \cdots, (\frac{q+1}{2})$-sets, for q is odd and $(\frac{q}{2})$-sets, for q is even in $PG(1, q)$.

Let K and K' be two 5-sets, to check they are equivalent as following : calculate the transformations between them. By using Theorem (1.8), Two 5-sets K and $K'$are equivalent if $K\beta = K'$ and β is given by a matrix T and $\beta = M(T)$ with $M(\lambda T) = M(T), \lambda \in \mathbb{F}_{23} - \{0\}$, where T is a non-singular

2×2 matrix. Also can be used to calculate the stabilizer group of each k-set.

2.2 The Projective Line $PG(1,23)$

On $PG(1,23)$, the projective line over Galois field of order 23, there are 24 points. The points of $PG(1,23)$ are elements of the set

$$\mathbb{F}_{23} \cup \{\infty\} = \{\infty, 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9, \pm 10, \pm 11\}.$$

The order of the projective group $PGL(2,23)$ is 24.23.22=12144. This is the number of ordered sets of three points.

The polynomial function of degree two, $f(x) = x^2 + x + 7$ is a primitive over $\mathbb{F}_{23}$, when $\mathbb{F}_{23} = \{1,2,\cdots,22; 23 = 0\}$. On $PG(1,23)$ there are 24 points. They are generated by a non-singular matrix of size 2×2; $T = C(f) = \begin{pmatrix} 0 & 1 \\ -7 & -1 \end{pmatrix}$. Such $P(i) = (1,0) \cdot T^i, i = 0, \cdots, 23$. The points of $PG(1,23)$ are given in Table 1 as the following.

$P(0) = (1,0)$	$P(1) = (0,1)$	$P(2) = (5,1)$	$P(3) = (18,1)$
$P(4) = (5,1)$	$P(5) = (4,1)$	$P(6) = (13,1)$	$P(7) = (9,1)$
$P(8) = (2,1)$	$P(9) = (16,1)$	$P(10) = (21,1)$	$P(11) = (10,1)$
$P(12) = (12,1)$	$P(13) = (14,1)$	$P(14) = (3,1)$	$P(15) = (8,1)$
$P(16) = (22,1)$	$P(17) = (15,1)$	$P(18) = (11,1)$	$P(19) = (20,1)$
$P(20) = (19,1)$	$P(21) = (6,1)$	$P(22) = (17,1)$	$P(23) = (1,1)$

Table 1: The Points of $PG(1,23)$

The group action of $< T >$ on $PG(1,23)$ is given as follows:

$$P(0) \xrightarrow{T} P(1) \xrightarrow{T} \cdots \xrightarrow{T} P(23) \xrightarrow{T} P(0).$$

So T cycles 24 points in $PG(1,23)$.

This gives the following conclusion.

<u>Theorem(1):</u> On $PG(1,23)$, $(< T >, \cdot)$ is a cyclic group of order 24.

The proof comes from $< T > = \{T, T^2, T^3, \cdots, T^{24} = I_{2\times 2}\}$.

2.2.1 The 3-sets

Let S be set of all different 3-sets in $PG(1,23)$. Then the order of S is $|S| = 24.23.22 = 12144$. Let $A = \{\infty, 0,1\}$ be a 3-set which is one of them. By computing the transformations between A and all the other 3-sets, we note that any 3-set is projectively equivalent to A. This gives the following conclusion.

Theorem(1): On $PG(1,23)$ there is precisely one 3-set, given with their stabilizer group in Table 2.

Symbol	The 3-set	Stabilizer
A	$\{\infty, 0,1\}$	$S_3 = \langle \frac{1}{t}, \frac{t-1}{t} \rangle$

Table 2: The 3-set on $PG(1,23)$

The proof comes from Theorem (1.8) .

2.2.2 The 4-sets

To construct the 4-set in $PG(1,23)$, it is enough to add one point from each orbit that comes from the action of the projective group of the 3-set G_A on the complement of A. All orbits of the 3-set in Table 2 are given in Table 3.

A	Partition of A^c
$\{\infty, 0,1\}$	1.$\{2,12,22\}$ 2.$\{3,8,11,13,16,21\}$ 3.$\{4,6,9,15,18,20\}$ 4. $\{5,7,10,14,17,19\}$

Table 3: Partition of $PG(1,23)$ by the projectivities of 3-set

The total numbers of all orbits is 4; therefore 4 inequivalent 4-sets can be constructed in $PG(1,23)$. Table 3 gives the following conclusion.

Theorem(2): On $PG(1,23)$, there are precisely four 4-sets, given with their stabilizer group in Table 4.

Symbol	The 4-set	Stabilizer
B_1	$A \cup \{2\}$	$D_4 = \langle \frac{t-1}{12t-1}, \frac{t-1}{12t} \rangle$
B_2	$A \cup \{3\}$	$Z_2 \times Z_2 = \langle \frac{3}{t} \rangle \times \langle \frac{t-3}{t-1} \rangle$
B_3	$A \cup \{4\}$	$Z_2 \times Z_2 = \langle \frac{4}{t} \rangle \times \langle \frac{t-4}{t-1} \rangle$
B_4	$A \cup \{5\}$	$Z_2 \times Z_2 = \langle \frac{5}{t} \rangle \times \langle \frac{t-5}{t-1} \rangle$

Table 4: Inequivalent 4-sets on $PG(1,23)$

2.2.3 The 5-sets

The projective group G_{B_i} splits ${B_i}^c, i = 1,2,3,4$ into a number of orbits. The 5-sets are constructed by adding one point from each orbit to the corresponding 4-sets. All orbits are listed in Table 5.

B_i	Partition ${B_i}^c$
B_1	{3,4,9,12,13,16,21,22} ,{5,7,18,20}, {6,8,10,11,14,15,17,19}
B_2	{2,13,20,22}, {4,8,9,18}, {5,6,12,19}, {7,16}, {10,11,17,21},{14,15}
B_3	{2,21}, {3,9,11,15}, {5,6,10,16}, {7,8,12,17}, {13,18},{14,19,20,22}
B_4	{2,6,14,20}, {3,17,18,22}, {4,7,8,15}, {9,10,12,21}, {11,13,16,19}

Table 5: Partition of $PG(1,23)$ by the projectivities of 4-set

The total numbers of all orbits is 20; therefore 20 5-sets can be constructed in $PG(1,23)$. In Table 6 all equivalent 5-sets with their projective equations are listed.

No.	Equivalent 5-sets	Projective equation
1	$B_1 \cup \{3\} \to B_2 \cup \{2\}$	t
2	$B_1 \cup \{3\} \to B_3 \cup \{2\}$	$(t+2)/(12t+11)$
3	$B_1 \cup \{5\} \to B_3 \cup \{14\}$	$(t-5)/(t-2)$
4	$B_1 \cup \{5\} \to B_4 \cup \{2\}$	t
5	$B_1 \cup \{6\} \to B_2 \cup \{5\}$	$(t-6)/(t-2)$
6	$B_1 \cup \{6\} \to B_3 \cup \{7\}$	$(t-6)/(6t-6)$
7	$B_1 \cup \{6\} \to B_4 \cup \{3\}$	$(t-6)/(t-2)$
8	$B_1 \cup \{6\} \to B_4 \cup \{9\}$	$(t-5)/(18t)$
9	$B_2 \cup \{4\} \to B_3 \cup \{3\}$	t
10	$B_2 \cup \{4\} \to B_3 \cup \{13\}$	$(t-3)/(16t+5)$
11	$B_2 \cup \{7\} \to B_2 \cup \{10\}$	$(t+16)/(t-3)$
12	$B_2 \cup \{7\} \to B_4 \cup \{11\}$	$(t-3)/(4)$
13	$B_2 \cup \{14\} \to B_3 \cup \{5\}$	$(t-1)/(t+9)$
14	$B_2 \cup \{14\} \to B_4 \cup \{4\}$	$(t-1)/(t+9)$

Table 6: The equivalence of 5-sets

Table 6 gives the following conclusion.

Theorem(3): On $PG(1,23)$, there are precisely six projectively distinct 5-sets summarized in Table 7.

Symbol	The 5-set	Stabilizer
C_1	$B_1 \cup \{3\}$	$Z_2 = \langle 3-t \rangle$
C_2	$B_1 \cup \{5\}$	$Z_2 = \langle 2/t \rangle$
C_3	$B_1 \cup \{6\}$	$I = \langle t \rangle$
C_4	$B_2 \cup \{4\}$	$Z_2 = \langle 4-t \rangle$
C_5	$B_2 \cup \{7\}$	$Z_2 = \langle 3/t \rangle$
C_6	$B_2 \cup \{14\}$	$Z_2 = \langle t-1/8t-1 \rangle$

Table 7: Inequivalent 5-sets on $PG(1,23)$

2.2.4 The 6-sets

The projective group G_{C_i} splits ${C_i}^c$, $i = 1,2,3,4,5,6$ into a number of orbits. The 6-sets are constructed by adding one point from each orbit to the corresponding 5-sets. All orbits are listed in Table 8.

C_i	Partition ${C_i}^c$
C_1	{4,22}, {5,21},{6,20}, {7,19},{8,18},{9,17},{10,16},{11,15},{12,14}, {13}
C_2	{3,16}, {4,12},{6,8}, {7,20},{9,13},{10,14},{11,19},{15,17},{18}, {21,22}
C_3	G_{C_3} splits ${C_3}^c$ into 19 orbits of single points
C_4	{2}, {5,22},{6,21}, {7,20},{8,19},{9,18},{10,17},{11,16},{12,15}, {13,14}
C_5	{2,13}, {4,18},{5,19}, {6,12},{8,9},{10,21},{11,17},{14,15},{16}, {20,22}
C_6	{2,20}, {4,9},{5,6}, {7,16},{8,18},{10,17},{11,21},{12,19},{13,22}, {15}

Table 8: Partition of $PG(1,23)$ by the projectivities of 5-sets

The total numbers of all orbits is 69; therefore 69 6-sets can be constructed in $PG(1,23)$. In Table 9 all equivalent 6-sets with their projective equations are listed.

Equivalent 6-sets	Projective equation	Equivalent 6-sets	Projective equation
$C_1 \cup \{4\} \to C_4 \cup \{2\}$	t	$C_1 \cup \{5\} \to C_1 \cup \{9\}$	$(t-2)/(12t+10)$
$C_1 \cup \{5\} \to C_2 \cup \{3\}$	t	$C_1 \cup \{5\} \to C_3 \cup \{21\}$	$(t-3)/(12t+11)$
$C_1 \cup \{5\} \to C_4 \cup \{7\}$	$(t-4)/(12t)$	$C_1 \cup \{5\} \to C_5 \cup \{4\}$	$(t-4)/(10t)$
$C_1 \cup \{6\} \to C_2 \cup \{21\}$	$(2)/(t-1)$	$C_1 \cup \{6\} \to C_3 \cup \{3\}$	$(6)/(t)$
$C_1 \cup \{7\} \to C_2 \cup \{9\}$	$(t-1)/(12t)$	$C_1 \cup \{7\} \to C_3 \cup \{11\}$	$(t)/(-2t+6)$
$C_1 \cup \{7\} \to C_3 \cup \{17\}$	$(t-2)/(4t)$	$C_1 \cup \{7\} \to C_3 \cup \{22\}$	$t-1$
$C_1 \cup \{7\} \to C_5 \cup \{2\}$	t	$C_1 \cup \{8\} \to C_2 \cup \{4\}$	$(t-2)/(12t+4)$
$C_1 \cup \{8\} \to C_3 \cup \{16\}$	$(2)/(t)$	$C_1 \cup \{8\} \to C_4 \cup \{5\}$	$(t-8)/(8t-8)$
$C_1 \cup \{8\} \to C_4 \cup \{13\}$	$(-3)/(t-3)$	$C_1 \cup \{8\} \to C_6 \cup \{4\}$	$(t-8)/(6t+5)$
$C_1 \cup \{10\} \to C_3 \cup \{12\}$	$(t-2)/(t-1)$	$C_1 \cup \{10\} \to C_5 \cup \{20\}$	$(t-1)/(t+13)$
$C_1 \cup \{11\} \to C_3 \cup \{9\}$	$(t-1)/(12t)$	$C_1 \cup \{11\} \to C_3 \cup \{13\}$	$(t-3)/(t-2)$
$C_1 \cup \{11\} \to C_4 \cup \{6\}$	$(t-2)/(4t+11)$	$C_1 \cup \{11\} \to C_5 \cup \{5\}$	$(t+12)/(12)$
$C_1 \cup \{11\} \to C_6 \cup \{13\}$	$(-3)/(t-3)$	$C_1 \cup \{12\} \to C_3 \cup \{4\}$	$(t)/(12)$
$C_1 \cup \{12\} \to C_6 \cup \{2\}$	$3-t$	$C_2 \cup \{6\} \to C_3 \cup \{5\}$	t
$C_2 \cup \{6\} \to C_6 \cup \{12\}$	$(t)/(2)$	$C_2 \cup \{7\} \to C_6 \cup \{15\}$	$(-6)/(t-7)$
$C_2 \cup \{10\} \to C_3 \cup \{18\}$	$(t-3)/(t-1)$	$C_2 \cup \{10\} \to C_4 \cup \{8\}$	$(t-5)/(3t-7)$
$C_2 \cup \{11\} \to C_3 \cup \{20\}$	$(t-1)/(12t)$	$C_2 \cup \{11\} \to C_4 \cup \{10\}$	$(t-1)/(7t-8)$
$C_2 \cup \{11\} \to C_5 \cup \{11\}$	$(t-5)/(6)$	$C_2 \cup \{11\} \to C_5 \cup \{14\}$	$(9)/(t-2)$
$C_2 \cup \{15\} \to C_3 \cup \{7\}$	$(t)/(t-1)$	$C_2 \cup \{15\} \to C_6 \cup \{5\}$	$(t)/(5)$
$C_2 \cup \{18\} \to C_3 \cup \{15\}$	$(t+5)/(5)$	$C_3 \cup \{8\} \to C_4 \cup \{12\}$	$(1)/(t)$
$C_3 \cup \{8\} \to C_6 \cup \{8\}$	$(t-2)/(t-6)$	$C_3 \cup \{14\} \to C_5 \cup \{6\}$	$(6)/(t)$
$C_3 \cup \{14\} \to C_6 \cup \{11\}$	$(t+9)/(t-1)$	$C_3 \cup \{19\} \to C_5 \cup \{16\}$	$(t+4)/(10t+9)$
$C_4 \cup \{11\} \to C_5 \cup \{8\}$	$(13)/(t+12)$	$C_4 \cup \{11\} \to C_6 \cup \{10\}$	$(t-3)/(7t)$

Table 9: The equivalence of 6-sets

Table 9 gives the following conclusion.

Theorem(5): On $PG(1,23)$, there are precisely 23 projectively distinct 6-sets summarized in Table 10.

Symbol	The 6-set	Stabilizer
E_1	$C_1 \cup \{4\}$	$Z_2 \times Z_2 = \langle 4-t \rangle \times \langle \frac{t-3}{12t-1} \rangle$
E_2	$C_1 \cup \{5\}$	$I = \langle t \rangle$
E_3	$C_1 \cup \{6\}$	$Z_2 = \langle 6/t \rangle$
E_4	$C_1 \cup \{7\}$	$I = \langle t \rangle$
E_5	$C_1 \cup \{8\}$	$I = \langle t \rangle$
E_6	$C_1 \cup \{10\}$	$Z_2 = \langle t+13/8t-1 \rangle$
E_7	$C_1 \cup \{11\}$	$I = \langle t \rangle$
E_8	$C_1 \cup \{12\}$	$Z_2 = \langle t-2/t-1 \rangle$
E_9	$C_1 \cup \{13\}$	$D_6 = \langle \frac{3}{t}, \frac{-3}{t-3} \rangle$
E_{10}	$C_2 \cup \{6\}$	$Z_2 = \langle t-1/14t-1 \rangle$
E_{11}	$C_2 \cup \{7\}$	$Z_2 \times Z_2 = \langle \frac{t}{t-1} \rangle \times \langle \frac{t-2}{10t-1} \rangle$
E_{12}	$C_2 \cup \{10\}$	$Z_2 = \langle 10/t \rangle$
E_{13}	$C_2 \cup \{11\}$	$I = \langle t \rangle$
E_{14}	$C_2 \cup \{15\}$	$Z_2 = \langle t+8/12t-1 \rangle$
E_{15}	$C_2 \cup \{18\}$	$Z_2 \times Z_2 = \langle \frac{2}{t} \rangle \times \langle \frac{t-2}{t-1} \rangle$
E_{16}	$C_3 \cup \{8\}$	$Z_2 = \langle 2/t \rangle$
E_{17}	$C_3 \cup \{10\}$	$S_3 = \langle \frac{15}{t+13}, \frac{t-2}{t-1} \rangle$
E_{18}	$C_3 \cup \{14\}$	$Z_2 = \langle t-1/12t-1 \rangle$
E_{19}	$C_3 \cup \{19\}$	$Z_2 \times Z_2 = \langle 2-t \rangle \times \langle \frac{t+4}{t-1} \rangle$
E_{20}	$C_4 \cup \{9\}$	$S_3 = \langle \frac{4}{t}, \frac{t-4}{t+14} \rangle$
E_{21}	$C_4 \cup \{11\}$	$Z_2 = \langle t-4/t-1 \rangle$
E_{22}	$C_5 \cup \{10\}$	$S_3 = \langle \frac{7}{t}, \frac{t+16}{t-3} \rangle$
E_{23}	$C_6 \cup \{7\}$	$I = \langle t \rangle$

Table 10: Inequivalent 6-sets on $PG(1,23)$

2.2.5 The 7-sets

The projective group G_{E_i} splits ${E_i}^c, i = 1, \cdots, 23$ into a number of orbits. The 7-sets are constructed by adding one point from each orbit to the corresponding 6-sets. All orbits are listed in Table 11.

E_i	Partition ${E_i}^c$
E_1	{5,9,18,22}, {6,13,14,21},{7,20}, {8,10,17,19},{11,12,15,16}
E_2	G_{E_2} splits ${E_2}^c$ into 18 orbits of single points
E_3	{4,13}, {5,15},{7,14}, {8,18},{9,16},{10,19}, {11},{12}, {17,22},{20,21}
E_4	G_{E_4} splits ${E_4}^c$ into 18 orbits of single points
E_5	G_{E_5} splits ${E_5}^c$ into 18 orbits of single points
E_6	{4,5}, {6,19},{7,15}, {8},{9,11},{12,16}, {13,17},{14,22}, {18,20},{21}
E_7	G_{E_7} splits ${E_7}^c$ into 18 orbits of single points
E_8	{4,16}, {5,18},{6,10}, {7,20},{8,14},{9,21}, {11,17},{13,22}, {15,19}
E_9	{4,6,8,9,11,12,14,15,17,18,20,22}, {5,7,10,16,19,21}
E_{10}	{3,18}, {4,8},{7,15}, {9,10},{11,16},{12,21}, {13,19},{14,20}, {17,22}
E_{11}	{3,4,9,13}, {6,15,18,20},{8,10,11,19}, {12,14,17,22},{16,21}
E_{12}	{3,11}, {4,14},{6,17}, {7,8},{9,19},{12,20}, {13,22},{15,16}, {18,21}
E_{13}	$G_{E_{13}}$ splits ${E_{13}}^c$ into 18 orbits of single points
E_{14}	{3,22}, {4,12},{6,7}, {8,13},{9,18},{10,16}, {11,17},{14,19}, {20,21}
E_{15}	{3,4,12,16}, {6,8,10,14},{7,20}, {9,13,21,22},{11,15,17,19}
E_{16}	{3,16}, {4,12},{5}, {7,20},{9,13},{10,14},

	{11,19},{15,17}, {18},{21,22}
E_{17}	{3,11,12,15,17,19}, {4,8,9,14,16,21},{5,7,13,18,20,22}
E_{18}	{3,4}, {5,18},{7}, {8,10},{9,22},{11,15}, {12,16},{13,21}, {17,19},{20}
E_{19}	{3,10,15,22}, {4,7,18,21},{5,8,17,20}, {9,16},{11,12,13,14}
E_{20}	{2,14,20}, {5,6,7,10,16,17},{8,12,13,18,19,22},{11,15,21}
E_{21}	{2,21}, {5,6},{7,12}, {8,17},{9,15},{10,16}, {13,18},{14,22}, {19,20}
E_{22}	{2,5,6,15,16,22}, {4,13,17,18,19,20},{8,9,11,12,14,21}
E_{23}	$G_{E_{23}}$ splits ${E_{23}}^c$ into 18 orbits of single points

Table 11: Partition of $PG(1,23)$ by the projectivities of 6-sets

There are 225 different orbits; therefore 225 7-sets can be constructed in $PG(1,23)$. The projectively distinct 7-sets with their stabilizer groups are given in the following theorem.

Theorem(6): On $PG(1,23)$, there are 38 projectively distinct 7-sets summarized in Table 12.

Symbol	The 7-set	Stabilizer	Symbol	The 7-set	Stabilizer
F_1	$E_1 \cup \{5\}$	$Z_2 = \langle 5 - t \rangle$	F_{20}	$E_3 \cup \{10\}$	$I = \langle t \rangle$
F_2	$E_1 \cup \{6\}$	$I = \langle t \rangle$	F_{21}	$E_3 \cup \{11\}$	$Z_2 = \langle 6/t \rangle$
F_3	$E_1 \cup \{7\}$	$Z_2 = \langle t - 1/12t - 1 \rangle$	F_{22}	$E_3 \cup \{12\}$	$Z_2 = \langle 6/t \rangle$
F_4	$E_1 \cup \{8\}$	$I = \langle t \rangle$	F_{23}	$E_4 \cup \{8\}$	$I = \langle t \rangle$
F_5	$E_1 \cup \{11\}$	$I = \langle t \rangle$	F_{24}	$E_4 \cup \{10\}$	$I = \langle t \rangle$
F_6	$E_2 \cup \{6\}$	$I = \langle t \rangle$	F_{25}	$E_4 \cup \{11\}$	$I = \langle t \rangle$
F_7	$E_2 \cup \{7\}$	$I = \langle t \rangle$	F_{26}	$E_4 \cup \{14\}$	$I = \langle t \rangle$
F_8	$E_2 \cup \{8\}$	$I = \langle t \rangle$	F_{27}	$E_4 \cup \{15\}$	$I = \langle t \rangle$
F_9	$E_2 \cup \{9\}$	$I = \langle t \rangle$	F_{28}	$E_5 \cup \{10\}$	$Z_2 = \langle t + 13/8t - 1 \rangle$

F_{10}	$E_2 \cup \{11\}$	$I = \langle t \rangle$	F_{29}	$E_5 \cup \{11\}$	$Z_2 = \langle t-2/3t-1 \rangle$
F_{11}	$E_2 \cup \{12\}$	$I = \langle t \rangle$	F_{30}	$E_5 \cup \{12\}$	$Z_2 = \langle 1/t \rangle$
F_{12}	$E_2 \cup \{13\}$	$Z_2 = \langle t-1/16t-1 \rangle$	F_{31}	$E_5 \cup \{18\}$	$Z_2 = \langle 3-t \rangle$
F_{13}	$E_2 \cup \{14\}$	$I = \langle t \rangle$	F_{32}	$E_7 \cup \{15\}$	$Z_2 = \langle 3-t \rangle$
F_{14}	$E_2 \cup \{15\}$	$I = \langle t \rangle$	F_{33}	$E_{10} \cup \{7\}$	$Z_2 = \langle -16-t \rangle$
F_{15}	$E_2 \cup \{16\}$	$Z_2 = \langle 2/t \rangle$	F_{34}	$E_{10} \cup \{14\}$	$I = \langle t \rangle$
F_{16}	$E_2 \cup \{18\}$	$I = \langle t \rangle$	F_{35}	$E_{12} \cup \{7\}$	$I = \langle t \rangle$
F_{17}	$E_2 \cup \{20\}$	$Z_2 = \langle t+3/t-1 \rangle$	F_{36}	$E_{13} \cup \{15\}$	$Z_2 = \langle t+12/14t-1 \rangle$
F_{18}	$E_3 \cup \{7\}$	$I = \langle t \rangle$	F_{37}	$E_{13} \cup \{19\}$	$Z_2 = \langle 2/t \rangle$
F_{19}	$E_3 \cup \{8\}$	$I = \langle t \rangle$	F_{38}	$E_{16} \cup \{11\}$	$I = \langle t \rangle$

Table 12: Inequivalent 7-sets on $PG(1,23)$

2.2.6 The 8-sets

The projective group G_{F_i} splits ${F_i}^c, i = 1, \cdots, 38$ into a number of orbits. The 8-sets are constructed by adding one point from each orbit to the corresponding 7-sets. We have 526 different orbits; therefore 526 8-sets can be constructed in $PG(1,23)$. The projectively distinct 8-sets with their stabilizer groups are given in the following theorem.

Theorem(7): On $PG(1,23)$, there are 89 projectively distinct 8-sets, the number of 8-sets and their stabilizer are summarized in Table 13.

Stabilizer	$\boldsymbol{I}$	$\boldsymbol{Z_2}$	$\boldsymbol{Z_2 \times Z_2}$	$\boldsymbol{D_4}$	$\boldsymbol{D_8}$
Number	50	29	7	2	1

Table 13: The stabilizers of the 8-sets

2.2.7 The 9-sets

The projective group splits the complements of 8-sets into 1080 orbits. The 9-sets are constructed by adding one point from each orbit to the corresponding 8-sets. therefore 1080 (9-sets) can be constructed in

$PG(1,23)$. The main computing tool is the mathematical programming language **GAP**. By help the computer to compute the transformations between 1080 (9-sets) and the stabilizer groups of them, we have the projectively distinct 9-sets with their stabilizer groups are given in the following theorem.

Theorem(8): On $PG(1,23)$, there are 127 projectively distinct 9-sets, the number of 9-sets and their stabilizer are summarized in Table 14.

Stabilizer	I	Z_2	Z_3	S_3
Number	95	27	2	3

Table 14: The stabilizers of the 9-sets

2.2.8 The 10-sets

The projective group splits the complements of 9-sets into 1660 orbits. The 10-sets are constructed by adding one point from each orbit to the corresponding 9-sets. therefore 1660 (10-sets) can be constructed in $PG(1,23)$. The main computing tool is the mathematical programming language **GAP**. By help the computer to compute the transformations between 1660 (10-sets) and the stabilizer groups of them, we have the projectively distinct 10-sets with their stabilizer groups are given in the following theorem.

Theorem(9): On $PG(1,23)$, there are 198 projectively distinct 10-sets, the number of 10-sets and their stabilizer are summarized in Table 15.

Stabilizer	I	Z_2	$Z_2 \times Z_2$
Number	134	54	10

Table 15: The stabilizers of the 10-sets

2.2.9 The 11-sets

The projective group splits the complements of 10-sets into 2309 orbits. The 11-sets are constructed by adding one point from each orbit to the corresponding 10-sets. therefore 2309 (11-sets) can be constructed in $PG(1,23)$. The main computing tool is the mathematical programming language **GAP**. By help the computer to compute the transformations between 2309 (11-sets) and the stabilizer groups of them, we have the projectively distinct 11-sets with their stabilizer groups are given in the following theorem.

Theorem(10): On $PG(1,23)$, there are 228 projectively distinct 11-sets, the number of 11-sets and their stabilizer are summarized in Table 16.

Stabilizer	I	Z_2	D_{11}
Number	186	41	1

Table 16: The stabilizers of the 11-sets

2.2.10 The 12-sets

The projective group splits the complements of 11-sets into 2707 orbits. The 12-sets are constructed by adding one point from each orbit to the corresponding 11-sets. therefore 2707 (12-sets) can be constructed in $PG(1,23)$. The main computing tool is the mathematical programming language **GAP**. By help the computer to compute the transformations between 2707 (12-sets) and the stabilizer groups of them, we have the projectively distinct 12-sets with their stabilizer groups are given in the following theorem.

Theorem(11): On $PG(1,23)$, there are 268 projectively distinct 12-sets, the number of 12-sets and their stabilizer are summarized in Table 17.

Stabilizer	I	Z_2	Z_3	Z_4	$Z_2 \times Z_2$	S_3	D_4	Z_{11}	D_6	A_4	D_{12}
Number	191	56	1	1	10	4	1	1	1	1	1

Table 17: The stabilizers of the 12-sets

2.2.11 The Partition of $PG(1{,}23)$

Let Z be a 12-set on $PG(1{,}23)$, the set $\{Z, Z^c\}$ contains Z and its complement Z^c which constructed of $PG(1{,}23)$. The stabilizer group $G(Z)$ is stabilized Z^c as well. The projectively equation between Z and Z^c with $G(Z)$ are generated $G(Z, Z^c)$ which stabilized Z and Z^c. The 12-sets which equivalent or inequivalent to its complement have been classified. Amongst the 268 (12-set) there are 178 of them which are not equivalent to their complement as given in the following theorem.

<u>Theorem(12):</u> On $PG(1{,}23)$, we have

1. 90 projectively distinct partitions into two equivalent 12-sets;
2. 178 projectively distinct partitions into two inequivalent 12-sets.

The numbers of equivalent and inequivalent 12-sets and their stabilizer are given in Tables 18 and 19.

Stabilizer	$\boldsymbol{Z_2}$	$\boldsymbol{Z_4}$	$\boldsymbol{Z_2 \times Z_2}$	$\boldsymbol{S_3}$	$\boldsymbol{D_4}$	$\boldsymbol{D_6}$	$\boldsymbol{D_8}$	$\boldsymbol{D_{11}}$	$\boldsymbol{D_{12}}$	$\boldsymbol{S_4}$	$\boldsymbol{D_{24}}$
Number	68	1	10	1	3	2	1	1	1	1	1

Table 18: Partitions of $PG(1{,}23)$ into two equivalent 12-sets

Stabilizer	$\boldsymbol{I}$	$\boldsymbol{Z_2}$	$\boldsymbol{Z_2 \times Z_2}$	$\boldsymbol{S_3}$
Number	126	42	8	2

Table 19: Partitions of $PG(1{,}23)$ into two inequivalent 12-sets

2.2.12 Splitting $PG(1{,}23)$ into six disjoint 4-sets

From Table 3, the projective group G_A partitions A^c into four orbits are as following:

$\Omega_1 = \{2,12,22\}$;

$\Omega_2 = \{3,8,11,13,16,21\}$;

$\Omega_3 = \{4,6,9,15,18,20\}$;

$\Omega_4 = \{5,7,10,14,17,19\}$.

According to the values of $\lambda \in \Omega_i$, $i = 1,2,3,4$. Such the Cross-ratio $\lambda = \frac{(P_1-P_3)(P_2-P_4)}{(P_1-P_4)(P_2-P_3)}$, when P_1, P_2, P_3, P_4 are four ordered points in $PG(1{,}23)$,

the projective line $PG(1,23)$ partitions into six disjoint 4-sets of Types Ω_i, $i = 1,2,3,4$.
The 4-sets of Type Ω_i, $i = 1,2,3,4$ with its Cross-ratio are given in Tables 20, 21, 22,23.

Number	The 4-sets	The Cross-ratio
1	$\{\infty, 0,1,2\}$	$\lambda = 2$
2	$\{3,14,17,19\}$	$\lambda = 12$
3	$\{4,5,12,16\}$	$\lambda = 12$
4	$\{6,7,8,15\}$	$\lambda = 12$
5	$\{9,10,11,18\}$	$\lambda = 12$
6	$\{13,20,21,22\}$	$\lambda = 12$

Table 20: The 4-sets of Type Ω_1

Number	The 4-sets	The Cross-ratio
1	$\{\infty, 0,1,3\}$	$\lambda = 3$
2	$\{2,4,5,6\}$	$\lambda = 13$
3	$\{7,8,9,22\}$	$\lambda = 8$
4	$\{10,11,12,20\}$	$\lambda = 11$
5	$\{13,14,15,17\}$	$\lambda = 13$
6	$\{16,18,19,21\}$	$\lambda = 11$

Table 21: The 4-sets of Type Ω_2

Number	The 4-sets	The Cross-ratio
1	$\{\infty, 0,1,4\}$	$\lambda = 4$
2	$\{2,3,5,6\}$	$\lambda = 4$
3	$\{7,8,9,10\}$	$\lambda = 9$
4	$\{11,12,13,14\}$	$\lambda = 9$
5	$\{15,16,17,18\}$	$\lambda = 9$
6	$\{19,20,21,22\}$	$\lambda = 9$

Table 22: The 4-sets of Type Ω_3

Number	The 4-sets	The Cross-ratio
1	$\{\infty, 0,1,5\}$	$\lambda = 5$
2	$\{2,3,4,8\}$	$\lambda = 17$
3	$\{6,7,9,20\}$	$\lambda = 17$
4	$\{10,11,12,18\}$	$\lambda = 19$
5	$\{13,14,15,21\}$	$\lambda = 19$
6	$\{16,17,19,22\}$	$\lambda = 7$

Table 23: The 4-sets of Type Ω_4

3. Conclusions

- The action of $<T>$ on $PG(1,23)$ is transitive since $P(i) = P(0) \cdot T^i, i = 0, \cdots, 23$; that is, given $x, y \in PG(1,23)$ there exists $g \in <T>$ such that $y = xg$.
- On projective plane $PG(2,23)$, the numbers of distinct k-arcs which is a set of k points no three of which are collinear on a conic with $5 \leq k \leq 12$ are given in Table 24 as following:

k-arc	5-arc	6-arc	7-arc	8-arc	9-arc	10-arc	11-arc	12-arc
Number	6	23	38	89	127	198	228	268

Table 24: The numbers of k-arcs on a conic

- The equations: How many projective inequivalent k-sets in $PG(1,23)$ are there and what is the stabilizer group of each one have been answered.
- The open problem, we can study the sets on $PG(1,q)$ when $q \geq 25$ (prime number or prime power).

REFERENCES

[1] Hirschfeld J.W.P, Korchmáros G and Torres F," *Algebraic Curves Over a Finite field"* Oxford University Press, Oxford, (2007)

[2] A.H. Ali, "Classification of Arcs in Galois Plane of Order Thirteen", Ph.D. Thesis, University of Sussex, (1993)

[3] Al-seraji N.A.M, "Classification of the projective line over Galois Field of order sixteen " journal of Science, College of Science, University of Al-Mustansiriyah, No. 6 Vol. 24. (2014)

[4] Al-seraji N.A.M, "On Almost Maximum Distance Separable codes" journal of Baghdad, College of Science, University of Baghdad, No. 3 Vol. 54. (2013)

[5] Hirschfeld J.W.P," *Projective Geometries Over Finite Fields* " S*econd Edition*, Oxford University Press, Oxford, (1998)

[6] Al-seraji N.A.M, " The Geometry of The Plane of order Seventeen and its Application to Error-correcting Codes" Ph.D. Thesis, University of Sussex, UK, (2010)

[7] Al-zangana E.B, " The Geometry Of The Plane Of Order Nineteen and Its Application To Error-Correcting Codes" Ph.D. Thesis,University of Sussex, UK,(2011)

[8] Hirschfeld J.W.P and Al-seraji N.A.M, "The Geometry of the line of order seventeen and its Applications to Error-correcting codes" journal of Science, College of Science, University of Al-Mustansiriyah, No. 5 Vol. 24. (2014)