

Generalized Chebyshev Algorithm for Solving Calculus of

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Abstract

In this paper, the generalized Chebyshev polynomials defined on the interval $[a,b]$ are presented. Some properties of such polynomials that are useful for numerical applications are discussed and derived. Then a direct method for solving variational problems is proposed which is based on the generalized Chebyshev polynomials to reduce a variational problem to a nonlinear mathematical programming problem.

The solution is obtained in terms of generalized Chebyshev polynomials method, illustrative examples is given.

Keywords: Generalized Chebyshev Polynomials, Variational Problem, Nonlinear Programming.

1. Introduction

Chebyshev polynomials have a wide variety of practical uses in numerical algorithms and are easy to compute and apply. Most areas of numerical analysis as well as many other of mathematics as a whole make use of the Chebyshev polynomials. In several areas of mathematics, polynomials approximation, numerical integration and pseudospectral methods for ordinary and partial differential equations, the Chebyshev polynomials take a significant role. Many authors and researchers studied the Chebyshev polynomials such as in [1,2] a method based on Chebyshev polynomials was proposed to solve constrained linear quadratic optimal control with the aid of spectral method, while higher order linear differential equations was solved by [3] using rational Chebyshev collocation method and others see [8]

In this paper, a generalized Chebyshev method defined on the interval $[a,b]$ is presented to solve calculus of variation problems. Some properties of generalized Chebyshev polynomials are derived which are very important in the proposed method.

2. General Chebyshev Polynomials With Some New Properties

To use the Chebyshev polynomial $T(x)$ with a variable t in the finite range $[a,b]$, we make the mapping into the range $[-1,1]$ of x by the linear

transformation $x = \frac{2t - (b + a)}{b - a}$, various nonlinear transformations can be used if

t is unbounded from below ($a \rightarrow -\infty$) or from above ($b \rightarrow \infty$) [6].

Then the general Chebyshev polynomials $T_n^g(t)$ of order n can be defined using the following formula:

$$T_n^g(t) = \frac{n}{2} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^n (n-r-1)! (2T_1^g)^{n-2r}}{r!(n-2r)!}$$

where $T_1^g = \frac{2t - (a+b)}{b-a}$

The first few of general Chebyshev polynomials are

$$\begin{aligned} T_0^g &= 1, \\ T_1^g &= \frac{2t - (a+b)}{b-a}, \\ T_2^g &= 2T_1^g{}^2 - T_0^g, \\ T_3^g &= 4T_1^g{}^3 - 3T_1^g, \\ T_4^g &= 8T_1^g{}^4 - 8T_1^g{}^2 + T_0^g, \\ T_5^g &= 16T_1^g{}^5 - 20T_1^g{}^3 + 5T_1^g \end{aligned}$$

Also the general Chebyshev polynomials can be obtained using the recurrence relation give throughout the following lemma.

Lemma (1):

The general Chebyshev polynomials can be obtained from the recurrence relation

$$T_n^g(t) = 2T_1^g T_{n-1}^g - T_{n-2}^g, \quad n = 2, 3, \dots \quad \dots(1)$$

where $T_0^g = 1, \quad T_1^g = \frac{2t - (a+b)}{b-a}$

Proof:

Using mathematical induction to show (1) is true for $n=2$

$$\begin{aligned} T_2^g &= 2T_1^g T_1^g - T_0^g \\ &= 2 \left[\left(\frac{1}{2} \right) (T_2^g + T_0^g) \right] - T_0^g = T_2^g \end{aligned}$$

Let eq. (1) true when $n=k$, i.e.

$$T_k^g = \left[(2T_1^g T_{k-1}^g) - T_{k-2}^g \right] \quad \dots(2)$$

To show it is true for $n=k+1$

$$\left[(T_1^g T_k^g) \right] = \left[\left(\frac{1}{2} \right) (T_{k+1}^g + T_{k-1}^g) \right]$$

Since,

$$T_{k+1}^g = 2T_1^g T_k^g - T_{k-1}^g, \quad \text{and since } n=k+1$$

therefore, $T_n^g = 2T_1^g T_{n-1}^g - T_{n-2}^g$ which is the required result.

The derivative of T_n^g of order n can be obtained using the following formula

$$\dot{T}_n^g(T_1) = \frac{N}{b-a} \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^n (n-r-1)! (2T_1)^{n-2r-1}}{r! (n-2r-1)!}$$

The first few of derivative general Chebyshev polynomials are

$$\dot{T}_0^g = 0,$$

$$\dot{T}_1^g = \frac{2}{b-a},$$

$$\dot{T}_2^g = \frac{2.2.2}{b-a} T_1^g,$$

$$\dot{T}_3^g = 3 \frac{2}{(b-a) (4(T_1^g)^2 - 1)},$$

$$\dot{T}_4^g = \left[4 \frac{2.2}{(b-a) (4(T_1^g)^2 - 1)} - 2T_1^g \right],$$

$$\dot{T}_5^g = 5 \frac{2}{(b-a) \left(16 [(T_1^g)^4 - 12 [(T_1^g)^2 + 1] \right)}$$

Now, $(T_1^g)^m$ can be obtained in term of T_n^g using the product relation

$$T_n^g T_m^g = \frac{1}{2} [T_{n+m}^g + T_{|n-m|}^g].$$

Therefore,

$$[(T_1^g)^2] = \frac{1}{2 [T_2^g + T_0^g]},$$

$$[(T_1^g)^3] = \frac{1}{2^2 [T_3^g + 3T_1^g]},$$

$$[(T_1^g)^4] = \frac{1}{2^3 [T_4^g + 4T_2^g + 3T_0^g]},$$

$$[(T_1^g)^5] = \frac{1}{2^4 [T_5^g + 5T_3^g + 10T_1^g]},$$

$$[(T_1^g)^6] = \frac{1}{2^5 [T_6^g + 6T_4^g + 15T_2^g + 20T_0^g]}$$

In general $(T_1^g)^m$ can be found using the following matrix

$$\begin{pmatrix} (T_1^g)^2 \\ \left[(T_1^g) \right]^3 \\ \left[(T_1^g) \right]^4 \\ \left[(T_1^g) \right]^5 \\ \left[(T_1^g) \right]^6 \\ \left[(T_1^g) \right]^7 \\ \left[(T_1^g) \right]^8 \\ \left[(T_1^g) \right]^9 \\ \left[(T_1^g) \right]^{10} \end{pmatrix} = \text{(EMBED Equatio.DSMT4)} \begin{pmatrix} T_0^g \\ T_1^g \\ T_2^g \\ T_3^g \\ T_4^g \\ T_5^g \\ T_6^g \\ T_7^g \\ T_8^g \\ T_9^g \\ T_{10}^g \end{pmatrix}$$

or $(T_1^g)^m =$
 ... (3)

where $(T_1^g)^m = \left(\left[(T_1^g) \right]^2 \quad \left[(T_1^g) \right]^3 \quad \left[(T_1^g) \right]^4 \quad \dots \quad T_1^m \right)^T$,

$$T_n^g = (T_0^g \quad T_1^g \quad T_2^g \quad T_3^g \quad \dots \quad T_n^g)^T$$

where $k = (k_{ij})$ is an $(m-1) \times (m+1)$ matrix and its elements can be obtained with the use of the following:

Using eq.(3), the relationship between T_n^g and T_n^g can be rewritten in the

following form:

$$T_0^g = 0,$$

$$T_1^g = \frac{2}{b-a},$$

$$T_2^g = \frac{2.2.2}{b-a} T_1,$$

$$T_3^g = 3 \left(\frac{2}{b-a} \right) (2T_2 + T_0),$$

$$T_4^g = 4.2. \left(\frac{2}{b-a} \right) (T_3 + T_1),$$

$$T_5^g = 5 \left(\frac{2}{b-a} \right) (2T_4 + 2T_2 + T_0)$$

or in matrix form as $T_n^g = DT_n^g$ where

$$T_n^g = (T_1^g T_2^g \dots T_n^g)^T, \quad T_n^g = (T_0 T_1 T_2 T_3 \dots T_n)$$

and D is the operational matrix of differentiation of general Chebyshev polynomials which is given by:

$$D_e = \frac{2n}{b-a} \text{ (EMBED Equationn.DSMT4)}$$

If n even, and

$$D_o = \frac{2n}{b-a} \text{ (EMBED Equationn.DSMT4)}$$

If n odd

3. Application of General Chebyshev Polynomials for Solving Variational Problem

The calculus of variations is concerned with finding the maxima and minima of certain functional. Functional minimization problems known as variational problems appears in engineering and science.

Several methods have been used to solve variational problems. For example, the homotopy-perturbation method [4], Bernstein Direct method [5] while Walsh-Hybrid method [7] was applied to solve variational problems.

Consider the problem of finding the minimum of the time-varying functional

$$J(x) = \int_0^1 (x^2 + tx + x^2) dt$$

where J is the functional whose extremum must be found. In order to find the extreme value of J, the boundary points of the admissible curves are given by

$$x(0) = 0, x(1) = \frac{1}{4}$$

$$J(x) = \int_0^1 [(x^2) + t\dot{x} + x^2] dt \quad \dots (4)$$

with boundary conditions $x(0) = 0$ $x(1) = \frac{1}{4}$... (5)

the exact solution is $x(t) = \frac{1}{2} + c_1 e^t + c_2 e^{-t}$ where $c_1 = \frac{2-e}{4(e^2-1)}$ and $c_2 = \frac{e-2e^2}{4(e^2-1)}$

Approximate the variable $x(t)$ using general Chebyshev polynomial

... (6)

or

where $a = [a_0 \ a_1 \ a_2 \ a_3]^T$ and $T^g = [T_0^g \ T_1^g \ T_2^g \ T_3^g]^T$
 $\dot{x}(t) = a^T D_0 T^g$... (7)

Here $D_0 = 2.3$ (EMBED Equatio.DSMT4)

Substitute eqs. (6) and (7) into (1) to get

$$J(x) = \int_0^1 (a^T D_0 T^g D_0 T^{gT} a + a^T t D_0 T^g + a^T T^g T^{gT} a) dt$$

$$= \int_0^1 (a^T (D_0 T^g D_0 T^{gT} + T^g T^{gT}) a + a^T t T^g) dt \quad \dots (8)$$

Let $H = 2 \int_0^1 [D_0 T^g D_0 T^{gT} + T^g T^{gT}] dt$

$c^T = \int_0^1 t T^g dt$

$$C = \begin{bmatrix} 0 \\ 1 \\ 4 \\ - \\ 3 \\ 1 \end{bmatrix}$$

Therefore; $H =$ (EMBED Equatio.DSMT4) ,

Eq.(8) can be rewritten as $J(x) = \frac{1}{2} a^T H a + c^T a$

Eq.(5) and Eq.(6) gives:

$x(0) = a^T T^g(0) = 0$,

$$x(1) = a^T T^g(1) = \frac{1}{4}$$

where $T^g(0) = [1 \ -1 \ 1 \ -1]$

and $T^g(1) = [1 \ 1 \ 1 \ 1]$

Then the quadratic programming problem

$$J(x) = \frac{1}{2} a^T H a + c^T a \quad \text{subject to } ,$$

where

The optimal values of unknown parameters $a_i, i = 0,1,2,3$ are:

$$a_0 = \frac{103}{704}, \quad a_1 = \frac{681}{5504}, \quad a_2 = \frac{-15}{704}, \quad a_3 = \frac{7}{5504},$$

The approximate solution is:

$$x(t) = \frac{103}{704} T_0^g + \frac{681}{5504} T_1^g - \frac{15}{704} T_2^g + \frac{7}{5504} T_3^g$$

Table (1) shows the approximate solution obtained by using generalized Chebyshev polynomials with M=3 and 4 and the exact solution, This table shows that by increasing m the accuracy of solution will increase.

Table (1) comparison between exact and approximate solution.

t	Exact	M=3	M=4
0	0	5.63785-018	0
0.1	0.04195073	0.04180603	0.04195416
0.2	0.07931716	0.07922622	0.07932535
0.3	0.11247325	0.11250476	0.11247145
0.4	0.14175084	0.14188584	0.14172635
0.5	0.16774429	0.16761363	0.16773902
0.6	0.18980672	0.18993235	0.18982955
0.7	0.20906597	0.20908615	0.209067705
0.8	0.22541345	0.22531924	0.225413318
0.9	0.23901278	0.23887579	0.23905895
1	0.25000007	0.25000000	0.25000000

4. Conclusion

The generalized Chebyshev method have been successfully applied to the problem of variational problem. The method is based upon reducing the variational problem by a quadratic programming one. The generalized Chebyshev method will be used to solve more complex variational problem.

References

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خوارزمية شيبشيف العامة لحل حساب التغيرات

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قسم العلوم التطبيقية

الجامعة التكنولوجية

المستخلص:

في هذا البحث، تم تقديم متعددات حدود شيبشيف العامة والمعرفة في الفترة $[a, b]$. بعض الخواص لهذه المتعددات والمفيدة في التطبيقات العددية تم مناقشتها واشتقاقها. ثم اقترحت طريقة مباشرة لحل مسائل التغيرات والمعتمدة على متعددات شيبشيف العامة لأختزال مسألة التغيرات الى مسألة البرمجة اللاخطية. تم الحصول على الحل بدلالة متعددات شيبشيف العامة وأعطى مثال توضيحي لذلك.