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On fuzzy Laplace transforms for fuzzy differential equations of the third order

حول تحويلات لابلاس الضبابية لمعادلات تفاضلية ضبابية من الرتبة الثالثة

Hawrra Fadel Muhammad Ali, Amal khalaf Haydar,
University of Kufa — College Education for Girls

Abstract

In this paper, we introduce a result for fuzzy derivative of the third-order in the sense of H-differentiability, and we find fuzzy Laplace transform for third-order derivative. In addition, we present examples of fuzzy initial value problems of the third-order .

المستخلص

في هذا البحث حصلنا على نتيجة حول المشتقة الضبابية من الرتبة الثالثة حسب مفهوم مشتقة -H, وأوجدنا تحويل لابلاس الضبابي للمشتقة من الرتبة الثالثة. بالإضافة إلى ذلك نقدم أمثلة على مسائل قيم ابتدائية ضبابية من الرتبة الثالثة.

1.Introduction

The concept of fuzzy derivative was first introduced by Chang and Zadeh in 1972 [1] while the term fuzzy differential equation was coined by Kandel and Bratt in 1978 [2]. Puri and Ralescu [3] introduced the concept of the differential of a fuzzy function. Also, many researchers have worked on the theoretical and numerical solutions of fuzzy differential equations such as [4],[5] and [6].

Recently, Allahviranloo and Ahmadi [7] have proposed the fuzzy Laplace transforms for solving first-order fuzzy differential equations. Salahshour and Allahviranloo [8] pointed out that under what conditions the fuzzy- valued functions can possess the fuzzy Laplace transform.

In this paper, we are going to find a result for fuzzy derivative of the third-order and fuzzy Laplace transform for that order are obtained. The structure of this paper is as follows: In Sect.2, some basic concepts are provided. In Sect.3, a result for fuzzy derivative of the third-order and Laplace transform for that order are presented. In Sect.4, we construct a system can be used for solving fuzzy initial value problems (FIVPs) of the third-order with examples. In Sect.5, a conclusion is drawn.

2.Preliminaries

In this section, we give some necessary definitions and concepts which will used in this paper.

Definition 2.1:[4] A fuzzy number u in parametric form is a pair $(\underline{u},\overline{u})$ of functions $\underline{u}(r),\overline{u}(r), 0 \le r \le 1,$ which satisfy the following requirements :

1. $\underline{u}(r)$ is a bounded non – decreasing left continuous function in $(0,1]$, and right continuous at 0,
2. $\overline{u}(r)$ is a bounded non – increasing left continuous function in $(0,1]$, and right continuous at 0,
3. $\underline{u}(r) \le \overline{u}(r),\ 0 \le r \le 1$.

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha,\ 0 \le r \le 1$. We recall that for $a < b < c$ which $a,b,c \in R$, the triangular fuzzy number $u = (a,b,c)$ determined by a,b,c is given such that $\underline{u}(r) = a + (b-c)r$ and $\overline{u}(r) = c - (c-b)r$ are the endpoints of the r-level sets, for all $r \in [0,1]$.

Definition 2.2 Let $x, y \in E$. If there exists $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y, and is denoted by $x \Theta y$. In this paper, the sign Θ always stands for H-difference, and it is denoted by $x \Theta y \neq x + (-y)$.

Definition 2.3:[3] A fuzzy function $F : U \to F_0(R^n)$ is called H-differentiable at $x_0 \in U$ if there exists $DF(x_0) = F'(x_0) \in F_0(R^n)$, such that the limits

$$\lim_{h \to 0^+} [(F(x_0 + h) \Theta F(x_0))/h]$$

and

$$\lim_{h \to 0^+} [(F(x_0) \Theta F(x_0 - h))/h]$$

both exist and are equal to $DF(x_0)$

Theorem 2.4:[9] Let $f(x)$ be a fuzzy-valued function on $[a, \infty)$ represented by $\left(\underline{f}(x,r), \overline{f}(x,r)\right)$. For any fixed $r \in [0,1]$, assume $\underline{f}(x,r)$ and $\overline{f}(x,r)$ are Riemann -integrable on $[a,b]$ for every $b \geq a$, and assume there are two positive $\underline{\text{M}}(\text{r})$ and $\overline{\text{M}}(\text{r})$ such that $\int_a^b \left|\underline{f}(x,r)\right| dx \leq \underline{\text{M}}(\text{r})$ and $\int_a^b \left|\overline{f}(x,r)\right| dx \leq \overline{\text{M}}(\text{r})$ for every $b \geq a$. Then $f(x)$ is improper fuzzy Riemann-integrable on $[a, \infty)$ and the improper fuzzy Riemann -integral is a fuzzy number. Furthermore, we have

$$\int_a^\infty f(x)dx = \left(\int_a^\infty \underline{f}(x,r)\, dx, \int_a^\infty \overline{f}(x,r)\, dx\right).$$

Definition 2.5:[7] Let $f(x)$ be continuous fuzzy-value function. Suppose that $f(x)e^{-sx}$ is improper fuzzy Rimann-integrable on $[0, \infty)$, then $\int_0^\infty f(x)\, e^{-sx} dt$ is called fuzzy Laplace transforms and is denoted as

$$L(f(x)) = \int_0^\infty f(x)\, e^{-sx} dx \text{ , } (s > 0 \text{ and integer }).$$

From theorem 2.4, we have

$$\int_0^\infty f(x)\, e^{-sx} dx = \left(\int_0^\infty \underline{f}(x,r)\, e^{-sx} dx, \overline{f}(x,r)\, e^{-sx} dx\right),$$

also by using the definition of classical Laplace transform: $l(\underline{f}(x,r)) = \int_0^\infty \underline{f}(x,r) e^{-sx} dx$ and

$$l(\overline{f}(x,r)) = \int_0^\infty \overline{f}(x,r) e^{-sx} dx$$

then, we follow:

$$L(f(x)) = (l(\underline{f}(x,r)), l(\overline{f}(x,r)))$$

3.Third order FIVPs

In this section, we have the following result for third-order derivative under H-differentiability:

Theorem 3.1 Let $F(t), F'(t), F''(t)$ are differentiable fuzzy-valued functions. Moreover, we denote α -cut representation of fuzzy-valued function $F(t)$ with $[F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$, then:

$$[F'''(t)]^\alpha = [f_\alpha'''(t), g_\alpha'''(t)]$$

Proof Since $F(t)$ and $F'(t)$ are differentiable then we get

$$[F''(t)]^\alpha = [f_\alpha''(t), g_\alpha''(t)],$$

since $F''(t)$ is differentiable then by definition 2.3 we get

$$[F''(t+h)\Theta F''(t)]^{\alpha} = [f''_{\alpha}(t+h), g''_{\alpha}(t+h)]\Theta[f''_{\alpha}(t), g''_{\alpha}(t)]$$
$$= [f''_{\alpha}(t+h) - f''_{\alpha}(t), g''_{\alpha}(t+h) - g''_{\alpha}(t)]$$

and

$$[F''(t)\Theta F''(t-h)]^{\alpha} = [f''_{\alpha}(t), g''_{\alpha}(t)]\Theta[f''_{\alpha}(t-h), g''_{\alpha}(t-h)]$$
$$= [f''_{\alpha}(t) - f''_{\alpha}(t-h), g''_{\alpha}(t) - g''_{\alpha}(t-h)]$$

and , multiplying by $\frac{1}{h}$, $h>0$ we get:

$$\frac{1}{h}[F''(t+h)\Theta F''(t)]^{\alpha} = [\frac{f''_{\alpha}(t+h) - f''_{\alpha}(t)}{h}, \frac{g''_{\alpha}(t+h) - g''_{\alpha}(t)}{h}]$$

and

$$\frac{1}{h}[F''(t)\ \Theta F''(t-h)]^{\alpha} = [\frac{f''_{\alpha}(t) - f''_{\alpha}(t-h)}{h}, \frac{g''_{\alpha}(t) - g''_{\alpha}(t-h)}{h}]$$

Finally, using the fact that $h \longrightarrow 0$ on both sides, the proof is completed .

Theorem 3.2 Suppose that $g(t), g'(t)$ and $g''(t)$ are continuous fuzzy-valued functions on $[0,\infty)$ and of exponential order and $g'''(t)$ is piecewise continuous fuzzy-valued function on $[0,\infty)$ with $g(t) = (\underline{g}(t,\alpha), \overline{g}(t,\alpha))$,then

$$L(g'''(t)) = s^3 L(g(t))\Theta s^2 g(0)\Theta s g'(0)\Theta g''(0) \tag{3.1}$$

Proof First, we state the notations carefully as follows : $\underline{g}'$, $\underline{g}''$ and $\underline{g}'''$ are the lower endpoints function's derivatives, $\overline{g}'$, $\overline{g}''$ and $\overline{g}'''$ are the upper endpoints function's derivatives. By using theorem 3.1 we have

$$L(g'''(t)) = L(\underline{g}'''(t,\alpha), \overline{g}'''(t,\alpha))$$
$$= (l(\underline{g}'''(t,\alpha)), l(\overline{g}'''(t,\alpha))), \tag{3.2}$$

for arbitrary fixed $\alpha \in [0,1]$. Now, by using the definition of classical Laplace transform we get

$$l(\underline{g}'''(t,\alpha)) = s^3 l(\underline{g}(t,\alpha)) - s^2 \underline{g}(0,\alpha) - s\underline{g}'(0,\alpha) - \underline{g}''(0,\alpha)$$
$$l(\overline{g}'''(t,\alpha)) = s^3 l(\overline{g}(t,\alpha)) - s^2 \overline{g}(0,\alpha) - s\overline{g}'(0,\alpha) - \overline{g}''(0,\alpha)$$

Then, equation (3.2) becomes

$$L(g'''(t)) = (s^3 l(\underline{g}(t,\alpha)) - s^2 \underline{g}(0,\alpha) - s\underline{g}'(0,\alpha) - \underline{g}''(0,\alpha), s^3 l(\overline{g}(t,\alpha)) - s^2 \overline{g}(0,\alpha)$$
$$- s\overline{g}'(0,\alpha) - \overline{g}''(0,\alpha))$$
$$= s^3 L(g(t))\Theta s^2 g(0)\Theta s g'(0)\Theta g''(0)$$

4. Constructing solutions via fuzzy initial value problems

In this section ,we consider the fuzzy initial value problem

$$y'''(t) = f\ (t, y(t), y'(t), y''(t))$$
$$y(0) = (\underline{y}(0,\alpha), \overline{y}(0,\alpha))$$
$$y'(0) = (\underline{y}'(0,\alpha), \overline{y}'(0,\alpha)) \tag{4.1}$$
$$y''(0) = (\underline{y}''(0,\alpha), \overline{y}''(0,\alpha)), 0 \le \alpha \le 1.$$

By using fuzzy Laplace transform method we have :

$$L(y'''(t)) = L(f(t, y(t), y'(t), y''(t)))\ . \tag{4.2}$$

By using theorem 3.2, equation (4.2) can be written as follows :

$$s^3 L(y(t))\Theta s^2 y(0)\Theta s y'(0)\Theta y''(0) = L(f(t, y(t), y'(t), y''(t)))$$

Thus

$$s^3 l(\underline{y}(t,\alpha)) - s^2\underline{y}(0,\alpha) - s\underline{y}'(0,\alpha) - \underline{y}''(0,\alpha) = l(\underline{f}(t,y(t),y'(t),y''(t),\alpha))$$
$$s^3 l(\overline{y}(t,\alpha)) - s^2\overline{y}(0,\alpha) - s\overline{y}'0,\alpha) - \overline{y}''(0,\alpha) = l(\overline{f}(t,y(t),y'(t),y''(t),\alpha)) \quad (4.3)$$

To solve the linear system (4.3), for simplicity we assume that:

$l(\underline{y}(t,\alpha)) = H(s,\alpha)$

$l(\overline{y}(t,\alpha)) = K(s,\alpha)$

where $H(s,\alpha)$ and $K(s,\alpha)$ are solutions of system (4.3). By using inverse Laplace transform, $\underline{y}(t,\alpha)$ and $\overline{y}(t,\alpha)$ are computed as follows

$\underline{y}(t,\alpha) = l^{-1}(H(s,\alpha))$

$\overline{y}(t,\alpha) = l^{-1}(K(s,\alpha))$

Example 4.1 Consider the following FIVP

$y'''(t) = 2y''(t) + 3y'(t)$

$y(0) = (3+\alpha, 5-\alpha)$

$y'(0) = (-3+\alpha, -1-\alpha)$

$y''(0) = (8+\alpha, 10-\alpha)$

We note that

$f(t,y(t),y'(t),y''(t)) = 2y''(t) + 3y'(t) = (2\underline{y}''(t,\alpha) + 3\underline{y}'(t,\alpha), 2\overline{y}''(t,\alpha) + 3\overline{y}'(t,\alpha))$

$\underline{f}(t,y(t),y'(t),y''(t),\alpha) = 2\underline{y}''(t,\alpha) + 3\underline{y}'(t,\alpha)$

and

$\overline{f}(t,y(t),y'(t),y''(t),\alpha) = 2\overline{y}''(t,\alpha) + 3\overline{y}'(t,\alpha)$

Thus

$$\begin{aligned} l(\underline{f}(t,y(t),y'(t),y''(t),\alpha) &= 2l(\underline{y}''(t,\alpha)) + 3l(\underline{y}'(t,\alpha)) \\ &= 2[s^2 l(\underline{y}(t,\alpha) - s\underline{y}(0,\alpha) - \underline{y}'(0,\alpha)] + 3[sl(\underline{y}(t,\alpha) - \underline{y}(0,\alpha)] \\ &= (2s^2+3s)l(\underline{y}(t,\alpha)) - (6+2\alpha)s - (3+5\alpha) \end{aligned} \quad (4.4)$$

$$\begin{aligned} l(\overline{f}(t,y(t),y'(t),y''(t),\alpha) &= 2l(\overline{y}''(t,\alpha)) + 3l(\overline{y}'(t,\alpha)) \\ &= 2[s^2 l(\overline{y}(t,\alpha) - s\overline{y}(0,\alpha) - \overline{y}'(0,\alpha)] + 3[sl(\overline{y}(t,\alpha) - \overline{y}(0,\alpha)] \\ &= (2s^2+3s)l(\overline{y}(t,\alpha) + (-10+2\alpha)s + (5\alpha-13) \end{aligned} \quad (4.5)$$

Substituting (4.4)and (4.5) in system (4.3) gives

$$l(\underline{y}(t,\alpha)(s^3-2s^2-3s) = (3+\alpha)s^2 + (-9-\alpha)s + (5-4\alpha)$$
$$l(\overline{y}(t,\alpha)(s^3-2s^2-3s) = (5-\alpha)s^2 + (-11+\alpha)s + (4\alpha-3) \quad (4.6)$$

The solution of system (4.6) is as follows:

$$l(\underline{y}(t,\alpha)) = \frac{(3+\alpha)s^2 + (-9-\alpha)s + (5-4\alpha)}{s^3-2s^2-3s} = \frac{(3+\alpha)s^2 + (-9-\alpha)s + (5-4\alpha)}{s(s-3)(s+1)}$$

$$l(\overline{y}(t,\alpha)) = \frac{(5-\alpha)s^2 + (-11+\alpha)s + (-3+4\alpha)}{s^3-2s^2-3s} = \frac{(5-\alpha)s^2 + (-11+\alpha)s + (-3+4\alpha)}{s(s-3)(s+1)}$$

After performing partition of fractions yields

$$l(\underline{y}(t,\alpha)) = \frac{4\alpha - 5}{3s} + \frac{17-2\alpha}{4(s+1)} + \frac{5+2\alpha}{12(s-3)}$$

$$l(\overline{y}(t,\alpha)) = \frac{-3+4\alpha}{-3s} + \frac{13+2\alpha}{4(s+1)} + \frac{9-2\alpha}{12(s-3)}$$

By using inverse Laplace transform, we get

$$\underline{y}(t,\alpha) = \frac{4\alpha-5}{3} l^{-1}(\frac{1}{s}) + \frac{17-2\alpha}{4} l^{-1}(\frac{1}{s+1}) + \frac{5+2\alpha}{12} l^{-1}(\frac{1}{s-3})$$

$$= \frac{4\alpha-5}{3} + \frac{17-2\alpha}{4} e^{-t} + \frac{5+2\alpha}{12} e^{3t}$$

$$\overline{y}(t,\alpha) = \frac{3-4\alpha}{3} l^{-1}(\frac{1}{s}) + \frac{13+2\alpha}{4} l^{-1}(\frac{1}{s+1}) + \frac{9-2\alpha}{12} l^{-1}(\frac{1}{(s-3})$$

$$= \frac{3-4\alpha}{3} + \frac{13+2\alpha}{4} e^{-t} + \frac{9-2\alpha}{12} e^{3t}$$

Example 4.2 Consider the following FIVP

$$y'''(t) = -y''(t) - 3y'(t)$$

$$y(0) = (\frac{3}{4} + \frac{1}{4} r, \frac{5}{4} - \frac{1}{4} r)$$

$$y'(0) = (\frac{3}{2} + \frac{1}{2} r, \frac{5}{2} - \frac{1}{2} r)$$

$$y''(0) = (\frac{15}{4} + \frac{1}{4} r, \frac{17}{4} - \frac{1}{4} r)$$

We note that

$$f(t, y(t), y'(t), y''(t)) = -y''(t) - 3y'(t) = (-\overline{y}''(t,r) - 3\overline{y}'(t,r), -\underline{y}''(t,r) - 3\underline{y}'(t,r))$$

$$\underline{f}(t, y(t), y'(t), y''(t), r) = -\overline{y}''(t,r) - 3\overline{y}'(t,r)$$

$$\overline{f}(t, y(t), y'(t), y''(t), r) = -\underline{y}''(t,r) - 3\underline{y}'(t,r)$$

Thus

$$l(\underline{f}(t, y(t), y'(t), y''(t), r) = -l(\overline{y}''(t,r)) - 3l(\overline{y}'(t,r))$$

$$= -[s^2 l(\overline{y}(t,r) - s\overline{y}(0,r) - \overline{y}'(0,r)] - 3[sl(\overline{y}(t,r) - \overline{y}(0,r)]$$

$$= (-s^2 - 3s) l(\overline{y}(t,r) + (\frac{5}{4} - \frac{1}{4} r)s + (\frac{25}{4} - \frac{5}{4} r) \qquad (4.7)$$

$$l(\overline{f}(t, y(t), y'(t), y''(t), r) = -l(\underline{y}''(t,r)) - 3l(\underline{y}'(t,r))$$

$$= -[s^2 l(\underline{y}(t,r) - s\underline{y}(0,r) - \underline{y}'(0,r)] - 3[sl(\underline{y}(t,r) - \underline{y}(0,r)]$$

$$= (-s^2 - 3s) l(\underline{y}(t,r) + (\frac{3}{4} + \frac{1}{4} r)s + (\frac{15}{4} + \frac{5}{4} r) \qquad (4.8)$$

Substituting (4.7)and (4.8) in system (4.3) gives

$$s^3 l(\underline{y}(t,r) + (s^2 + 3s) l(\overline{y}(t,r)) = (\frac{3}{4} + \frac{1}{4} r)s^2 + (\frac{11}{4} + \frac{1}{4} r)s + (10 - r)$$

$$s^3 l(\overline{y}(t,r) + (s^2 + 3s) l(\underline{y}(t,r)) = (\frac{5}{4} - \frac{1}{4} r)s^2 + (\frac{13}{4} - \frac{1}{4} r)s + (8 + r) \qquad (4.9)$$

The solution of system (4.9) is as follows:

$$l(\underline{y}(t,r)) = \frac{(r+3)s^4 + (2r+6)s^3 + 12s^2 - (r+71)s - 12r - 96}{4s^5 - 4s^3 - 24s^2 - 36s}$$

$$=\frac{(r+3)s^4+(2r+6)s^3+12s^2-(r+71)s-12r-96}{4s(s^2+s+3)(s^2-s-3)}$$

$$l(\overline{y}(t,r))=\frac{(5-r)s^4+(10-2r)s^3+12s^2+(r-73)s+12r-120}{4s^5-4s^3-24s^2-36s}$$

$$=\frac{(5-r)s^4+(10-2r)s^3+12s^2+(r-73)s+12r-120}{4s(s^2+s+3)(s^2-s-3)}$$

After performing partition of fractions yields

$$l(\underline{y}(t,r))=\frac{4r+32}{12s}-\frac{2s}{s^2+s+3}-\frac{(r-1)}{12}\frac{s-7}{s^2-s-3}$$

$$l(\overline{y}(t,r)=\frac{40-4r}{12s}-\frac{2s}{s^2+s+3}+\frac{(r-1)}{12}\frac{s-7}{s^2-s-3}$$

By using inverse Laplace transform, we get

$$\underline{y}(t,r)=\frac{4r+32}{12}l^{-1}(\frac{1}{s})-l^{-1}(\frac{2s}{s^2+s+3})-\frac{r-1}{12}l^{-1}(\frac{s-7}{s^2-s-3})$$

$$=\frac{8+r}{3}-2e^{-\frac{t}{2}}[\cos\frac{\sqrt{11}}{2}t-\frac{\sqrt{11}}{11}\sin\frac{\sqrt{11}}{2}t]-\frac{r-1}{12}e^{\frac{t}{2}}[\cosh\frac{\sqrt{13}}{2}t-\sqrt{13}\sinh\frac{\sqrt{13}}{2}t]$$

$$\overline{y}(t,r)=\frac{40-4r}{12}l^{-1}(\frac{1}{s})-l^{-1}(\frac{2s}{s^2+s+3})+\frac{r-1}{12}l^{-1}(\frac{s-7}{s^2-s-3})$$

$$=\frac{10-r}{3}-2e^{-\frac{t}{2}}[\cos\frac{\sqrt{11}}{2}t-\frac{\sqrt{11}}{11}\sin\frac{\sqrt{11}}{2}t]+\frac{r-1}{12}e^{\frac{t}{2}}[\cosh\frac{\sqrt{13}}{2}t-\sqrt{13}\sinh\frac{\sqrt{13}}{2}t]$$

5.Conclusion

A result for fuzzy derivative of the third-order in the sense of H- differentiability and fuzzy Laplace transform for third-order derivative are presented ,then we used these results for solving fuzzy initial value problems of the third-order .

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