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Ehrhart Polynomials of a Cyclic Polytopes

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Abstract

Computing the volume of a polytope in R^n is a very important subject in different areas of mathematic. A pplications range from the very pure (number theory, toric Hilbert functions, Kostant's partition function in representation theory) to the most applied (cryptography, integer programming, contingency tables).

In this work, the cyclic polytopes with some methods for finding their volumes are given.

Moreover, the Ehrhart polynomial of cyclic polytope is computed with some methods. One of these methods is modified and gives a theorem for computing the coefficients of the Ehrhart polynomials.

Keywords: cyclic polytopes; Ehrhart polynomial

متعدد حدود ايرهارت للمتعدد السطوح الدوري

الخلاصة

حساب حجم متعدد السطوح في المجا ل R^n هو موضوع مهم جدا في فروع الرياضيا ت المختلفة. مدى تطبيقاته من النظري جدا" (نظرية الأعداد، دوال هلبرت، تجزئة كاستنت في نظرية التمثيل) الى الأكثر تطبيقا" (التشفير، برمجة الأعداد الصحيحة، جداول الأحتمالات)
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1-Introduction

One of the most easily accessible combinatorials among the d-dimensional polytopes with n vertices is a cyclic polytope. The Ehrhart polynomial of a cyclic polytope was given by [1] as conjecture, which gave a simple formula of the Ehrhart polynomial of an integral cyclic polytope. Fu Liu [2] proved the conjecture about the coefficients of the Erhart polynomial of a cyclic polytope.

In this paper, some properties of the cyclic polytope are presented, and a theorem for computing the Ehrhart polynomial of a cyclic polytope

Acomparison between the coefficients which are obtained by the given theorem and the other which were founded using another method are made, same results are obtained.

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2-The Cyclic Polytope

In this section some definitions concerning the cyclic polytope are given.

Definition 2.1: [3]

The moment curve in R^d is defined by $m : R \to R^d$, where

$$m(t) = \begin{pmatrix} t \\ t^2 \\ \vdots \\ t^d \end{pmatrix} \in R^d .$$

Definition (2.2): [3]

The cyclic polytope of dimension d with n vertices is the convex hull $C_d(t_1, t_2, ..., t_n) = \mathrm{conv}\{m(t_1), m(t_2), ..., m(t_n)\}$ of distinct points $m(t_i)$ where n>d, with $t_1 < t_2 < \ldots < t_n$ on the moment curve.

Note: the cyclic polytope is sometimes denoted by $C_d(n)$, where n is the number of vertices and d is the dimension of cyclic polytope.

For example the cyclic polytope $C_2(4)$ is the convex hull of the vertices (-4,16), (-2,4), (1,1) and (4,16) as illustrates in figure (2.1).

3-Properties of a Cyclic Polytope

Theorem (3.1): [4]

For n>2 and d≥2, the cyclic polytope $C_d(n) = \mathrm{conv}\{m(t_1), ..., m(t_n)\}$, is a simplicial d-polytope.

Proof:

The proof that every facet of $C_d(n)$ has exactly d vertices is needed. Since their facets are d-1 dimensional polytopes, this mean that all the facets are simplices. This is done by showing that there is no hyperplane, that contains d+ 1 points on the moment curve.

Let $v^{(1)}, ..., v^{(d+1)}$ be the d+1 vertices of $C_d(n)$, since $v^{(1)}, ..., v^{(d+1)}$ are distinct vertices of $C_d(n)$, then $t_{i1}, ..., t_{id+1}$ are exists such that $v^{(j)} = m(t_{ij})$, where $i, j = 1, ..., d+1$. In particular, if $v_k^{(j)}$ denote the k^{th} coordinate of $v^{(j)}$, then $v_k^{(j)} = t_{ij}^k$, where $k = 1, ..., d$.

From the definition of a hyperplane which is a set $\{(x_1, ..., x_d) \in R^d : c_0 + c_1 x_1 + ... + c_d x_d = 0\}$ for some fixed numbers $c_0, ..., c_d$, not all zero, therefore a hyperplane contains v_j if

$$c_0 + c_1 v_1^{(j)} + \ldots + c_d v_d^{(j)} = 0 \ldots (3.1)$$

To find a hyperplane containing all $v^{(j)}$, the numbers $c_0, c_1, ..., c_d$ satisfying equation (3.1) for all $v^{(j)}$ must be found. Recall that $v_k^{(j)} = t_{ij}^k$,

therefore, the problem is to find a nonzero solution to the system of linear equations which is given by:

$$\begin{aligned} c_0 + c_1 t_{i1} + c_2 t_{i1}^2 + \cdots + c_d t_{i1}^d &= 0 \\ c_0 + c_1 t_{i2} + c_2 t_{i2}^2 + \cdots + c_d t_{i2}^d &= 0 \\ \vdots \qquad\qquad\qquad \vdots &= 0 \\ c_0 + c_1 t_{id+1} + c_2 t_{id+1}^2 + \cdots + c_d t_{id+1}^d &= 0 \end{aligned}$$

t

his means that

$$[c_0 c_1 \ldots c_d]\begin{bmatrix} 1 & \cdots & 1 \\ t_{i1} & \cdots & t_{id+1} \\ t_{i1}^2 & \cdots & t_{id+1}^2 \\ \vdots & & \vdots \\ t_{i1}^d & \cdots & t_{id+1}^d \end{bmatrix} = [0 \ldots 0] \quad (3.2)$$

The matrix T is $(d+1)\times(d+1)$ square matrix, then by [5. p.49] the determinant of this matrix is nonzero, and it's equal to

$$\det\begin{bmatrix} 1 & \cdots & 1 \\ t_{i1} & \cdots & t_{id+1} \\ t_{i1}^2 & \cdots & t_{id+1}^2 \\ \vdots & & \vdots \\ t_{i1}^d & \cdots & t_{id+1}^d \end{bmatrix} = \prod_{1 \le j < k \le d+1} (t_{ik} - t_{ij}).$$

This mean that, the only solution of the equation (3.2) is $c_0 = c_1 = \ldots = c_d = 0$, therefore there is no hyperplane contains all d+1 vertices $v^{(1)},\ldots,v^{(d+1)}$ of $C_d(n)$, so no facet of $C_d(n)$ have d+1 vertices, therefore $C_d(n)$ is simplicial.

Definition (3.1): [6]

Let $f_i(P)$ be the number of i-faces of a d-polytope P, for $i = 0,1,\ldots, d-1$. The face vector associated with a d-polytope is defined as $f(P) = (f_0(P), f_1(P),\ldots, f_{d-1}(P))$ where $n = f_0(P)$ is the number of vertices of d-polytope.

Theorem (3.2), [McMullen's upper bound theorem], [7]:

Let $f_i(P)$ denote the number of i-faces of a d-polytope P with n vertices then,

$$f_i(P) \le f_i(C_d(n)), \qquad \forall\, i = 1,\ldots,d-1$$

.

The number of i-faces of a cyclic polytope $C_d(n)$ can be given as, [8].

$$f_i(C_d(n)) = \begin{cases} \displaystyle\sum_{j=0}^{(d-1)/2} \frac{i+2}{n-j}\binom{n-j}{j+1}\binom{j+1}{i+1-j}, & d \text{ odd}, \\ \displaystyle\sum_{j=0}^{d/2} \frac{n}{n-j}\binom{n-j}{j}\binom{j}{i+1-j}, & d \text{ even}. \end{cases}$$

For example,

P	f_0	f_1	f_2	f_3	f_4
$C_5(10)$	10	45	100	105	42
$C_5(20)$	20	190	580	680	272
$C_5(30)$	30	435	1460	1755	702

4-The Ehrhart polynomial of a Cyclic Polytope

The Ehrhart polynomial of a cyclic polytope count the number of lattice points in a dilation of a cyclic polytope by positive integer t^* is equal to its volume plus the number of lattice points in its lower envelope, [9].

The following theorem appears in [2] without proof. Here, we prove it for the sake of completeness.

Theorem (4.1): [2]

For any integral cyclic polytope $C_d(T)$ where $T=\{t_1,t_2,...,t_n\}$ for $t_i=1,2,...,n$, and $i=1,...,n$.

$$L(C_d(T),t^*)=vol(t^*C_d(T))+L(C_{d-1}(T),t^*)...(4.1)$$

Hence,

$$L(C_d(T),t^*)=\sum_{k=0}^{d} vol(t^*C_k(T))=\sum_{k=0}^{d} vol_k(C_k(T)t^{*k},$$

where $vol_k(t^*C_k(T))$ is the volume of $t^*C_k(T)$ in k-dimensional space, and $vol_0(t^*C_0(T))=1$.

Proof:

By using a recurrence relation on equation (4.1) the following result is obtained,

$$L(C_{d-1}(T),t^*)=vol(t^*C_{d-1}(T))+L(C_{d-2}(T),t^*),$$

And

$$L(C_{d-2}(T),t^*)=vol(t^*C_{d-2}(T))+L(C_{d-3}(T),t^*).$$

Then similarly, we get

$$L(C_d(T),t^*)=\sum_{k=0}^{d} vol(t^*C_k(T))=\sum_{k=0}^{d} vol_k(C_k(T)t^{*k}.$$

Theorem (4.2): [2]

For any integral set T with $n=|T|=d+1$, which mean that $C_d(n)$ is simplex, then

$$vol(t^*C_d(T))=\frac{t^{*d}}{d!}\prod_{1\le i\,\mathbf{p}\,j\le d+1}(t_j-t_i)...(4.2)$$

From the above theorems we get the following theorem.

Theorem (4.3):

For any cyclic polytope $C_d(T)$ where $T=\{t_1,t_2,...,t_n\}$ for $t_i=1,2,...,n$, and $i=1,...,n$. we get the following:

(i) If $C_d(T)$ is not simplex, one can decomposed it into simplices, then using equation (4.2) to compute its volume.

(ii) In one dimensional space the volume of cyclic polytope $(t^*C_1(T))$ in the interval $[t_1,t_n]$ is equal to t_n-t_1.

5- Examples about Ehrhart Polynomial of Cyclic Polytope

Example (5.1):

Let us consider the cyclic polytope $C_d(T)$, where d=2 and T={1,2,3}, The convex hull of the vertices (1,1), (2,4) and (3,9) is a triangle, as illustrates in figure (5.1), [10].

The area of P is equal to the area of two triangles, which is computed using usual mathematical operation as

$$\Delta_1=(0.5)(5.1662)(0.3)$$
$$=0.8493.$$

$$\Delta_2 = (0.5)(3.0805)(0.3)$$
$$= 0.46207.$$

Therefore the area of them is $\Delta = \Delta_1 + \Delta_2$

$$= 0.8493 + 0.46207 = 1.31137.$$

From theorem (4.1) we get

$$L(C_2(T,t^*) = \sum_{k=0}^{2} vol_k(C_k(1,2,3)t^{*k}.$$

since $C_2(1,2,3)$ is simplex, then by using theorem (4.2), yields

$$vol_2(C_2(1,2,3)) = \frac{1}{2!}\prod_{1\le i\le j\le 3}(j-i)$$

$$= \frac{1}{2!}(2-1)(3-1)(3-2)$$
$$= 1$$

and using theorem (4.3)(ii), $vol_1(C_1(1,2,3))$ is just an interval [1,3], so $vol_1(C_1(1,2,3)) = 3-1 = 2$

The Ehrhart polynomial of $C_2(1,2,3)$ is

$$L(C_2(1,2,3),t^*) = t^{*2} + 2t^* + 1.$$

Example (5.2):

In this example we compute the Ehrhart polynomial of $C_3(1,2,3,4)$ as illustrates in figure (5.2), [10]. by theorem (4.1) we get

$$L(C_3(T),t^*) = \sum_{k=0}^{3} vol_k(C_k(1,2,3,4)t^{*k}.$$

since $C_3(1,2,3,4)$ is simplex, then using theorem (4.2) the following is obtained

$$vol_3(C_3(1,2,3,4)) = \frac{1}{3!}\prod_{1\le i\le j\le 4}(j-i)$$

$$= \frac{1}{3!}[(2-1)(3-1)(3-2)(4-1)(4-2)(4-3)]$$
$$= 2$$

To compute $vol_2(C_2(1,2,3,4))$, since $C_2(1,2,3,4)$ is not simplex, then by using theorem (4.3)(i), $C_2(1,2,3,4)$ can be decomposed into simplices $C_2(1,2,3)$ and $C_2(1,3,4)$

Then,

$$vol(C_2(1,2,3,4)) = vol_2(C_2(1,2,3)) + vol_2(C_2(1,3,4))$$

$$= \frac{1}{2!}[(2-1)(3-1)(3-2) + (3-1)(4-1)(4-3)]$$
$$= 4$$

and using theorem (4.3) (ii), $vol_1(C_1(1,2,3,4))$ is just an interval [1,4], is equal to $4-1=3$.

The Ehrhart polynomial of $C_3(1,2,3,4)$ is

$$L(C_3(1,2,3,4),t^*) = 2t^{*3} + 4t^{*2} + 3t^* + 1.$$

Now for n=5 with the same dimension, the Ehrhart polynomial of $C_3(T)$ when T={1,2,3,4,5}as illustrated in figure (5.3), [10], Was computed by using theorem (4.1), as follow,

$$L(C_3(T),t^*) = \sum_{k=0}^{3} vol_k(C_k(1,2,3,4,5)t^{*k}.$$

Since $C_3(5)$ is not simplex, then by using theorem (4.3)(i), can be decomposed it into simplices $C_3(1,2,3,5)$ and $C_3(1,3,4,5)$.

Then,

$$vol(C_3(1,2,3,4,5)) = vol_3(C_3(1,2,3,5)) + vol_3(C_3(1,3,4,5))$$

$$= \frac{1}{3!}[(2-1)(3-1)(3-2)(5-1)(5-2)(5-3) + (3-1)(4-1)(4-3)(5-1)(5-3)(5-4)]$$

$$= 16.$$

To compute $vol_2(C_2(1,2,3,4,5))$, since $C_2(1,2,3,4,5)$ is not simplex, then using theorem (4.3)(i), can be decomposed into simplices which are $C_2(1,2,3)$, $C_2(1,3,4)$ and $C_2(1,4,5)$.

Then,

$$vol(C_2(1,2,3,4,5)) = vol_2(C_2(1,2,3)) + vol_2(C_2(1,3,4)) + vol_2(C_2(1,4,5))$$

$$= \frac{1}{2!}[(2-1)(3-1)(3-2) + (3-1)(4-1)(4-3) + (4-1)(5-1)(5-4)]$$

$$=10$$

and using theorem (4.3)(ii), $vol_1(C_1(1,2,3,4,5))$ is just an interval [1,5], which is equal to $5-1=4$.

Therefore, the Ehrahart polynomial of $C_3(1,2,3,4,5)$ is

$$L(C_3(1,2,3,4,5), t^*) = 16t^{*3} + 10t^{*2} + 4t^* + 1.$$

Also the obtained Ehrhart polynomials of a cyclic polytopes are compared with a table contains the Ehrhart polynomial that is given in [9]. The same coefficient of the Ehrhart polynomial is obtained.

6-Conclousions

In this paper, a theorem that can compute the volume of any cyclic polytope of dimension two and three with any vertices was proposed depending on the graph of the shape.

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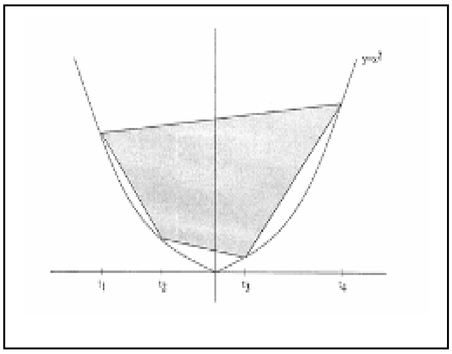

Figure (2.1) cyclic polytope in R^2 **; convex hull of vertices is shaded**

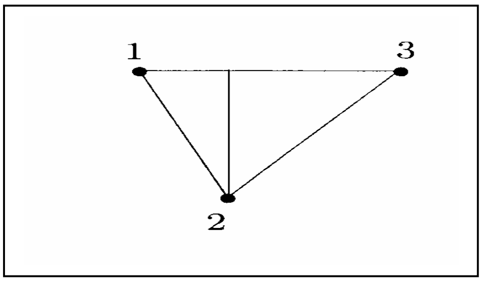

Figure (5.1) cyclic polytope $C_2(3)$

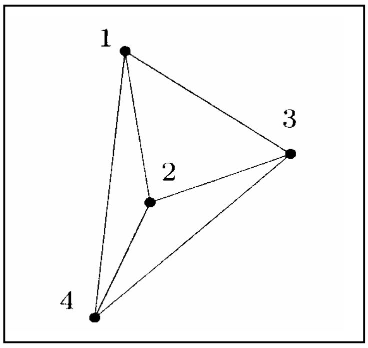

Figure (5.2) cyclic polytope $C_3(4)$ **.**

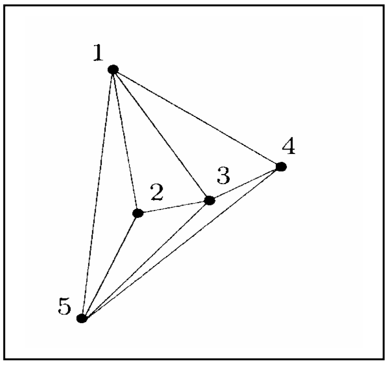

Figure (5.3) cyclic polytope $C_3(5)$ **.**