Solving Linear Delay Volterra Integro-Differential Equations by Using Galerkin's Method with Bernstien Polynomial

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Abstract

This paper present a method to find the approximate solution of linear delay Volterra integro differential equation which contains three kinds of equations(Retarded, Neutral and mixed) by using Galerkin's method with Bernsien polynomial as a basic function to .Three examples are given for determining the result for this method.

Keywords: Delay Volterra integro differential equation, method of weighted residual, Galerkin's method , Bernstein polynomials, Least square error .

الخلاصة

في هذا البحث نقدم طريقة لإيجاد الحل التقريبي لمعادلات فولتيرا التكاملية التفاضلية التباطؤية الخطية والتي تحتوي على ثلاثة أنواع(التراجعية،المتعادلة والمختلطة) وباستخدام طريقة كالركن مع الدوال التواصلية كدالة أساسية لإيجاد الحل التقريبي لهذه المعادلات . أعطينا ثلاثة امثلة لتحديد وإيجاد الحل التقريبي ولبيان كفاءة الطريقة أعلاه وتحديد النتائج النهائية.

Introduction

Delay Integro– differential equation is an equation involving one (or more) unknown function u(x) together with both differential and integral operations on u(x)[zhang2004] This means that it is an equation containing derivative of the unknown function u(x) which is appearing outside the integral sign. Delay integro-differential equations are equations having delay argument. They arise in many realistic models of problems in science, medicine, biology, economic and engineering[Ali2006].

Three types of delay integro- differential equations as following:

1. Retarded integro- differential equation.

$$u'(x) = f(x) + \int_a^{b(x)} k(x,t)u(t-\tau)dt \qquad \dots (1)$$

2. Neutral integro- differential equation.

$$u'(x-\tau) = f(x) + \int_a^{b(x)} k(x,t)u(t)dt \qquad \dots (2)$$

3. Mixed integro- differential equation.

$$u'(x-\tau_1) = f(x) + \int_a^{b(x)} k(x,t)u(t-\tau_2)dt \qquad \dots (3)$$

Where f and k are given function ,u(x) unknown function .The above three equations are Delay integro differential equations . ($\tau > 0$) usually represents here the time lag, and in special case if ($\tau = 0$) then the delay differential equation reduces to an ordinary differential equation.

If b(x)=b ,$a \le x \le b$ (b is constant) (1)(2)and(3)Fredholm integro differential equation respectively.

If b(x)=x ,$a \le x$, (1)(2)and(3) Volterra integro differential equation respectively.
In this paper, we considered linear delay Volterra integro differential equation .
The Bernstein polynomials of degree n are defined by

$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ for i=0,….,n and t=0,…,n-1

Where $t \in [a,b]$ and

$\binom{n}{i} = \frac{n!}{i!(n-i)!}$

, n is the degree of polynomials, i is the index of polynomials . The exponents on the (t) term increase by one as i increases, and the exponents on the (1-t) term decrease by one as (i) increases.

The Bernstein polynomial of degree two defined as

$B_0^2(x) = (1-x^2)$

$B_1^2(x) = 2x(1-x)$

$B_2^2(x) = (x^2)$

Weighted Residual method:

There existed an approximation technique for solving delay integro differential equations called the Method of Weighted Residuals, The Galerkin's Method of Weighted Residuals. The Galerkin's Method is very popular for finding numerical solutions to differential equations[Ali2006]. In this paper we will use Galerkin's method which it is one of the weighted residual method to find the approximated solution of the linear delay Volterra integro –differential equation .

To do this first

We will present these methods by considering the following functional equation.

$L[u(x)] = g(x) \quad x \in D \; \ldots \ldots (4)$

Where L denotes an operator which maps a set of functions U into a set G such that u ∈ U, g ∈ G, and D is a prescribed domain. The epitome of the weighted residual method is approximate the solution u(x) of equation(4) by the form

$$u(x) \cong u_n(x) = \sum_{i=0}^{n} c_i \varphi_i(x) \qquad \ldots \ldots \ldots (5)$$

Where the parameters $c_i's$ and the functions $\varphi_i(x)$ are algebraic or orthogonal functions. An approximate solution $u_n(x)$ given by equation(5) will not in general satisfy equation(4) exactly and associated with such an approximate solution is the residual defined by

$E(x) = L[u(x)] - g(x) \; \ldots \ldots \ldots \ldots (6)$

The residue E(x) depends on x as well as on the way that the parameters (c_i's) are chosen. It is certain that when E(x) = 0, then the exact solution is obtained which is difficult to be achieved; therefore we will try to minimize E(x) in some sense.

In the weighted residual method the unknown parameters (c_i's) are chosen to minimize the residual E(x) by setting its weighted residual integral equal to zero, i.e.

$$\int_D w_i E(x) dx = 0 \quad , i = 0, \ldots, n \qquad \ldots \ldots \ldots (7)$$

Where w_i is prescribed weighting function, the technique based on equation(7) is called weighted residual method. Choices of w_i will yield different methods with different approximate solutions.

One of weighted residual methods is Galerkin's method will be discussed below.

It is one of weighted residual methods that makes the residue E(x) of equation(7) orthogonal to n+1 given linear independent functions on the domain D. In this method the weighting function w_i is chosen to be

$w_{i=\frac{\partial u_n(x)}{\partial c_i}} \qquad i = 0, \ldots\ldots, n$

Where $u_n(x)$ is the approximate solution of the problem, then equation(7) becomes

$$\int_D \frac{\partial u_n(x)}{\partial c_i} E(x)dx = 0 \qquad \ldots\ldots\ldots(8)$$

This equation will provide us n+1 simultaneous equations for determination of c_i's , $i = 0,1,2,\ldots, n$

To solve linear delay Volterra integro differential equation by using the above method , now using operator forms for each type of these equations as:

$$L[u(x)] = g(x)$$

Where the operator L is defined for each type of delay integro-differential equations as:

1. Retarded Volterra integro- differential equation.

$$L[u(x)] = u'(x) - \int_a^x k(x,t)u(t-\tau)dt$$

2. Neutral Volterra integro- differential equation.

$$L[u(x)] = u'(x-\tau) - \int_a^x k(x,t)u(t)dt$$

3. Mixed Volterra integro- differential equation.

$$L[u(x)] = u'(x-\tau_1)\int_a^x k(x,t)u(t-\tau_2)dt$$

The unknown function u(x) is approximating by the form in equation(4) and by substituting equation(4) into equation(5) yields

$$L[u_n(x)] = g(x)$$

Where

$$L[u_n(x)] = \sum_{i=0}^{n} c_i \left[\varphi_i'(x) - \int_a^x k(x,t)\varphi_i(t-\tau)dt\right]$$

$$L[u_n(x)] = \sum_{i=0}^{n} c_i \left[\varphi_i'(x-\tau) - \int_a^x k(x,t)\varphi_i(t)dt\right]$$

$$L[u_n(x)] = \sum_{i=0}^{n} c_i \left[\varphi_i'(x-\tau_1) - \int_a^x k(x,t)\varphi_i(t-\tau_2)dt\right]$$

For which we have the residue equation

$$E_n(x) = L[u_n(x)] - g(x)$$

Substituting equation(5) into equation(6) to get:

$$E_n(x) = L\left[\sum_{i=0}^{n} c_i\varphi_i(x)\right] - g(x)$$

$$E_n(x) = \sum_{i}^{n} c_i\, [L(\varphi_i(x))] - g(x)$$

Where

$$L[\varphi_i(x)] = \varphi_i'(x) - \int_a^x k(x,t)\varphi_i(t-\tau)dt$$

$$L[\varphi_i(x)] = \varphi_i'(x-\tau) - \int_a^x k(x,t)\varphi_i(t)dt$$

$$L[\varphi_i(x)] = \varphi_i'(x-\tau_1) - \int_a^x k(x,t)\varphi_i(t-\tau_2)dt$$

For all i= 0,1,2, ...,n.
Since

$w_i = \frac{\partial u_n(x)}{\partial c_i}$ and $u_n(x) = \sum_{i=0}^{n} c_i \varphi_i(x)$

This yield to

$$w_i(x) = \varphi_i(x) \qquad \text{.....}(9)$$

Substituting equation(9) and (6) into equation(8) we get

$$\sum_{i=0}^{n} c_i \int_D \varphi_j L[\varphi_i(x)] = \int_D \varphi_j g(x) \qquad \text{....}(10)$$

Where φ_i is Bernstien polynomial .

By evaluating the above equation for each i=0,2,…,n ,we can get a linear system of n+1 equations with n+1 unknowns $c_0,c_1,\ldots,c_n$ which can be solved by any suitable method. By substituting these values into equation (5) we can get the approximated solution of the linear Delay Volterra integro –differential equation .

The approximated solution will be given in this steps

Step 1:

Select

$B_i^n(x) and\ B_i^n(x-\tau_j)\,, \qquad j = 1,2 \quad and\ n\ the\ degree\ of\ polynomial\,, \qquad i = 0,\ldots,n$

Step 2:

Find

$$f_i(x) = B_i^{n'}(x) - \int_0^x k(x,t) B_i^n(t-\tau)dt$$

$$f_i(x) = B_i^{n'}(x-\tau) - \int_0^x k(x,t) B_i^n(t)dt$$

$$f_i(x) = B_i^{n'}(x-\tau_1) - \int_0^x k(x,t) B_i^n(t-\tau_2)dt$$

Step 3:

Evaluate

$$\int_0^x f_i(x) B_i^n(x)dx \quad i = 0,\ldots,n \qquad \text{......}(11)$$

Step 4:

Evaluate

$$\int_0^x g(x) B_i^n(x)dx \quad i = 0,\ldots,n \qquad \text{......}(12)$$

Step 5:

Using (11) and (12) in

$$\sum_{i=0}^{n} c_i \int_0^x f_i(x) B_i^n(x)dx = \int_0^x g(x) B_i^n(x) \qquad \text{......}(13)$$

we get a system of n+1 equations

Step 6:

Solve the n+1 system to get c_i's

Step 7:
Substitute c_i's in (5) to get the approximate solution
To illustrate this method, consider the following example:
Examples:
1-Consider the linear Delay volterra integro differential equation of Retarded type

$$u'(x) = 1 - \frac{x^4}{3} + \int_0^x xtu(t-1)dt \qquad 0 \le x \le 1$$

. $u(x) = x + 1$ With exact solution

$c_i's$ After applying the above steps we will get two

$c_0 = 0.9861, \qquad c_1 = 2.1244$

in (5) we get the approximate solution $c_i's$ By using the two

$u(x) = 0.9861 + 1.1383x$

From the table (1)you can see the step size 0.1 between zero and one , Also in the same table you can see the difference between the exact solution and approximate solution ,They appeare close to each other.

2- Consider the linear Delay volterra integro differential equation of Neutral type

$$u'(x-1) = 1 - \frac{x^3}{6} + \int_0^x (x-t)u(t)dt \qquad 0 \le x \le 1$$

$u(x) = x$ With exact solution
After applying the above steps we will get the approximate solution
$u(x) = x$
With $c_0 = 0, \qquad c_1 = 1$
You can see the table (2) and figure (2).
3- Consider the linear Delay volterra integro differential equation of Mixed type

$$u'(x-1) = 2x - 2 - \frac{x^5}{4} + \frac{x^4}{3} - \frac{x^3}{8} + \int_0^x xtu\left(t - \frac{1}{2}\right)dt$$

$0 \le x \le 1$, $u(x) = x^2$ With exact solution
Table (3) lists the results obtained by achieving Galerkin's method with the aid of Bernstein polynomial.
The approximate solution is
$u(x) = x^2$

Conclusion:

The numerical solutions of delay Volterra integro differential equations are introduced using Galerkin's method. Some important formulas concerning the derivative of orthogonal polynomials: Bernstein polynomial have been derived which are essential to our numerical computations. Three examples were solved and good results are achieved.

From the table (1) and the figure (1) we can compare between the exact solution and the approximate solution ,they seems closed to each others, also L.S.E. is too small.

From the tables (2),(3) and the figures (2),(3) we can see the approximate solution equal to exact solution , L.S.E. equal to zero.

Finally from the last result we recommended this method to solve Volterra integro differential equation by using the above steps.

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Table (1)
The result of example (1)

x	exact solution	approximate solution
0	1	0.9861
0.1	1.1	1.09993
0.2	1.2	1.21376
0.3	1.3	1.32759
0.4	1.4	1.44142
0.5	1.5	1.55525
0.6	1.6	1.66908
0.7	1.7	1.78291
0.8	1.8	1.89674
0.9	1.9	2.01057
1	2	2.1244
L.S.E. 0.054618		

Figure (1)
Approximation solution of Volterra integro-differetial equation of example (1)

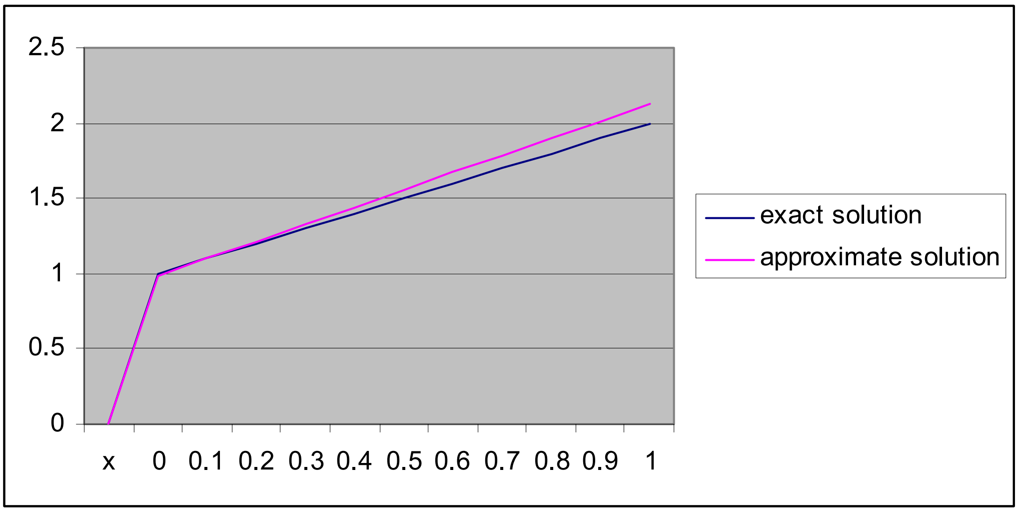

Table (2)
The result of example (2)

x	exact solution	approximate solution
0	0	0
0.1	0.1	0.1
0.2	0.2	0.2
0.3	0.3	0.3
0.4	0.4	0.4
0.5	0.5	0.5
0.6	0.6	0.6
0.7	0.7	0.7
0.8	0.8	0.8
0.9	0.9	0.9
1	1	1
L.S.E. 0.000		

Figure (2)
Approximation solution of Fredholm integro-differetial equation of example (2)

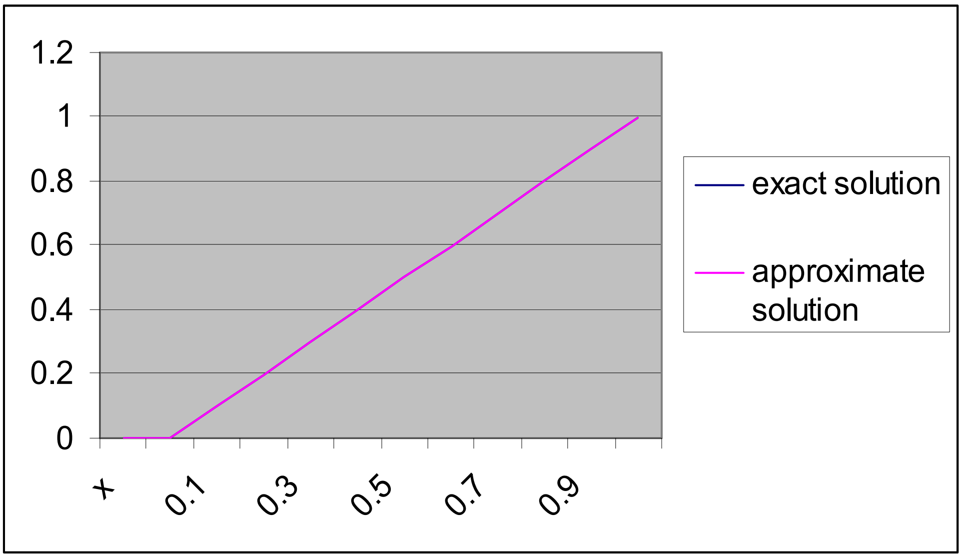

Table (3)
The result of example (3)

x	exact solution	approximate solution
0	0	0
0.1	0.01	0.01
0.2	0.04	0.04
0.3	0.09	0.09
0.4	0.16	0.16
0.5	0.25	0.25
0.6	0.36	0.36
0.7	0.49	0.49
0.8	0.64	0.64
0.9	0.81	0.81
1	1	1
L.S.E. 0.000		

Figure (3)
Approximation solution of Fredholm integro-differetial equation of example (3)