Equivalent between Weighted Earliness and Weighted Tardiness Problems On A single Machine

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الخلاصة

في هذا البحث تناولنا مسألة جدولة n من الاعمال على ماكنة واحده لمناقشة العلاقة بين مسألة التبكيروالتأخير الموزونين .وبما ان هاتان المسألتان من النوع Np الصعب ، برهنا نتيجه جيده إن قاعدة EDD والتي فيها $E_i \le P_i$ تعطي حل امثل للمسألة $1/C_i \le d_i / \Sigma W_i Ei$. وكذلك برهنا ان E^W_{max} و T^W_{max} متكافئتان لمسألتي $1/C_i \le d_i/ E^W_{max}$ و $1/C_j \ge d_j/T^W_{max}$.الخواص بين مسألتي التبكير الموزون والتأخير الموزون اعطي مع بعض الامثلة .

ABSTRACT

In this paper we consider the problem of scheduling n jobs on a single machine to discuss the relationship between weighted earliness and weighted tardiness problems (i.e., the problems $1/c_i \le d_i/\sum W_i E_i$ and $1/C_j \ge d_j/\sum W_j T_j$). These two problems are NP-hard ,for special case we proved a good result that EDD rule with $E_i \le P_i$ is optimal for $1/c_i \le d_i/\sum W_i E_i$ problem .

Also we proved that E^W_{max} and T^W_{max} are equivalent for $1/C_i \le d_i/ E^W_{max}$ problem and $1/C_j \ge d_j/ T^W_{max}$ problem. The properties between weighted earliness and weighted tardiness problems are given with some examples.

INTRODUCTION

Scheduling problems are one of the most studied problems in combinatorial optimization. It can be defined as a decision making process that is used on a regular basis in many manufacturing and services industries. It deals with the allocation of resources to task over given time periods and its goal is to minimize one or more objectives [1]. In the scheduling literature, the objective is generally to minimize functions such as makespan, tardiness, flow time, etc.

The two objectives $\sum E_i$ and $\sum T_i$ that are important in practice as well . The $1//\sum T_i$ problem has received an enormous amount of attention in the literature [1] ,[2],[3] . It is well know that the problems $1//\sum W_i E_i$ and $1//\sum W_i T_i$ and their generalizations $1//\sum W_i E_i$ and $1//\sum W_i T_i$ are NP-hard problems. Since the earliness objectives are non regular functions, hence there are a few studies to earliness problems.

The scheduling problem under consideration can be described as follows: there are n jobs to be scheduled on a single machine, which can handle one job at a time , each job i has positive integer processing time P_i and positive integer due date d_i .Job i (i=1,2,..,n) becomes available for processing at time zero . The main object of this paper to prove the equivalence of the following problems :

$1/C_i \le d_i/ \sum W_i E_i$ and $1/C_i \ge d_i/ \sum W_i Ti$, $1/C_i \le d_i/ E^W_{max}$ and $1/C_i \ge d_i/ T^W_{max}$ where

$E^{W}_{\max}$ = W_i Max{E_i} = W_i Max{d_i-c_i ,0 } and $T^{W}_{\max}$ = W_i Max {T_i} = W_i Max {C_i-d_i ,0 }.

In this paper in section one, we proved that $1/c_i \le d_i/E^{W}_{\max}$ is equivalent to $1/c_j \ge d_j/T^{W}_{\max}$ and we proved that EDD schedule with $E_i \le P_i$ is optimal for $1/c_i \le d_i/\sum W_i E_i$ .In section two we show some properties of weighted earliness and weighted tardiness problems with some examples for each case is given. In section three we show the conclusion and future work.

The following lemma shows that the total weighted earliness problem is equivalent to the total weighted tardiness problem.

Lemma (1)

The following measures are equivalent:

$$1\text{-} \sum_{i=1}^{n} W_i E_i \,, \quad 2\text{-} \sum_{j=1}^{n} W_j T_j \,.$$

Proof

Let $\mathbf{C}=\sum_{j=1}^{n} p_j$, consider an instance of the total weighted tardiness ($\sum_{j=1}^{n} W_j T_j$) problem where $p'_j = p_i$ and $d'_j = C - d_i + p_i$ for j = 1, 2,…,n. Suppose S is an optimal schedule for this instance. Define a new schedule S' as follows:

If a job j is the k-th job scheduled in S, then j' is the (n-k+1)th job scheduled in S'. Clearly, we have $C'_j = C - C_j + p_j$ and hence

$W_j T_j$ = W_j max { $C'_j - d'_j$,0 } = W_j max {(C–C_j+P_j) –(C – d_j + P_j), 0 }

= W_j max { d_j – C_j, 0 } = W_j E_j .

Therefore, the minimum total weighted earliness is the same as the minimum total weighted tardiness. Hence, as we know that the total weighted tardiness problem on one machine is NP-hard [5], then the total weighted earliness must also NP-hard.

It should be noted that in our problem $1/c_i \le d_i/\sum_{i=1}^{n} W_i E_i$ every job i is either early or on time, but in the total weighted tardiness problem $1/c_j \ge d_j/\sum_{j=1}^{n} W_j T_j$, every job j is either tardy or on time.

Lemma (2)

$E^{W}_{\max}$ is equivalent to $T^{W}_{\max}$ in case of complexity

Proof:

Let $\sum_{j=1}^{n} p_j$ **,**consider an instance of the maximum weighted tardiness Max{W_iT_i }problem where $p'_j = p_j$ and $d'_j = C - d_j + p_j$ for j = 1, 2,…,n. Suppose S is an optimal schedule for this instance. Define a new schedule S' as follows:
If a job j is the k-th job scheduled in S, then j' is the (n-k+1)th job scheduled in S'. Clearly, we have $C'_j = C - C_j + p_j$ and hence

$T^W_{\max}$ = W_j max{T_j} = W_j max{max{ $c'_j - d'_j$, 0 } } = W_j max {max(C– C_j+P_j) –(C – d_j + P_j), 0 } = W_j max { max{d_j – C_j , 0 }} = $E^W_{\max}$.
Hence, as we know that the $1//T^W_{\max}$ problem is solved by Lawler algorithm , then the $1//E^W_{\max}$ problem is also solved by sequencing the jobs in non –decreasing order of
$w_is_i = w_i(d_i\text{-}p_i)$.

The following result shows the equivalent of $1/c_i \le d_i/\sum_{i=1}^{n} W_iE_i$ problem and $1\ /c_i \ge d_i/\sum_{j=1}^{n} W_iT_i$ problem.

Theorem (1)

If the EDD schedule with $T_j \le P_j$ for each job j is optimal for $1/c_j \ge d_j/\sum_{j=1}^{n} W_jT_j$, then the EDD schedule with $d_i = C - d_j + P_j$ and with $E_i \le P_i$ is optimal for $1\ /c_i \le d_i/\ \sum_{i=1}^{n} W_iE_i$ problem.

Proof

Let S be the optimal schedule for $1\ /c_j \ge d_j/\ \sum_{j=1}^{n} W_jT_j$ problem obtained by EDD rule for the due date d_j and with $Tj \le P_j$ for each j.
Now construct a schedule S' by EDD rule for $d_i = C - d_j + P_j$, where $C = \sum_{j=1}^{n} P_j$ and the completion time for each job i is given by $C_i = C - C_j + P_j$. From lemma (1) the measures $\sum_{i=1}^{n} W_iE_i$ and $\sum_{j=1}^{n} W_jT_j$ are equivalent.

For the optimal schedule S, we have $T_j \le P_j$. Hence $C_j - d_j \le P_j$, using the definition of d_i and C_i in the schedule S' we have for each job i
$T_j = C_j – d_j =(C+ P_j – C_i) –(C+ P_j – d_i) = d_i – C_i = E_i \le P_j$ and $P_i = P_j$
Then $E_i \le P_i$.

Hence the EDD schedule with $E_i \leq P_i$ for each job i is optimal for 1 $/c_i \leq d_i/ \sum_{i=1}^{n} W_i E_i$.

Now we will give some examples to show some of the important properties for the weighted earliness and tardiness problem which is given in the above results.

Example (1) Consider the weighted earliness problem with four job.

We now show that the weighted earliness and weighted tardiness equivalence with the following four -job, for which the processing times and due dates are shown in the following table (1). The jobs are already numbered in EDD order.

Table-1:data for $1 /c_i \leq d_i/ \sum W_i E_i$ problem is arbitrary.

EDD	1	2	3	4
P_i	6	5	3	4
d_i	10	14	15	18
W_i	3	4	5	1
C_i	6	11	14	18
E_i	4	3	1	0
W_i Ei	12	12	5	0

is clear from table(1) that C =18 and the minimum $\sum_{i=1}^{4} W_i E_i$ = 29 and $E_i \leq P_i$ for each i , $E_{\max}^{W}$ = 12 .

Table-2: data for $1/c_j \geq d_j/ \sum W_j T_j$ problem.

EDD	4	3	2	1
P_j	4	3	5	6
D_j	4	6	9	14
W_j	1	5	4	3
C_j	4	7	12	18
T_j	0	1	3	4
W_j	1	5	4	3
$W_j T_j$	0	5	12	12

It is clear from table(2) that minimum $\sum_{j=1}^{6} W_j T_j$ = 29 and $T_j \leq P_j$ for each j , $T_{\max}^{W}$ = 12 .

Example (2) shows that, if the EDD rule is optimal for 1 $/c_i \leq d_i/ \sum W_i E_i$ problem, but there exists a job i with $E_i > P_i$, then there exists an optimal schedule for 1 /$c_j \geq d_j$/ $\sum W_j T_j$ problem with $\sum W_i E_i$ = $\sum W_j T_j$ and with same job j with $T_j > P_j$.

Table-3: data for 1 /$c_i \leq d_i$/ $\sum W_i E_i$ problem

EDD	1	2	3	4
P_i	4	3	6	2
d_i	8	12	13	16
W_i	2	3	1	4
C_i	4	7	13	15
E_i	4	5	0	1
W_i Ei	8	15	0	4

It is clear from table (3) that EDD rule is optimal with $E_2 = 5 > P_2 = 3$,

C= 15, $\sum_{i=1}^{4} W_i E_i = 27$.

Table -4:data for 1 /$c_j \geq d_j$/ $\Sigma W_j T_j$ problem.

J	4	3	2	1
P_j	2	6	3	4
d_j	1	8	6	11
W_j	4	1	3	2
C_j	2	8	11	15
T_j	1	0	5	4
$W_j T_j$	4	0	15	8

It is clear from table(4) that the schedule(4,3,2,1) is optimal, but it is not EDD schedule, and $\Sigma W_j T_j = \Sigma W_i E_i = 27$ and $T_2 = 5 > P_2 = 3$.

<u>Properties</u>

(1) If SWPT rule gives maximum value for 1 /$c_i \leq d_i$/ $\Sigma W_i E_i$ problem, then LWPT rule gives maximum value for 1 /$c_j \geq d_j$/ $\Sigma W_j T_j$ problem.

<u>Example (3)</u> Shows that SWPT rule is maximum for 1 /$c_i \leq d_i$/ $\Sigma w_i E_i$ problem and LWPT rule is maximum for 1 /$c_j \geq d_j$/ $\Sigma W_j T_j$ problem.

Table-5:data for the example (3) for 1 /$c_i \leq d_i$/ $\Sigma W_i E_i$ problem

SWPT	1	2	3	4	5
P_i	2	3	4	5	6
d_i	6	5	12	20	22
W_i	4	6	2	2	2
C_i	2	5	9	14	20
E_i	4	0	3	6	2
$W_i E_i$	16	0	6	12	4

It is clear from table (5) that C = 20, and $\sum_{i=1}^{5} W_i E_i$ = 38 (maximum)

Table-6:data for $1 / c_j \geq d_j / \Sigma W_j T_j$ problem

LWPT	5	4	3	2	1
P_j	6	5	4	3	2
d_j	4	5	12	18	16
W_j	2	2	2	6	4
C_j	6	11	15	18	20
T_j	2	6	3	0	4
$W_j T_j$	4	12	6	0	16

It is clear from table (6) that $\sum_{j=1}^{5} W_j T_j$ = 38 (maximum).

The other feasible schedules for $1 / c_j \geq d_j / \Sigma W_j T_j$ are: (5,4,3,1,2) with $\Sigma W_j T_j$ = 38, (4,5,3,2,1) with $\Sigma W_j T_j$ = 36,(4,5,3,1,2) with $\Sigma W_j T_j$ =36.

It is clear from theorem(1) and property(1) above, if LWPT rule gives minimum value for $1 / c_i \leq d_i / \Sigma W_i E_i$ problem, then SWPT rule gives minimum value for $1 / c_j \geq d_j / \Sigma W_j T_j$ problem, see property (2).

(2) If for each job i $\mathbf{d_i = d} = \sum_{j=1}^{n} P_j$, then LWPT rule is optimal for $1 / c_i \leq d_i / \Sigma W_i E_i$ problem, and the optimal schedule for $1 / c_j \geq d_j / \Sigma W_j T_j$ problem is obtained directly by setting $d_j = P_j$ for each job j.

Example(4) Shows that if $\mathbf{d_i = d} = \sum_{j=1}^{n} P_j$ for each job i for the $1 / c_i \leq d_i = d / \Sigma W_i E_i$ problem and LWPT schedule is optimal and if $\mathbf{d_j = P_j}$ for each job j, then SWPT schedule is optimal for the $1 / c_j \geq d_j / \Sigma W_j T_j$ problem.

Table-7: data for $1 / c_i \leq d_i / \Sigma W_i E_i$ problem

i	1	2	3	4	5
P_i	5	4	3	2	1
d_i	15	15	15	15	15
W_j	2	2	2	6	4
C_i	5	9	12	14	15
E_i	10	6	3	1	0
$W_i E_i$	20	12	6	6	0

It is clear that C = 15, and $\sum_{i=1}^{5} W_i E_i$ =44.

Table-8:data for $1 / c_j \geq d_j / \Sigma W_j T_j$ problem

j	5	4	3	2	1
P_j	1	2	3	4	5
d_j	1	2	3	4	5
W_j	4	6	2	2	2
C_j	1	3	6	10	15
T_j	0	1	3	6	10
$W_j T_j$	0	6	6	12	20

It is clear from table (8) that $d_j = P_j$, and $\sum_{j=1}^{5} W_j T_j$ =44.

(3) If for each job i $\mathbf{P_i = P}$, then LWPT is optimal for $1 / c_i \leq d_i / \Sigma W_i E_i$ problem and SWPT is optimal for $1 / c_j \geq d_j / \Sigma W_j T_j$ problem .

Example (5) Shows that if $\mathbf{P_i = P}$ for each job i for $1 / c_i \leq d_i / \Sigma W_i E_i$ problem and $\mathbf{P_j = P}$ for $1 / c_j \geq d_j / \Sigma W_j T_j$ problem .

Table-9:data for $1 / c_i \leq d_i / \Sigma W_i E_i$ problem

I	1	2	3	5	4
P_i	2	2	2	2	2
d_i	5	8	7	12	10
W_i	1	2	3	4	5
C_i	2	4	6	8	10
E_i	3	4	1	2	2
$W_i E_i$	3	8	3	16	0

It is clear that C= 10, and $\sum_{i=1}^{5} W_i E_i$ =30.

Table (10) data for $1 / c_j \geq d_j / \Sigma W_j T_j$ problem

j	4	5	3	2	1
P_j	2	2	2	2	2
d_j	2	0	5	4	7
W_j	5	4	3	2	1
C_j	2	4	6	8	10
T_j	0	4	1	4	3
$W_j T_j$	0	16	3	8	3

It is clear from table (10) that $\sum_{j=1}^{5} W_j T_j$ =30.

(4) If in the optimal solution for $1 / c_i \leq d_i / \Sigma Wi\ Ei$ problem, all the jobs completed on times (i.e., $C_i = d_i$ the ideal solution) then the optimal solution for $1 / c_j \geq d_j / \Sigma W_j T_j$ problem all the jobs completed on times.

Example (6) Shows that all jobs completed on their due dates(ideally solution) for $1 / c_i \leq d_i / \Sigma Wi\ E_i$ and $1 / c_j \geq d_j / \Sigma Wj\ T_j$ problems.

Table-11:data for $1 / c_i \leq d_i / \Sigma W_i E_i$ problem

i	1	2	3	4	5
P_i	4	3	1	2	5
d_i	4	7	8	10	15
W_i	1	6	2	8	7
C_i	4	7	8	10	15
E_i	0	0	0	0	0
$W_i E_i$	0	0	0	0	0

Hence it is clear that C =15, and $\sum_{i=1}^{5} W_i E_i = 0$.

Table-12: data for $1 / c_j \geq d_j / \Sigma W_j T_j$ problem

j	5	4	3	2	1
P_j	5	2	1	3	4
d_j	5	7	8	11	15
W_j	7	8	2	6	1
C_j	5	7	8	11	15
T_j	0	0	0	0	0
$W_j T_j$	0	0	0	0	0

Hence it is clear that $\sum_{j=1}^{5} W_j T_j = 0$.

Conclusions and Future Work

The study shows the relationship between earliness and tardiness problems. These two problems are NP-hard , then we proved a very good result that the EDD rule with $E_i \leq P_i$ is optimal for $1 / c_i \leq d_i / \Sigma W_i E_i$ problem.

An interesting future research topic would involve experimentation discuss the relationship between

$F2/C_i \leq d_i/\sum W_i E_i$ and $F2/C_i \geq d_i/\sum W_i T_i$

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