On Non-Singular Plane Cubic Curves Over $F_q,\ q = 2,3,5,7$

حول المنحنيات التكعيبية غير المنفردة في المستوي على F_q , q = 7,5,3,2

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Abstract

In this paper, the number of non-singular plane cubic curves over the finite fields $F_q,\ q = 2,3,5,7$ is determined in both complete and incomplete cases, and the maximum size of a complete arc of degree three that can be constructed from each incomplete arc are found.

المستخلص

في هذا البحث تم تحديد عدد المنحنيات التكعيبية غير المنفردة في المستوي على الحقول المنتهية F_q, q = 7,5,3,2 في الحالتين الكاملة وغير الكاملة. وكذلك تم ايجاد اقصى سعة لقوس كامل من الدرجة الثالثة والتي من الممكن بنائها من كل قوس غير كامل.

Key words and phrases: Plane cubic curves, inflexion, arc, finite geometry.

1- Introduction

Questions about non-singular plane cubic curves over a finite field F_q:

1- How many inequivalent non-singular plane cubic curves are there?
2- How many complete and incomplete non-singular plane cubic curves are there?
3- What is the maximum size of a complete arc of degree three that can be constructed from each incomplete arc?

Question one has been investigated in [1] for $F_q,\ 2 \le q \le 13$ and also answered for $q = 17$ and 19 [2] [3]. The largest size of an $(n;\ r)$-arc of $PG(2,q)$is indicated by $m_r(2,q)$. In [4] and [5], bounds for $m_r(2,q)$ are given. In particular, $m_r(2,q) \le 2q+1$ for $q \ge 4$; see [6].
Question three is a part of another question which is*: what is the value of* $m_3(2,q)$ *in the projective plane* $PG(2,q)$? General results related to this question have been given in [4]. Another question arises here which is*: does there a complete non-singular plane cubic curve or a complete arc constructed from the incomplete ones of size equal to* $m_3(2,q)$*?*

The value of $m_3(2,q)$ has been given in [1] for $2 \le q \le 13$. In [7], a full classification of $(n;3)$-arc have been given for $q = 7$. For $q = 2,3,5,7$ the value of $m_3(2,q)$ is

$m_3(2,2) = 7,\quad m_3(2,3) = 9,\quad m_3(2,5) = 11 \quad m_3(2,7) = 15.$

The aim of this paper is to answer question two and three over $F_q,\ q = 2,3,5,7.$

This paper consists of seven sections. Sections 1, 2, gave the necessary background to the research. In Sections 3,4,5,6, a full classification of inequivalent non-singular plane cubic curve into complete and incomplete arc of degree three has been given. Finally, section 7 is devoted to give a summary of this work and the suggestion idea from this work.

Definition 1.1: A projective plane curve $\mathcal{F}$ is the set

$$\mathcal{F} = \{(X_0,X_1,X_2) \in PG(2,q) |\ F(X_0,X_1,X_2) = 0\},$$

where F is a form. The curve is irreducible if F does not factor over $\bar{F}_q$ (the algebraic closure of F_q). An irreducible plane cubic curve is called non-singular if there is a unique tangent line at each point of the curve considered over $\bar{F}_q$. This means that, over $\bar{F}_q$, it is impossible to find a point P on $\mathcal{F}$ such that the three partial derivatives of F with respect to X_0, X_1, X_2 are all zero at P.

Definition 1.2: A rational inflexion point P of a non-singular cubic curve $\mathcal{F}$ is one for which the unique tangent at P has three-point contact; that is, the unique tangent at an inflexion P of $\mathcal{F}$ has no other point in common with the curve $\mathcal{F}$.

Definition 1.3: An *arc* K of degree n denoted by $(k;r)$ in $PG(2,q), q = p^h$ and p prime, is a set of k points no $r+1$ of them are collinear but some r collinear. A $(k;r)$-arc is called complete if it is not contained within $(k+1;r)$-arc.

Most of the non-singular plane cubic curve with k rational points can be regarded as a $(k;3)$-arc.

Definition 1.4: The *class* $\kappa = \kappa(\mathcal{F})$ of a plane cubic curve $\mathcal{F}$ defined over F_q is the number of distinct tangents to $\mathcal{F}$ through an arbitrary point of $PG(2, \bar{F}_q)$.

Lemma [1] 1.5: When q is odd, the following properties holds.

(i) The maximum class κ of $\mathcal{F}$ is six.

(ii) If $\mathcal{F}$ is a non-singular plane cubic curve and $\kappa = 6$, then there are four tangents to $\mathcal{F}$ from a point P of $\mathcal{F}$, other than the tangent at P, and the cross-ratio of the four tangents is constant.

Definition 1.6: An unordered set of four distinct points (tetrad) with cross-ratio λ is called harmonic, denoted by $\mathcal{H}$, if

$\lambda = 1/\lambda$, or $\lambda = \lambda/(\lambda - 1)$or $\lambda = 1 - \lambda$, and called equianharmonic, denoted by $\mathcal{E}$, if $\lambda = (1 - \lambda)$ or, equivalently $\lambda = (\lambda - 1)/\lambda$.

Definition 1.7: The non-singular plane cubic curve $\mathcal{F}$ called harmonic or equianharmonic if the four tangents through a point form a harmonic or equianharmonic set. A non-singular cubic curve which is not harmonic or equianharmonic is called general**.**

Lemma [1] 1.8: Over $\bar{F}_q$, $q \not\equiv 0\ (mod\ 3)$, a non-singular plane cubic curve $\mathcal{F}$ has nine rational inflexions.

Definition 1.9: A rational inflexional triangle is a set of three lines over F_q through the nine inflexions of $\mathcal{F}$.

A non-singular plane cubic curve $\mathcal{F}$over F_q , $q \not\equiv 0\ (mod\ 3)$, is denoted by $\mathcal{F}_n^r$, where n is the number of rational inflexions and r is the number of rational inflexional triangles. Also, $\mathcal{F}_n^r = \mathcal{G}_n^r, \mathcal{H}_n^r, \mathcal{E}_n^r$ when $\mathcal{F}$ is respectively general, equianharmonic, harmonic.

2- Classification of non-singular plane cubic curves

In this section the necessary theorems have been stated to determine the number of rational points on the non-singular plane cubic curves $\mathcal{F}_n^r$ over F_q, $q = 2,3,5,7$ and to give the canonical form of each curve.

Let N_1 be the number of rational points on a plane curve $\mathcal{F}$, $N_1(q)$ denote the maximum number of rational points on any non-singular plane cubic curves over F_q and $L_1(q)$ the minimum number. The Serre bound states that

$q + 1 - \lfloor 2\sqrt{q} \rfloor \le N_1 \le q + 1 + \lfloor 2\sqrt{q} \rfloor$,
where $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$ [1]. So; the following hold:

$1 \le N_1 \le 5$ if $q = 2$;
$1 \le N_1 \le 7$ if $q = 3$;
$2 \le N_1 \le 10$ if $q = 5$;

$3 \leq N_1 \leq 13 \quad$ if $\quad q = 7$.

Theorem [1] 2.1: The number N_1 takes every value between $q + 1 - \lfloor 2\sqrt{q} \rfloor$ and $q + 1 + \lfloor 2\sqrt{q} \rfloor$ if and only if (a)
$q = p$ or (b) $q = p^2$ with $p = 2$ or $p = 3$ or $p \equiv 11 \ (mod\ 12)$.

Let n_i for $i = 0,1,3,9$ be the number of projective equivalence classes with exactly i rational inflexions and let P_q be the total number of projective equivalence classes. Hence

$$P_q = n_0 + n_1 + n_3 + n_9.$$

Theorem [1] 2.2: $P_q = 3q + 2 + \left(\frac{-4}{q}\right) + \left(\frac{-3}{q}\right)^2 + 3\left(\frac{-3}{q}\right)$.

Here, the following *Legendre and Legendre-Jacobi* symbols are used:

$$\left(\frac{-4}{c}\right) = \begin{cases} 1 & if\ c \equiv 1 \quad (mod\ 4), \\ 0 & if\ c \equiv 0 \quad (mod\ 2), \\ -1 & if\ c \equiv -1 \quad (mod\ 4); \end{cases}$$
$$\left(\frac{-3}{c}\right) = \begin{cases} 1 & if\ c \equiv 1 \quad (mod\ 3), \\ 0 & if\ c \equiv 0 \quad (mod\ 3), \\ -1 & if\ c \equiv -1 \quad (mod\ 3). \end{cases}$$

Corollary [1] 2.3: $P_2 = 6, \ P_3 = 10, \ P_5 = 16, \ P_7 = 26$.

Theorem [1] 2.4: The number of rational inflexions of a non-singular plane cubic curve over F_q is $0,1,3$ or 9. The possibilities are as follows:

$$\begin{array}{lll} q \equiv 0 & (mod\ 3): & 0, 1, 3; \\ q \equiv 2 & (mod\ 3): & 0, 1, 3; \\ q \equiv 1 & (mod\ 3): & 0, 1, 3, 9. \end{array}$$

Corollary 2.5:

(i) Over F_q , $q = 2,3,5$ the number of rational inflexions of a non-singular plane cubic is $0,1$ or 3.

(ii) Over F_7, the number of rational inflexions of a non-singular plane cubic is $0,1,3$ or 9.

2.1 Non- singular plane cubics with nine rational inflexions

Theorem [1] **2.6:** A non-singular plane cubic curve with form F and nine rational inflexions exists over F_q if and only if $q \equiv 1 \ (mod\ 3)$, and then F has canonical form
$F = X_0^3 X_1^3 X_2^3 - 3cX_0X_1X_2$.

Corollary 2.7: There is a non-singular plane cubic curve with nine rational inflexions over F_7 but not over F_2, F_3, F_5.

2.2 Non- singular plane cubics with three rational inflexions

Theorem [1] **2.8:** A non-singular plane cubic curve with form F and three rational inflexions exists over F_q for all q. The inflexions are necessary collinear.

(i) If the inflexional tangent are concurrent, the canonical forms are as follows:

(a) $q \equiv 0, 2 \ (mod\ 3)$,

$$F = X_0X_1(X_0+X_1) + X_3^3;$$

(b) $q \equiv 1 \ (mod\ 3)$,

$$F = X_0X_1(X_0+X_1) + X_3^3;$$

$$F = X_0X_1(X_0+X_1) + cX_3^3;$$

$$F = X_0X_1(X_0+X_1) + cX_3^3;$$

where c is a primitive of F_q . Here, F will denote by $\bar{\mathcal{E}}$.

(ii)If the inflexional tangent are not concurrent, the canonical form is as follows:

$$F = X_0X_1X_2 + e(X_0+X_1+X_2)^3, \qquad e \neq 0, 1/27 .$$

2.3 Non- singular plane cubics with one rational inflexion

Theorem [1**] 2.9:** A non-singular plane cubic curve with form F defined over F_q , $q = p^h$ and at least one rational inflexion has one of following canonical forms.

(i) $p \neq 2, 3,$

$$F = X_2^2X_1 + X_0^3 + cX_0X_1^2 + dX_1^3,$$

where $4c^3 + 27d^2 \neq 0.$

(ii)$p = 3,$

(a) $F = X_2^2X_1 + X_0^3 + bX_1X_0^2 + dX_1^3,$

where $bd \neq 0.$

(b)
$F = X_2^2X_1 + X_0^3 + cX_0X_1^2 + dX_1^3,$

where $c \neq 0.$

(iii) $p = 2,$

(a) $F = X_1X_2^2 + X_0X_1X_2 + X_0^3 + bX_0^2X_1 + cX_0X_1^2,$

where $b = 0$ or a fixed element of trace 1, and $c \neq 0;$

(b) $F = X_2^2X_1 + X_2X_1^2 + eX_0^3 + cX_0X_1^2 + dX_1^3,$

where $e = 1$ when $(q-1, 3) = 1$ and $e = 1, \alpha, \alpha^2$ when $(q-1, 3) = 3$, with α a primitive element of F_q; also, $d = 0$ or a particular element of trace 1.

Remark 2.10: From Theorem 2.9(ii) (b) the curve $F = X_2^2 X_1 + X_0^3 + cX_0X_1^2 + dX_1^3$, where $c \neq 0$ is always harmonic. All the inflexions lie on $\boldsymbol{u_1}$ (with equation $\boldsymbol{X_1} = \boldsymbol{0}$), and so $U_2 = [0;0;1]$ is the only inflexion. The line $\boldsymbol{u_2}$ (with equation $\boldsymbol{X_2} = \boldsymbol{0}$) meets F in zero, one or three rational points as $X^3 + cX + d$ has zero, one or three roots in F_q. The curve F is accordingly described as S_1^0, S_1^1, S_1^2.

2.4 Non- singular plane cubics with no rational inflexions

Theorem [1] **2.11:** A non-singular plane cubic curve with form F defined over F_q, $q = p^h$, with no rational inflexion has one of following canonical forms.

(i) $q \equiv -1 \ (mod\ 3)$,

$$F = X_2^3 - 3c(X_0^2 - dX_0X_1 + X_1^2)X_2 - (X_0^3 - 3X_0X_1^2 + dX_1^3),$$

where $X^3 - 3X + d$ is irreducible.

(ii) $q \equiv 1 \ (mod\ 3)$,

(a) $F = X_0^3 + \alpha X_1^3 + \alpha^2 X_2^3 - 3cX_0X_1X_2$,

with α a primitive element of F_q.

(b) $F = X_0X_1^2 + X_0^2X_2 + eX_1X_2^3 - c(X_0^3 + eX_1^3 + e^2X_2^3 - 3eX_0X_1X_2)$,

with α a primitive element of F_q and $e = \alpha, \alpha^2$. Here, when $c \neq 0$, and $\mathcal{F}$ is equianharmonic, write $\mathcal{F} = \mathcal{E}_0^4$; when $c = 0$, and $\mathcal{F}$ is equianharmonic, write $\mathcal{F} = \bar{\mathcal{E}}_0^4$.

(iii) $q \equiv 0 \ (mod\ 3)$,

$$F = X_0^3 + X_1^3 + cX_2^3 + dX_0^2X_2 + dX_0X_1^2 + d^2X_0X_2^2 + dX_1X_2^2,$$

where $c \neq 1$ and $X^3 + dX - 1$ is a fixed irreducible polynomial.

3- Complete and incomplete plane cubic curves over F_2

In $PG(2,2)$; which consist of seven points, there are only six non-singular plane cubic curves and $1 \leq N_1 \leq 5$; that is, $P_2 = 6$ and $L_1(2) = 1$, $N_1(2) = 5$. Since $L_1(2) = 1$, so there are some cubic curves over F_2 which are not arc of degree three. In Table 3.1, the columns give the symbol of each type of $\mathcal{F}_n^r$, the canonical form, the number of rational points $|\mathcal{F}_n^r|$, the description, the maximum value $\boldsymbol{M}(\mathcal{F}_n^r)$ of k for a $(k;3)$-arc containing the curve.

Table 3.1: Non-singular plane cubic curves over F_2

$\mathcal{F}_n^r$	No.	Canonical form	$\lvert\mathcal{F}_n^r\rvert$	Description	$M(\mathcal{F}_n^r)$
$\bar{\mathcal{E}}_3$	1	$X_0X_1(X_0+X_1)+X_2^3$	3	Incomplete	7
$\mathcal{G}_1$	2	$X_1X_2^2+X_0X_1X_2+X_0^3+X_0^2X_1$	4	Incomplete	7
	3	$X_1X_2^2+X_0X_1X_2+X_0^3+X_0^2X_1+X_0X_1^2$	2	–	–
$\mathcal{E}_1$	4	$X_1X_2^2+X_1^2X_2+X_0^3+X_0X_1^2$	5	Incomplete	7
	5	$X_1X_2^2+X_1^2X_2+X_0^3+X_0X_1^2+X_1^3$	1	–	–
$\mathcal{E}_0$	6	$X_2^3-(X_0^3-X_0X_1^2+X_1^3)$	3	–	–

Since any line in $PG(2,2)$ is consist of three points, so all complete arc of degree three that constructed from the incomplete one are $PG(2,2)$.

4- Complete and incomplete plane cubic curves over F_3

In $PG(2,3)$; which consist of thirteen points, there are only ten non-singular plane cubic curves and $1 \le N_1 \le 7$; thats, $P_2 = 10$ and $L_1(3) = 1, N_1(3) = 7$. Since $L_1(3) = 1$, so there are some cubic curves over F_3 which are not arc of degree three as given in Table 4.1.

Table 4.1: Non-singular plane cubic curves over F_3

$\mathcal{F}_n^r$	No.	Canonical form	$\lvert\mathcal{F}_n^r\rvert$	Description	$M(\mathcal{F}_n^r)$
$\mathcal{G}_3$	1	$X_0X_1X_2+(X_0+X_1+X_2)^3$	6	Incomplete	7
	2	$X_0X_1X_2-(X_0+X_1+X_2)^3$	3	Incomplete	9
$\mathcal{G}_1$	3	$X_1X_2^2+X_0^3+X_0^2X_1+X_1^3$	2	–	–
	4	$X_1X_2^2+X_0^3+X_0^2X_1-X_1^3$	5	Incomplete	9
$\mathcal{S}_1^0$	5	$X_1X_2^2+X_0^3-X_1^2X_0+X_1^3$	1	–	–
	6	$X_1X_2^2+X_0^3-X_1^2X_0-X_1^3$	7	Incomplete	9
$\mathcal{S}_1^1$	7	$X_1X_2^2+X_0^3+X_1^2X_0+X_1^3$	4	Incomplete	9
$\mathcal{S}_1^3$	8	$X_1X_2^2+X_0^3-X_1^2X_0$	4	Incomplete	9
$\mathcal{G}_0$	9	$X_0^3+X_1^3-X_2^3-X_0^2X_2-X_0X_1^2+X_0X_2^2-X_1X_2^2$	3	–	–
	10	$X_0^3+X_1^3-X_0^2X_2-X_0X_1^2+X_0X_2^2-X_1X_2^2$	6	Incomplete	9

5- Complete and incomplete plane cubic curves over F_5

In $PG(2,5)$; which consist of thirty one points, there are sixteen non-singular plane cubic curves and $2 \le N_1 \le 10$; that is, $P_2 = 16$ and $L_1(5) = 2, N_1(5) = 10$. Since $L_1(5) = 2$, so there are some cubic curves over F_5 which are not arc of degree three as given in Table 5.1.

Table 5.1: Non-singular plane cubic curves over F_5

$\mathcal{F}_n^r$	No.	Canonical form	$\lvert\mathcal{F}_n^r\rvert$	Description	$M(\mathcal{F}_n^r)$
$\mathcal{G}_3^2$	1	$X_0X_1X_2 - (X_0 + X_1 + X_2)^3$	9	Complete	–
	2	$X_0X_1X_2 - 2(X_0 + X_1 + X_2)^3$	3	Incomplete	11
$\bar{\mathcal{E}}_3^2$	3	$X_0X_1(X_0 + X_1) + X_2^3$	6	Incomplete	11
$\mathcal{E}_3^2$	4	$X_0X_1X_2 + (X_0 + X_1 + X_2)^3$	6	Incomplete	11
$\mathcal{G}_1^0$	5	$X_1X_2^2 + X_0^3 - 2X_1^2X_0 - 2X_1^3$	5	Incomplete	11
	6	$X_1X_2^2 + X_0^3 - X_1^2X_0 - X_1^3$	8	Incomplete	11
	7	$X_1X_2^2 + X_0^3 + X_1^2X_0 - 2X_1^3$	4	Incomplete	11
	8	$X_1X_2^2 + X_0^3 + 2X_1^2X_0 - X_1^3$	7	Incomplete	11
$\mathcal{H}_1^0$	9	$X_1X_2^2 + X_0^3 + X_0X_1^2$	4	Incomplete	11
	10	$X_1X_2^2 + X_0^3 + 2X_0X_1^2$	2	–	–
	11	$X_1X_2^2 + X_0^3 - 2X_0X_1^2$	10	Incomplete	11
	12	$X_1X_2^2 + X_0^3 - X_0X_1^2$	8	Incomplete	10
$\mathcal{G}_0^2$	13	$X_2^3 - 3(X_0^2 - X_0X_1 + X_1^2)X_2 - (X_0^3 - 3X_0X_1^2 + X_1^3)$	3	–	–
	14	$X_2^3 - (X_0^2 - X_0X_1 + X_1^2)X_2 - (X_0^3 - 3X_0X_1^2 + X_1^3)$	9	Incomplete	10
$\mathcal{E}_0^2$	15	$X_2^3 - (X_0^3 - 3X_0X_1^2 + X_1^3)$	6	Incomplete	11
	16	$X_2^3 + 3(X_0^2 - X_0X_1 + X_1^2)X_2 - (X_0^3 - 3X_0X_1^2 + X_1^3)$	6	Incomplete	11

6- Complete and incomplete plane cubic curves over F_7

In $PG(2,7)$; which consist of fifty seven points, there are twenty six non-singular plane cubic curves and $3 \le N_1 \le 13$; that is, $P_7 = 26$ and $L_1(7) = 3, N_1(7) = 13$. There only one cubic curves over F_7 which is not arc of degree three which is of type $\mathcal{E}_0^1$ as given in Table 6.1.

Table 6.1: Non-singular plane cubic curves over F_7

$\mathcal{F}_n^r$	No.	Canonical form	$\lvert\mathcal{F}_n^r\rvert$	Description	$M(\mathcal{F}_n^r)$
$\mathcal{E}_9^4$	1	$X_0^3+X_1^3+X_2^3$	9	Complete	–
$\mathcal{G}_3^1$	2	$X_0X_1X_2-2(X_0+X_1+X_2)^3$	6	Incomplete	15
	3	$X_0X_1X_2+3(X_0+X_1+X_2)^3$	6	Incomplete	15
	4	$X_0X_1X_2-3(X_0+X_1+X_2)^3$	12	Complete	–
	5	$X_0X_1X_2-(X_0+X_1+X_2)^3$	9	Incomplete	12
$\bar{\mathcal{E}}_3^1$	6	$X_0X_1(X_0+X_1)+3X_2^3$	3	Incomplete	15
	7	$X_0X_1(X_0+X_1)+2X_2^3$	12	Complete	–
$\mathcal{G}_1^0$	8	$X_1X_2^2+X_0^3-3X_1^2X_0-X_1^3$	5	Incomplete	15
	9	$X_1X_2^2+X_0^3-3X_1^2X_0+X_1^3$	11	Incomplete	13
$\mathcal{H}_1^0$	10	$X_1X_2^2+X_0^3+X_0X_1^2$	8	Incomplete	14
	11	$X_1X_2^2+X_0^3+3X_0X_1^2$	8	Incomplete	13
$\mathcal{G}_1^1$	12	$X_1X_2^2+X_0^3-3X_1^2X_0+3X_1^3$	10	Incomplete	12
	13	$X_1X_2^2+X_0^3-2X_1^2X_0+2X_1^3$	7	Incomplete	15
	14	$X_1X_2^2+X_0^3-2X_1^2X_0+X_1^3$	4	Incomplete	15
	15	$X_1X_2^2+X_0^3-2X_1^2X_0+X_1^3$	10	Incomplete	12
$\mathcal{E}_1^1$	16	$X_1X_2^2+X_0^3-3X_1^3$	13	Incomplete	14
	17	$X_1X_2^2+X_0^3+X_1^3$	4	Incomplete	15
$\mathcal{E}_1^4$	18	$X_1X_2^2+X_0^3+2X_1^3$	7	Incomplete	14
$\mathcal{G}_0^1$	19	$X_0X_1^2+X_0^2X_2+3X_1X_2^2-(X_0^3+3X_1^3+2X_2^3-2X_0X_1X_2)$	6	Incomplete	15
	20	$X_0X_1^2+X_0^2X_2+2X_1X_2^2-(X_0^3+2X_1^3-3X_2^3+X_0X_1X_2)$	6	Incomplete	15
	21	$X_0X_1^2+X_0^2X_2+2X_1X_2^2-3(X_0^3+2X_1^3-3X_2^3+X_0X_1X_2)$	9	Incomplete	13
	22	$X_0X_1^2+X_0^2X_2+3X_1X_2^2-3(X_0^3+3X_1^3+2X_2^3-2X_0X_1X_2)$	12	Complete	–
$\mathcal{E}_0^1$	23	$X_0X_1^2+X_0^2X_2+3X_1X_2^2$	3	–	–
	24	$X_0X_1^2+X_0^2X_2+2X_1X_2^2$	12	Complete	–
$\mathcal{E}_0^4$	25	$X_0^3+3X_1^3+2X_2^3-3X_0X_1X_2$	9	Incomplete	13
$\bar{\mathcal{E}}_0^4$	26	$X_0^3+3X_1^3+2X_2^3$	9	Incomplete	13

7- Summary and suggestion

Theorem 7.1: In $PG(2,q)$, $q = 2,3,5,7,$ the total number of inequivalent non-singular plane cubic curve P_q has been divided into complete, incomplete and not $(k;3)$-arc as given in Table 7.1.

Table 7.1: Partition of P_q, $q = 2,3,5,7,$

F_q	P_q	No. incomplete	No. complete	Not $(k;3)$-arc
F_2	6	3	0	3
F_3	10	7	0	3
F_5	16	13	1	2
F_7	26	20	5	1