Some properties of the Oscillatory and Nonoscillatory Solutions Of Second Order Nonlinear Neutral Differential Equation

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Abstract

In this paper sufficient conditions for oscillation of all solutions of nonlinear second order neutral differential equation and sufficient conditions for nonoscillatory solutions to converge to zero are obtained.

Introduction

Consider the second order nonlinear neutral differential equation

(1.1) $[x(t)+p(t)x(\tau(t))]''+q(t)f(x(\sigma(t)))=0, \quad t\geq t_0$

under the standing hypotheses:

(1) $p\in C[(t_0,\infty), R]$;

(2) $\tau,\sigma\in C[(t_0,\infty),R]$, τ,σ are strictly increasing and $\lim_{t\to\infty}\tau(t)=\infty$, $\lim_{t\to\infty}\sigma(t)=\infty$

(3) $q\in C[(t_0,\infty),R]$, $q(t)$ not equivalent to zero.

Our aim is to obtain new sufficient conditions for the oscillation of all solutions of equation (1.1) and sufficient conditions for nonoscillatory solutions to converge to zero. By a solution of equation (1.1) we means a continuous function $x:[t_x,\infty)\to R$ such that $x(t)+p(\tau)x(t))$ is two times continuously differentiable, and $x(t)$ satisfies equation (1.1) for all sufficiently large $t\geq t_x$. A solution of (1.1) is said to be oscillatory if it has an infinite sequence of zero tending to infinity, otherwise a solution is said to be nonoscillatory. The problem of oscillation for neutral differential equations has received considerable attention in recent years, see e.g. [1-6] and the references cited therein, however many of these papers discuss the cases when the coefficients and the arguments are constants and a few of them investigate the cases of variable coefficients and arguments. In this paper we improve some results of [3], [5], and give some other new theorems.

Main Results

In this sections we studied the oscillation of all solutions of equation (1.1), and obtained some new sufficient conditions to the bounded and all solutions of (1.1). Let

$u(t)=x(t)+p(t)x(\tau(t))$

So, equation (1.1) reduce to

(2.1) $u''(t)=-q(t)f(x(\sigma(t)))$

The next theorem concerns bounded oscillatory solutions of equation (1.1).

Theorem (2.1) : Suppose that $p(t)\geq 0$ is bounded. $q(t)\leq 0$ for $t\geq t_0$, f is an increasing function, such that $u f(u)>0$ for $u\neq 0$ and

(2.2) $\int^{\infty} q(s)\,ds=-\infty$

Then all bounded solutions of (1.1) are oscillatory.

Proof. Without loss of generality we may assume that $x(t)$ is an eventually

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positive and bounded solution of equation (1.1), and $x(\sigma(t)) > 0$ for $t \ge t_0$, then $u''(t) \ge 0$ for all large t. For sufficiently large t_0 we have two cases :

1. $u'(t) > 0, \quad t \ge t_1 \ge t_0$
2. $u'(t) < 0, \quad t \ge t_1 \ge t_0$

Case (1) : Since $x(t)$ and $p(t)$ are bounded, then also $u(t)$ is bounded, this is impossible according to $u''(t) \ge 0$ and $u'(t) > 0$ for all large t.

Case (2) : We can consider one possibility $u(t) > 0$, for $t \ge t_1 \ge t_0$ where t_1 is sufficiently large, integrating the equation $u''(t) = -q(t) f(x(\sigma(t)))$ from t_1 to t we ge

$$u'(t) - u'(t_1) = \int_{t_1}^{t} q(s) f(x(\sigma(s))) ds \ge$$

and so

$$f(x(\sigma(t_1))) \int_{t_1}^{t} -q(s)\, ds$$

$$-u'(t_1) \ge f(x(\sigma(t_1))) \int_{t_1}^{t} -q(s)\, ds$$

as $t \to \infty$ We get a contradiction. The proof is complete.

Example (2.2): Consider the nonlinear neutral differential equation

$$(2.3) \qquad \frac{d^2}{dt^2}[x(t)+4x(t+\pi)] - \frac{3}{2} f(\sin(t+2\pi)) = 0$$

Which satisfied all conditions of theorem 2.1 so all solutions of equation (2.3) are oscillatory for instance $x(t) = \sin t$, is such solution.

Remark (2.3): We can replace the conditions of theorem 2.1 by the conditions: $p(t) \ge 0$ is bounded, $q(t) \ge 0$, f is decreasing function such that $u f(u) < 0$ for $u \ne 0$ and

$$\int^{\infty} q(s)\, ds = \infty$$

Then every bounded solution of equation (1.1) is oscillatory.

Theorem (2.4) : Suppose that $p(t) \ge 0$, $q(t) \ge 0$, and f is an increasing function such that $u f(u) > 0$, for $u \ne 0$,

and (2.4) $$\int^{\infty} q(s)\, ds = \infty$$

Then every solution of (1.1) is oscillatory.

Proof : Assume that $x(t)$ is an eventually positive solution of (1.1), and $x(\sigma(t)) > 0$ for $t \ge t_0$. Then $u''(t) \le 0$ for all large t. We consider two cases : 1. $u'(t) < 0$, $t \ge t_0$; 2. $u'(t) > 0$, $t \ge t_0$.

Cases (1) : One can find that $u(t) < 0$ for $t \ge t_1 \ge t_0$, and $\lim_{t\to\infty} u(t) = -\infty$, which is Impossible ,since $u(t) > 0$.

Cases (2) :- We have only the possibility $u(t) > 0$ for $t \ge t_1 \ge t_0$, and we have, $u''(t) = -q(t) f(x(\sigma(t)))$, by integrating this equation from t_1 to t we get

$$u'(t) - u'(t_1) = \int_{t_1} -q(s) f(x(\sigma(s)))\, ds$$

$$\le f(x(\sigma(t_1))) \int_{t_1} -q(s)\, ds$$

Then

$$-u'(t_1) \le f(x(\sigma(t_1))) \int_{t_1} -q(s)\, ds$$ as $t \to \infty$ We get a contradiction . The proof is complete.

Example (2.5) : Consider the nonlinear neutered differential equation