Some Results on S-best Coapproximation in Linear 2-Normed Spaces

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ABSTRACT

The purpose of this paper is to discuss the new concept (S-best coapproximation in linear 2-normed spaces), we introduce the notions S-best coapproximation and S-orthogonality in the setting of linear 2-normed spaces. And then, some characterization and important theorems about existence of S-best coapproximation in convex subset of 2-normed linear spaces are proved.

INTRODUCTION

The concepts of linear 2-normed spaces were initially introduced by Gahler [1] in 1964, hence many researchers (see also [2-3]) have been studied the geometric structure of 2-normed spaces and obtained various results. In the literature, most studies on best coapproximation consider with normed linear spaces like [4-5-6], recently Vijayaraavan (2013) [7] extended this problem and dealt with some fundamental properties of the set of strongly unique best coapproximation in linear 2-normed spaces. Throughout this paper we introduced and study new concept namely, S-best coapproximation in linear 2-normed spaces, where we introduce the notions S-best coapproximation and S-orthogonality in 2-normed spaces and the relation between these concepts are obtained in convex subset of 2-normed spaces. We conclude this section with the following definitions:

Definition (1) [6]: Let $X$ be a linear space over real numbers with dimension $d$, where $2 \leq d \leq \infty$ and let $\|\cdot,\cdot\|: X \times X \to [0,\infty)$ be a non-negative real valued function on $X \times X$ satisfying the following properties for all $x, y, z \in X$:

1. $\|x, y\| = 0 \iff x = y$
2. $\|x, y\| = \|y, x\|$ (symmetry)
3. $\|\alpha x, y\| = |\alpha| \|x, y\|$ where $\alpha \in R$
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

Then $\|\cdot,\cdot\|$ is called 2-norm and the pair $(X, \|\cdot,\cdot\|)$ linear space $X$ is called a linear 2-normed space.

A standard example of a 2-normed space is $R^2$ equipped with the following 2-norm, $\|x, y\| := $ the area of the triangle having vertices $0, x$ and $y$. Observe that in any 2-normed space $(X, \|\cdot,\cdot\|)$ we have $\|x, y\| \geq 0$ and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and $\alpha \in R$. Also, if $x, y, z$ are linearly dependent (this happens for instance, when $d = 2$), then $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ [6].

Every subspace $G$ of 2-normed spaces $X$ is convex. In particular every 2-normed spaces $X$ is convex. Since, if $G$ is a subspace of $X$ and $g_1, g_2 \in G$ then $\alpha g_1 + \beta g_2 \in G$, for all scalars $\alpha, \beta$, thus in particular if we put $\alpha = 1 - \lambda$ and $\beta = \lambda$, for all $\lambda \in [0,1]$, then we have $(1 - \lambda)g_1 + \lambda g_2 \in G$, and so $G$ is convex.
Definition (2) [7]: Let $G$ be a subset of real linear 2-normed space $X$ and $x \in X$, then $g_* \in G$ is said to be a best coapproximation to $x \in X$ from the element of $G$, if for every $g \in G$
$$\forall z \in X \setminus \text{V}(x,G), \text{where } \text{V}(x,G) \text{ is the subspace generated by } x \text{ and } G. \text{ The set of all elements of best coapproximation to } x \in X \text{ from } G \text{ with respect to } z \text{ is denoted by } R_G(x, z) \text{ where } R_G(x, z) = \{ g_* \in G \mid \| g - g_* \| \leq \| x - g \| \}$$

The set of all elements of best coapproximation of $X$ from $G$ is denoted by $S\text{R}_G(x, z)$, this means $S\text{R}_G(x, z) = \bigcap_{i=1}^{d} R_G(x, z_i)$. Also if each $x \in X$ has at least (respectively exactly) one S-best coapproximation in $G$, then $S\text{R}_G(x, z)$ is called S-best coapproximantal (respectively S-coChebyshev) set.

Example (1): Suppose $X = R^2$ with usual basis $e = \{ e_1, e_2 \}$ and the norm $\| x, z \| = \sqrt{x_1^2 + x_2^2}$, for any $g = (g_1, g_2) \in G$, we have, $g* = (0, 1) \in \bigcap_{i=1}^{2} R_{(1,1)}((-1,1), e_i) = S\text{R}_{(1,1)}((-1,1), e_i)$.

MAIN RESULTS

Suppose that $(X, \| \cdot \|)$ is a 2-normed space, with dimension $d$, where $2 \leq d < \infty$, and $\{ z_1, \ldots, z_d \}$ be its basis. We start with:

Definition (3): let $G$ be a nonempty subset of linear 2-normed spaces $X$. An element $g_* \in G$ is said to be an $S$-best coapproximation of $x \in X$ from $G$ if $g_* \in \bigcap_{i=1}^{d} R_G(x, z_i)$

The set of all elements of best coapproximation of $X$ from $G$ is denoted by $S\text{R}_G(x, z)$, this means $S\text{R}_G(x, z) = \bigcap_{i=1}^{d} R_G(x, z_i)$. Also if each $x \in X$ has at least (respectively exactly) one $S$-best coapproximation in $G$, then $S\text{R}_G(x, z)$ is called $S$-best coapproximantal (respectively $S$-coChebyshev) set.

Example (2): Suppose $x = R^2$ with usual basis $e = \{ e_1, e_2 \}$ and the norm $\| x, z \| = \sqrt{x_1^2 + x_2^2}$, for any $g = (g_1, g_2) \in G$, we have, $g* = (0, 1) \in \bigcap_{i=1}^{2} R_{(1,1)}((-1,1), e_i) = S\text{R}_{(1,1)}((-1,1), e_i)$.

and so $g* = (0, 1) \in R_{(1,1)}((-1,1), e_i)$, and so $S\text{R}_G(x, z)$ is $S$-coChebyshev set.

Example (3): Suppose that $X$ as in the previous example and $G = \{ g = (g_1, g_2) \mid -1 \leq g_1 \leq 1, g_2 \leq |g_1| \}$ is a subset of $X$, then with a simple calculus can be shown that $(-1, 1), (1, 0) \in \bigcap_{i=1}^{2} R_{(-1,1)}((-2,2), e_i) = S\text{R}_{(-1,1)}((-2,2), e_i)$ and so $S\text{R}_G(x, z)$ is not $S$-coChebyshev set.

Theorem (1): Let $G$ be convex subset in linear 2-normed space $X$. then $S\text{R}_G(x, z)$ is convex nonempty subset of $G$.

Proof: Let $g_1, g_2 \in S\text{R}_G(x, z)$, then for all $i = 1, \ldots, d$, $g \in G$, we have $\| g - g_i, z_i \| \leq \| g - x, z_i \|$ and $\| g - g_2, z_i \| \leq \| g - x, z_i \|$.
\[ \lambda \in [0,1], \quad \text{we have} \quad r_i = \lambda r_i^1 \quad \text{then} \quad r_i > 0, \quad \text{let} \quad t_i \in \bigcap_{i=1}^{d} B_{z_i} (t, r_i^1), \quad \text{then} \quad \|t - t_i, z_i\| < r_i^1 \quad \text{and} \quad r_i^2 = \frac{1}{1 - \lambda} [r_i - \lambda r_i^1]. \]

then \((1 - \lambda) g_1 + \lambda g_2 \in SR_G (x, z)\) and so \(SR_G (x, z)\) is convex.

**Theorem (2):** Let \( X \) be linear 2-normed space, \( G \) is subspace of \( X \) then for any \( x \in X \), \( SR_G (x, z) \) is a subspace of \( G \).

**Proof:** The proof follows immediately from convexity and theorem (1).

**Remark (1):** For a 2-normed space with S-best coapproximation, we consider the following subsets:

\[
\bigcap_{i=1}^{d} \bigcap_{i=1}^{d} B_{z_i} (a, r_i) = \bigcap_{i=1}^{d} \{ x \mid \|x - a, z_i\| < r_i \} \]

for \( (a, r_i) \in (0,1), (i = 1, \ldots, d) \) since \( g_0 \notin \partial (G) \) and \( g_0 \in \partial (G) \) for all \( g_0 \in SR_G (x, z) \).

**Proof:** Suppose that \( g_0 \notin \partial (G) \) and \( x \notin G \), let \( t = \lambda g_0 + (1 - \lambda) x \) for \( \lambda \in (0,1) \), then \( g_0 \in \partial (G) \) and \( g_0 \in \text{int} (G) \) and since \( g_0 \in SR_G (x, z) \) then \( g_0 \in \text{int} (SR_G (x, z)) \), then there exist \( r_i^1 > 0 \) such that \( r_i > 0, \) let \( t_i \in \bigcap_{i=1}^{d} B_{z_i} (t, r_i) \) then \( \|t - t_i, z_i\| < r_i^1 \), let \( r_i^2 = \frac{1}{1 - \lambda} [r_i - \lambda r_i^1]. \)

Now, let \( t = \lambda g_0 + (1 - \lambda) x \) then \( g_0 = t - (1 - \lambda) x \) \( \lambda \) and so \( t_2 = \frac{1}{\lambda} [t_1 - (1 - \lambda) a] \), since \( t = \lambda g_0 + (1 - \lambda) x \)

\[
\bigcap_{i=1}^{d} B_{z_i} (a, r_i) \quad \text{is open ball and} \quad \bigcap_{i=1}^{d} \bigcap_{i=1}^{d} B_{z_i} (a, r_i) \quad \text{is a closed ball.}
\]

**Theorem (3):** Let \( X \) be linear 2-normed space and \( G \) be subspace of \( X \), if \( SR_G (x, z) \) is convex subset then \( g_0 \in \partial (G) \) for all \( g_0 \in SR_G (x, z) \).

This implies that \( t_2 \in SR_G (x, z), \) since \( t_2, a \in SR_G (x, z) \) and \( SR_G (x, z) \) is convex subset, then \( \lambda t_2 + (1 - \lambda) a \in SR_G (x, z) \) for \( \lambda \in [0,1] \), but \( t_1 = \lambda t_2 + (1 - \lambda) a \) this implies that \( t_2 \in SR_G (x, z) \), then we have \( \bigcap_{i=1}^{d} B_{z_i} (t, r_i) \subset SR_G (x, z) \) and so \( t \in \text{int} (SR_G (x, z)) \) this implies that \( t \in SR_G (x, z) \), then \( t \in G \) and so
\[ \lambda g_\circ + (1 - \lambda) x \in G \]

This implies that \( x \in G \), since \( G \) is a subspace, then \( x \in G \) which is a contradiction, hence \( g_\circ \in \partial(G) \). \( \blacksquare \)

**Theorem (4):** Let \( X \) be linear 2-normed space and \( G \) be subset of \( X \), for each \( x \in X / G \), we have

\[ SR_G(x, z) = \bigcap_{i=1}^{d} [G \cap \bigcap_{g \in G} [g, \|x - g, z_i\|]] \]

**Proof:** By definition (3) of \( SR_G(x, z) \) for each \( g \in G \), we have \( SR_G(x, z) \subseteq G \), and if \( g_\circ \in SR_G(x, z) \), for \( (i = 1, ..., d) \) we have

\[ \|g_\circ - g, z_i\| \leq \|x - g, z_i\| \]

and so by remark(1)

\[ g_\circ \in \bigcap_{i=1}^{d} [G \cap \bigcap_{g \in G} [g, \|x - g, z_i\|]] \]

we have

\[ SR_G(x, z) \subseteq \bigcap_{i=1}^{d} [G \cap \bigcap_{g \in G} [g, \|x - g, z_i\|]] \]

and

\[ SR_G(x, z) \subseteq \bigcap_{i=1}^{d} [G \cap \bigcap_{g \in G} [g, \|x - g, z_i\|]] \] conversely.

Let \( g_\circ \in \bigcap_{i=1}^{d} [G \cap \bigcap_{g \in G} [g, \|x - g, z_i\|]] \)

then we have \( g_\circ \in G \) and for each \( g \in G \),

\[ (i = 1, ..., d) \]

\[ \|g_\circ - g, z_i\| \leq \|x - g, z_i\| \]

which implies that

\[ g_\circ \in SR_G(x, z) \].

So

\[ \bigcap_{i=1}^{d} [G \cap \bigcap_{g \in G} [g, \|x - g, z_i\|]] \subseteq SR_G(x, z) \]

which completes the proof. \( \blacksquare \)

**Theorem (5):** Let \( G \) be non-empty subset of linear 2-normed spaces \( X \), then for each \( x \in X \),

i. for every \( x, y \in X \), \( i = 1, ..., d \), \( g_\circ \in SR_{G+y}(x + y, z) \) (by definition (3)) if

\[ d \]

and only if

\[ g_\circ \in \bigcap_{i=1}^{d} R_{G+y}(x + y, z_i) \]

and only if

\[ \|g_\circ - (g + y), z_i\| \leq \|(g + y) - (x + y), z_i\| \]

which implies that \( g_\circ \in SR_{G+y}(x + y, z) \) if and only if

\[ \|g_\circ - y - g, z_i\| \leq \|g - x, z_i\| \]

if and only if

\[ g_\circ - y \in \bigcap_{i=1}^{d} R_{G+y}(x, z_i) \]

if and only if

\[ g_\circ \in SR_{G+y}(x, z) \]

\( \blacksquare \)

ii. for every \( g_\circ \in SR_{\alpha G}(\alpha x, |\alpha| z) \), \( i = 1, ..., d \) by definition (3) if and only if

\[ \|g_\circ - \alpha g, |\alpha| z_i\| \leq \|\alpha g - \alpha x, |\alpha| z_i\| \]

if and only if

\[ \|\alpha^{-1} g_\circ - g, z_i\| \leq \|\alpha(g - x), |\alpha| z_i\| \]

if and only if

\[ \|\alpha^{-1} g_\circ - g, z_i\| \leq |\alpha|\|g - x, z_i\| \]

if and only if

\[ \|\alpha^{-1} g_\circ - g, z_i\| \leq |\alpha|\|g - x, z_i\| \]

if and only if

\[ g_\circ \in \alpha SR_{G}(x, z) \]

\( \blacksquare \)

**Theorem (6):** Let \( G \) be convex subset of linear 2-normed spaces \( X \), then for each \( x \in X \), \( g_\circ \in SR_G(x, z) \) if and only if

\[ g_\circ \in SR_G(\alpha^m x + (1 - \alpha^m) g_\circ, z) \]

for all \( \alpha \in R \).

**Proof:** Suppose \( g_\circ \in SR_G(x, z) \) then

\[ g_\circ \in \bigcap_{i=1}^{d} R_G(x, z_i) \]

and so for all \( g \in G \),
 Define $z$, $z'$ be two subsets of $\mathcal{G}$ be convex subset of linear 2-normed spaces, and for all $i \in \{1, \ldots, d\}$, we have

$$\|g_i - g_i', z_i\| \leq \|x - x_i, z_i\|$$

for all scalar $\alpha$, $(i = 1, \ldots, d)$, sybomatically $x \perp_{SB} y$ if and only if

$$\|x + \alpha y, z_i\| \geq \|x, z_i\|.$$ 

**Theorem (7):** Let $G$ be convex subset of linear 2-normed spaces $X$, then for each $x \in X$, $g_i \in SR_G(x, z)$ if and only if $G \perp_{SB} (x - g_i).$

**Proof:** Suppose $g_i \in SR_G(x, z)$ and $g \in G$, for $(i = 1, \ldots, d)$, $\alpha \in R$, $\alpha \neq 0$, put $g_1 = g_i - \frac{1}{\alpha} g_i$, since $g_i \in SR_G(x, z)$ so,

$$\|g_i - g_1, z_i\| \leq \|g_1 - x, z_i\|$$

therefore

$$\frac{1}{\alpha} g_i, z_i \in \|x - g_i + \frac{1}{\alpha} g_i, z_i\|,$$

equivalently $\|g_i, z_i\| \leq \|x - g_i, z_i\|$, and so

$$g_1 \perp_{SB} (x - g_i), \zeta \in G$$

thus, $g \perp_{SB} (x - g_i)$ and so

$$G \perp_{SB} (x - g_i).$$

Then for all $(i = 1, \ldots, d)$, $\alpha \in R$ and $g \in G$, we have

$$\|g, z_i\| \leq \|g + \alpha(x - g_i), z_i\|.$$ 

**Definition (5):** Let $X$ be linear 2-normed space, and let $G$ and $H$ be two subsets of $X$. Define:

$$SR_G(x, z) = \bigcap_{i=1}^{d} R_G(h, z_i).$$

**Theorem (8):** Let $G, G'$ be convex subset of linear 2-normed spaces $X$, such that $G \subseteq G'$, then for each $x \in X$,

$$SR_G(SR_G'(x, z), z) \subseteq SR_G(x, z).$$

**Proof:** Suppose $g_i \in SR_G(G_i, z)$, then $g'_i \in SR_G(g_i', z)$, for some $g_i' \in SR_G'(x, z)$, so $G_i \perp_{SB} (x - g_i)$ and $G_i' \perp_{SB} (x - g_i)$ (by theorem (7)). Then for each $(i = 1, \ldots, d)$, $\alpha \in R$ and $g, g_i' \in G$, we have

$$\|g' + \alpha(x - g_i'), z_i\| \geq \|g', z_i\|$$

and

$$\|g + \alpha(x - g_i), z_i\| \geq \|g, z_i\|.$$ 

Now since $g + \alpha(g_i' - g_i) \in G'$ for each $\alpha \in R$ and $g \in G \subseteq G'$, therefore

$$\|g + \alpha(x - g_i), z_i\| = \|g + \alpha(g_i' - g_i) + \alpha(x - g_i'), z_i\|$$

by (*). Since $\|g + \alpha(x - g_i), z_i\| \geq \|g, z_i\|$, then

$$\|g + \alpha(x - g_i), z_i\| \leq \|x - g_i + \frac{1}{\alpha} g_i, z_i\|.$$
so \( g \perp_{SB}^{\perp}(x - g_o) \) for some \( g \in G \). Then
\[ G \perp_{SB}^{\perp}(x - g_o), \]
this means \( g_o \in SR_{G}(x, z) \), hence
\[ SR_{G}(SR_{G}(x, z), z) \subseteq SR_{G}(x, z). \]

**Conclusion:** This paper concludes S-best coapproximation in linear 2-normed spaces and two examples to illustrate this concept. Many results have been proved about S-best coapproximation, S-orthogonality. This paper can be extended to other setting, such as S-best coapproximation in n-normed spaces using any type of orthogonality since there are different kinds of orthogonality such as Birkhoff, Isosceles, Roberts, and Singer orthogonality.

**References**


