Efficient Approximate Method for Solutions of Linear Mixed Volterra-Fredholm Integro-Differential Equations

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ABSTRACT

In this paper, we present approach of Boubaker polynomials to find approximate solution of the mixed linear Volterra-Fredholm integro-differential equation (MLV-FID). We employ variational formulation method to obtain approximate solution of (MLV-FID) which is based on Boubaker polynomials. This method presents a computational technique through converting this integral equation into a system of linear equations which can be easily solved by the known methods. Some examples illustrate the efficiency and accuracy of this method are given.

INTRODUCTION

Currently, considerable interest in mixed integro-differential equations has been stimulated due to their numerous applications in the areas of engineering, mechanics, physics, chemistry, biology, economics, potential theory, and electrotechnics. The mixed linear Volterra-Fredholm integro-differential equation (MLV-FID), the unknown function appears to be under integration sign, and it may also include the derivatives and functional arguments of the unknown function. This type of equations can be grouped into the mixed linear Volterra-Fredholm integro-differential equation (MLV-FID). The upper bound of the integral part of Volterra type is variable, while it is a fixed number for that of Fredholm type [10]. Since (MLV-FID) are usually difficult to solve in an analytical manner, or to obtain closed form solution, so the numerical method is needed to solve this kind of (MLV-FID).

In this paper, we propose an efficient method, namely variation formulation method, to obtain the numerical solution of mixed linear Volterra-Fredholm integro-differential equation of the form

\[ u^n(x) = f(x) + \int_0^x \int_a^b K(r,t)u(t)dt\,dr \]  \hspace{1cm} (1)

Where \( u \) is an unknown function, the functions \( f(x) \) and \( K(r,t) \) are analytic on \( D = [0,T] \times \Omega \) and \( 0 \leq r, t \leq T \) and \( \Omega = [a,b] \). Where the unknown function \( u(x) \) appears inside the integral and the derivative \( u^n(x) \) appears outside integral [1, 9].

Many research studies the Boubaker polynomials M. Agida and A.S. Kumar [6], used Boubaker polynomials schema for solving to random Love's integral equation (Fredholm integral equation of the second kind) of a rational kernel. Salih Y. and Tugce A.[8].

The aim of this study is to get approximate solution of the equation (1) as the truncated Boubaker series defined

\[ y(x) = \sum_{i=0}^{N} a_i B_i(x) \] \hspace{1cm} (2)

Where \( B_i(x) \), \( i = 0, 1, 2, ..., N \) denote the Boubaker polynomials and \( a_i \) are unknown coefficient, and \( N \) is chosen any positive integer.

2. Prelimianries:

The paper is organized as follows: In section 2, we give some definitions. In section 3, we review variational formulation method and their definition. In section 4, describe Boubaker polynomials and their properties. In section 5, we propose the variational formulation by using basis Boubaker polynomials for approximate solution of mixed linear Volterra-Fredholm integro-differential equation (MLV-FID) via variational formulation method. Finally in section 6, we illustrate some examples to show the accuracy and efficiency of this method is given.

2.1 Definition [3]:- The bilinear \( \langle u, v \rangle \) is called non-degenerate on \( U \) and \( V \) (where \( U \) and \( V \) are linear space) if the following two conditions are satisfied:

- \( \langle u, v \rangle = 0 \) Then \( \vec{v} = 0 \) for every \( u \in U \)
- \( \langle u, v \rangle = 0 \) Then \( \vec{u} = 0 \) for every \( v \in V \)

For instance some example of non-degenerate bilinear forms is given by:-

\[ \langle u, v \rangle = \int_0^T u(x,t)v(x,t)\,dt\,dx , \text{ where } 0 \leq x \leq T \] \hspace{1cm} (3)

2.2 Definition [5]:- The operator \( L \) is said to be symmetric with respect to chosen bilinear form \( \langle u, v \rangle \), if the condition: \( \langle Lu_1, u_2 \rangle = \langle Lu_2, u_1 \rangle \), is satisfied for every pair of elements \( u_1, u_2 \) of \( D(L) \) (where \( D \) the domain of \( L \)). This shows that the symmetry of an operator is relation to the chosen bilinear form.
3. Variation formulation method [5,10]:

Calculus variations, is the solution of optimization problems over functions of one or more variables. A fundamental problem in calculus of variations and its applications is the problem of the finding minimum or maximum value of given functional $F(u)$. Therefore we construct a variational formulation for the linear mixed Volterra-Fredholm integro-differential equations and illustrate the solution of these equations by finding the critical point for it corresponding variational formulation formal. Where the critical point $u_0 \in U$ is meaning that $\delta F[u_0] = 0$, where $\delta$ is the customary symbol of variation of functional. The most important difficulty of the subject of calculus of variation is to find variational formulation which corresponds to the linear operator equation $Lu = f$ ...

Where $u$ denotes a scalar-vector valued functional and $L$ denotes a linear operator.

3.1 Theorem: If the given linear operator $L$ is symmetric with respect to the chosen non-degenerated bilinear form $<u,v>$ then the solution to the equation $Lu = f$ is critical points of the functional $F(u) = \frac{1}{2} < u, v > - f, u >$.

Now, in this work the bilinear operator form equation (3) is used since it makes the linear operator symmetric with respect to it. This could be done after the following transformation:

$Lu = f$ is critical points of the functional $F(u) = \frac{1}{2} < u, v > - f, u >$ ...

Where $v \in V$ and $u \in D(L)$. The bilinear form equation (5) makes the given linear operator symmetric since:

$<Lu_1, u_2> = <Lu_1Lu_2> = <Lu_2Lu_1> = Lu_2u_1$ ...

Therefore, in general the bilinear form equation (5) will be used to find a variational formulation because of the symmetry of $L$. Since $L$ is symmetric and by using theorem (3.1) the solution of equation (4) is a critical point of functional $F(u) = \frac{1}{2} < Lu, u > - f, Lu >$ ...

The functional (7) is variational formulation of linear equation $Lu = f$.

4. Definition [2,4,7]: A monomial definition of the Boubaker polynomial is

$$B_n(x) = \sum_{p=0}^{\lfloor n/2 \rfloor} \binom{n-4p}{n-p} \binom{n-p}{p} (-1)^p X^{n-2p}$$

Where $\lfloor \frac{n}{2} \rfloor = 2n + (\lfloor \frac{n-1}{2} \rfloor)$

The symbol: $\lfloor \cdot \rfloor$ designates the floor function $B_n(x)$ = $\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \left( b_{n,j} \right) X^{n-2j}$

$b_{n+1,j} = (n - 2j)(n - 4j - 4) b_{n,j}$

$b_{n,0} = 1$ if $n$ even

$b_{n,0} = \left( \frac{1}{2} \right) n^2 \cdot (n - 2)$ if $n$ odd

$B_m(x) = xB_{m-1}(x) - B_{m-2}(x)$ for $m > 2$

5. Variation formulation for approximate solution linear mixed Volterra-Fredholm integro-differential equation using Boubaker polynomials:

Consider the linear mixed Volterra-Fredholm integro-differential equation of the form:

$L = u^{(o)}(x) + \int_0^x \int_a^b K(r, t) u(t) dt dr$ ...

The operator $L$ is linear since it is easily seen that:

$Lw_1 u_1 + w_2 u_2 = w_1 \int_0^x \int_a^b K(r, t) u_1(t) dt dr + w_2 \int_0^x \int_a^b K(r, t) u_2(t) dt dr$ = $w_1 Lu_1 + w_2 Lu_2$

Therefore, the operator $L$ if linear [5].

The variational formulation of the given linear operator $L$ could be found as follows: $F(u) = \frac{1}{2} < Lu, Lu > - f, Lu >$ ...

$$F(u) = \frac{1}{2} \int_0^x \left( Lu \right)^2 dx - \int_0^x f(x) Lu dx$$

$$F(u) = \frac{1}{2} \int_0^x \left[ u^n(x) - \int_0^x \int_a^b K(r, t) u(t) dt dr \right]^2 dx - \left[ \int_0^x f(x) \left[ u^n(x) - \int_0^x \int_a^b K(r, t) u(t) dt dr \right] \right]$$ ...

To solve the above variational formulation one must approximate the solution (8) as linear combination of $N$ linearly independent Boubaker polynomials $\{B_i(x)\}_{i=0}^{N}$ such that $u_n(x) = \sum_{i=0}^{N} a_i B_i(x)$

Where $\{a_i(x)\}_{i=0}^{N}$ are the unknown parameters that must be determined.

Then after substituting this approximated solution we can get:

$$F(u) = \frac{1}{2} \int_0^x \left[ u^n(x) - \int_0^x \int_a^b K(r, t) u(t) dt dr \right]^2 dx - \frac{1}{2} \int_0^x f(x) \left[ u^n(x) - \int_0^x \int_a^b K(r, t) u(t) dt dr \right]$$

$$F(u) = \frac{1}{2} \int_0^x \sum_{i=0}^{N} a_i \left[ B_i^n(x) - \int_0^x \int_a^b K(r, t) B_i(t) dt dr \right]$$

Then we can solving the linear system equation by using Matlab language, we get tabulated results.
value of \( \{a_i\}_{i=0}^{n} \) are obtained by solution the linear system by getting \( \frac{\partial F}{\partial a_i} = 0 \), \( i=0, 1, 2, \ldots n \)

### 6. Application By Examples:

**Example (1):** Consider linear mixed Volterra-Fredholm integro-differential equation of Second kind

\[
\dot{u}(x) = -\frac{10}{3} + 2x^3 + \int_0^x \int_{-1}^1 (rt^2 - r^2t)u(t) \, dt \, dr
\]

Where \( f(x) = -\frac{10}{3} + 2x^3 \) and \( K(r,t) = (rt^2 - r^2t) \), with the exact solution \( u(x) = 1 + 9x - \frac{2}{3}x^2 \)

**Example (2):** Consider linear mixed Volterra-Fredholm integro-differential equation of Second kind

\[
\dot{u}(x) = -\frac{10}{3} + 2x^3 + \int_0^x \int_{-1}^1 (rt^2 - r^2t)u(t) \, dt \, dr
\]

Where \( f(x) = -\frac{10}{3} + 2x^3 \) and \( K(r,t) = (rt^2 - r^2t) \), with the exact solution \( u(x) = 1 + 9x - \frac{2}{3}x^2 \)

### Table 1: Numerical results for example (1):

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<th>Exact solution</th>
<th>Approximate solution using Boubaker polynomial(N=3)</th>
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### Table 2: Numerical results for example (2):

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### Figure 1: Compare of Approximate Solution Using Boubaker Polynomial and Exact Solution

### Figure 2: Compare of Approximate Solution Using Boubaker Polynomial and Exact Solution

### 7. Conclusion:

1. The Boubaker polynomials coefficients of the solution (MLV-FID) are found very easily by using a program with written Mathlab12
2. To obtain the best approximating solution (MLV-FID), we take more form from the Boubaker expansion of the function, which is the truncation limit N must be chosen large enough.
3. Suggested approximations by using Boubaker variational formulation is very attractive and contributed to solution (MLV-FID) to get good agreement between approximate and values in the numerical examples.

### 8. Reference:


