On $S_s$-Compact Functions

Haider J. Ali
Department of Mathematics, College of Science, Al-Mustansiriyh University

ABSTRACT

The purpose of the present paper is to extend the notion of semi compact functions by using $s_s$-compact spaces. We introduce further types of this concept namely $s_s$-compact, $s_s^*$-compact and $s_s^{**}$-compact functions. Furthermore theorems, facts and several examples have been given to illustrate our results.

INTRODUCTION

In 1963 Levine introduced new notion of sets called semi-open set in a topological spaces. In 2014 J. M. J. defined new class of semi-open sets and gives several properties about this set. In 2000 Ressen D. A. introduced the notion semi-compact functions these are the function in which the inverse image of every compact set in $Y$ is semi-compact set in $X$. In this work we continue to study this notion by using $s_s$-compact set which introduced by [2]. Also we introduce some types of these functions.

PRELIMINARIES

Definition 1[1]: A subset $A$ of a topological space $X$ is said to be semi-open set if there exist open set $U$ with $U \subseteq A$ and $A \subseteq cl(U)$, that is $U \subseteq A \subseteq cl(U)$.

Definition 2[1]: A subset $A$ of a space $X$ is said to be $\alpha$-open if $A \subseteq int(cl(int(A)))$.

Definition 3[2]: A semi-open subset $A$ of a space $X$ is called $s_s$-open if for each $x \in A$ there exist semi-closed set $F$ such that $x \notin F \cap A$. The family of all $s_s$-open subsets of topological space $X$ is denoted by $S_sO(X)$.

Proposition 1[2]: A subset $A$ of a space $X$ is $S_s$-open iff $A = \bigcup F_i$, where $A$ is semi-open and $F_i$ is semi-closed for each $i$.

Remark 1: Every $S_s$-open set is semi-open set, but the converse may be not true, as in the following example.

Example 1[2]: Let $X=\{a, b, c\}$ and $T=\{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$, then we have $SO(X)=\{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$, and hence $SC(X)=\{\emptyset, X, \{a\}, \{b\}, \{b, c\}\}$ so $S_sO(X)=\{\emptyset, X\}$ implies that $\{a\} \notin SO(X)$ but $\{a\} \notin S_sO(X)$.

Proposition 2[2]: Let $\{A_i: i \in I\}$ be collection of $S_s$-open sets in topological space $X$, then $\bigcup \{A_i: i \in I\}$ is $S_s$-open set in $X$.

Remark 2[2]: The intersection of two $S_s$-open sets need not be $S_s$-open set in general, as in the following example.

Example 2[2]: Consider the intervals $[0, 1]$ and $[1, 2]$ in $R$ with usual topology. Since $R$ is $T_1$ space and hence it is semi-$T_1$, so both the intervals are $S_s$-open and we have $[0, 1] \cap [1, 2] = \{1\}$ which is not $S_s$-open.

Remark 3:

i. there is no relation between $S_s$-open and open set.

ii. the sets $\{a, b\}$ and $\{a, c\}$ are open set but not $s_s$-open set.

iii. in example 2 the sets $[0, 1] \cap [1, 2]$ are $S_s$-open sets which are not open sets.

iv. in discreet space each open set is $S_s$-open set.

Proposition 3[2]: Let A, B be two subsets of a space $X$. If $A$ is $S_s$-open set and $B$ is both $\alpha$-open and semi-open, then $A \cap B$ is $S_s$-open set.
In this section we introduce other types of semi-compact functions namely \( S^*_c \)-compact, \( S^* \)-compact, \( S^{**}_c \)-compact and \( S^{**} \)-compact functions by using the concept of \( S_c \)-compact sets. Certain facts, example and theorems have been given to explain our results.

**Proposition 4**[2]: Let \((Y, T)\) be an open subspace of space \((X, T)\). If \(A \subseteq S_c O(X, T)\) and \(A \subseteq Y\) then \(A \subseteq S_c O(Y, T_d)\).

**Proposition 5**[2]: Let \(Y\) be semi-regular set in a space \((X, T)\). If \(A \subseteq S_c O(Y, T)\), then \(A \subseteq S_c O(X, T)\) where \(A \subseteq Y\).

**Definition 4**[2]: Let \(A\) be a subset of topological space \((X, T)\). A point \(x \in A\) is said to be \( S_c \)-interior point of \(A\) if there exists an \( S_c \)-open set \( U\) containing \(x\) such that \(U \subseteq A\). The set of all \( S_c \)-interior point of \(A\) is said to be \( S_c \)-interior of \(A\) and denoted by \( S_c \text{Int}(A)\).

**Proposition 6**[2]: For any subset \(A\) of topological space \(X\). The following statements are true:
1. The \( S_c \)-interior of \(A\) is the union of all \( S_c \)-open sets which are contained in \(A\).
2. \( S_c \text{Int}(A)\) is \( S_c \)-open set in \(X\).
3. \( S_c \text{Int}(A)\) is the largest \( S_c \)-open set contained in \(A\).
4. \(A\) is \( S_c \)-open set iff \(A = S_c \text{Int}(A)\).

Finally from (4) we get \( S_c \text{Int}(A) = S_c \text{Int}( S_c \text{Int}(A))\).

**Definition 5**[2]: The intersection of all \( S_c \)-closed sets containing \(F\) is called \( S_c \)-closure of \(F\) and we denote it by \( S_c \text{Cl}(F)\).

**Proposition 7**[2]: For a subset \(F\) of a space \(X\), the following statement are true:
1. \( S_c \text{Cl}(F)\) is \( S_c \)-closed set in \(X\) containing \(A\).
2. \( S_c \text{Cl}(F)\) is the smallest \( S_c \)-closed set containing \(A\).
3. \(F\) is \( S_c \)-closed iff \(F = S_c \text{Cl}(F)\). So \( S_c \text{Cl}(S_c \text{Cl}(F)) = S_c \text{Cl}(F)\).

**Definition 6**[3]: Let \(X\) and \(Y\) be topological spaces, then the function \(f: X \rightarrow Y\) is said to be \( S \)-compact (resp. \( S^* \)-compact, \( S^{**} \)-compact functions) if the inverse image of each compact set (s-compact set, s-compact set) in \(Y\) is \( S \)-compact set (compact set, s-compact set) in \(X\).

**Definition 7**[4]: A space \(X\) is called semi-compact if every semi-open cover of \(X\) admits a finite subcover.

**Definition 8**[2]: A space \(X\) is called \( S_c \)-compact if every \( S_c \)-open cover \(\{U_y; y \in \Delta\}\) of \(X\), there exists a finite subset \(\Delta \subseteq \Delta\) such that \(X = \bigcup_{y \in \Delta} U_y\).

**Remark 4**[2]: Every semi-compact space is \( S_c \)-compact but the converse is not true in general, as in example 4.1.15[2].

**SOME TYPES OF SEMI-COMPACT FUNCTIONS**
In this section we introduce other types of semi-compact functions namely \( S^*_c \)-compact, \( S^*_c \)-compact and \( S^{**}_c \)-compact functions by using the concept of \( S_c \)-compact sets. Certain facts, example and theorems have been given to explain our results.

**Definition 9**: Let \(X\) and \(Y\) be topological spaces, then the function \(f: X \rightarrow Y\) is said to be \( S^*_c \)-compact function if the inverse image of semi-compact set is semi-compact set.

**Definition 10**: Let \(X\) and \(Y\) be topological spaces, then the function \(f: X \rightarrow Y\) is said to be \( S^*_c \)-compact function if the inverse image of semi-compact set is semi-compact set.

**Definition 11**: Let \(X\) and \(Y\) be topological spaces, then the function \(f: X \rightarrow Y\) is said to be \( S^{**}_c \)-compact function if the inverse image of semi-compact set is semi-compact set.

**Example 3**: The identity function \(f: (X, T_d) \rightarrow (Y, T_d)\) is \( S_c \)-compact, \( S^*_c \)-compact and \( S^{**}_c \)-compact functions.

**Proposition 8**: Let \(X\) and \(Y\) be topological spaces and let \(f: X \rightarrow Y\) be a function then
1. Every \( S^*_c \)-compact function is \( S_c \)-compact function.
2. Every \( S^*_c \)-compact function is \( S^{**}_c \)-compact function.
3. Every \( S^{**}_c \)-compact function is \( S_c \)-compact function.

Proof (1): Let \(f\) be \( S^*_c \)-compact function, to prove \(f\) is \( S_c \)-compact function. Let \(K\) be semi-compact in \(Y\), thus \(K\) is \( S_c \)-compact set, but \(f\) is \( S^*_c \)-compact function, thus \(f^{-1}(K)\) is semi-compact in \(Y\) so \(f^{-1}(K)\) is \( S_c \)-compact set. Further \(f\) is \( S_c \)-compact function. By the same way can prove the other cases.

The following diagram explain the relationships of our concepts.

**Remark 5**: The converse of above proposition is no true in general.

**Definition 12**[5]: A space \(X\) is called locally indiscrete if every open subset of \(X\) is closed.

**Proposition 9**[2]: If a space \(X\) is locally indiscrete then the following are equivalent:
1. \(X\) is \( S_c \)-compact space.
2. \(X\) is semi-compact space.
3. \(X\) is compact space.

By above proposition we can make the converse proposition is true.

**Proposition 10**:
1. Every \( S_c \)-compact function from locally indiscrete space into locally indiscrete space is \( S^*_c \)-compact function.
2. Every \( S^{**}_c \)-compact function from locally indiscrete space into any space is \( S^*_c \)-compact function.
3. Every $S_c$-compact function from any space into locally indiscrete space is $S_c^{**}$-compact function.

**Definition 13:** A function $f: X \rightarrow Y$ is said to be $S_c^{**}$-continuous function, if $f^{-1}(K)$ is $S_c$-open (closed) subset in $X$, whenever $F$ is $S_c$-closed (open) subset in $Y$.

**Proposition 11:** Every $S_c^{**}$-continuous image of $S_c$-compact set is $S_c$-compact set.

**Proof:** Let $f: X \rightarrow Y$ be $S_c^{**}$-continuous, and $K$ be an $S_c$-compact subset of $X$. To show that $f(K)$ is $S_c$-compact subset of $Y$. Let $W = \{G_i : \forall i \in I\}$ be $S_c$-open cover of $f(K)$, since $f$ is $S_c^{**}$-continuous function, thus $f^{-1}(W) = \{f^{-1}(G_i) : \forall i \in I\}$ be $S_c$-open cover of $K$, and since $K$ is $S_c$-compact set, there is finite sub cover of $K$ such that $K \subseteq \cup_{i \in I} f^{-1}(G_i)$ so $f(K) \subseteq (\cup_{i \in I} f^{-1}(G_i))$. Thus $f^{-1}(G_i) \subseteq \cup_{i \in I} G_i$. Then $f(K) \subseteq \cup_{i \in I} G_i$. Therefore $f(K)$ is $S_c$-compact set.

**Proposition 12:** Every $S_c$-compact subset of locally indiscrete $T_2$-space is closed.

**Proof:** Let $X$ be locally indiscrete $T_2$-space and let $A$ be $S_c$-compact set in $X$. To show that $A^C$ open, let $x \in A^C$ then for each $a \in A$ there exist two open sets $U_{x,a}$ and $V_a$ such that $x \in U_{x,a}$ and $a \in V_a$ and $U_{x,a} \cap V_a = \emptyset$ (since $X$ $T_2$-space) the collection $\{V_a : a \in A\}$ be open cover of $A$. Since $X$ is locally indiscrete thus every $S_c$-compact set is compact. Therefore there exists a finite subcollection $V_{a_1}, V_{a_2}, \ldots , V_{a_m}$ that cover of $A$. Let $U_i = U_{x,a_1} \cap U_{x,a_2} \cap \ldots \cap U_{x,a_m}$ thus $U_i \in T$, $x \in U_i$ and $U_i \cap K = \emptyset$ thus $U_i \cap A^C = \emptyset$ so $x \notin U_i \subseteq A^C$. Therefore $x$ is interior point of $A^C$ so $A^C$ is open set in $X$, then $A$ is closed.

**Proposition 13:** Every $S_c$-closed subset of $S_c$-compact space is $S_c$-compact set.

**Proof:** Let $(X, T)$ be $S_c$-compact space. And let $A$ be $S_c$-closed sub set of $X$. to prove $A$ is $S_c$-compact set. Let $\{G_i : i \in I\}$ be $S_c$-open cover of $A$; that is $A \subseteq \cup_{i \in I} G_i$. Thus $X = A^C \cup U_{i \in I} G_i$ Since $X$ is $S_c$-compact space. Then $X = A^C \cup (\cup_{i \in I} G_i)$. Therefore $A = (\cup_{i = 1}^n G_i)$ that is $A$ is $S_c$-compact set.

**Proposition 14:** Every $S_c^{**}$-continuous function from $S_c$-compact into locally indiscrete $T_2$-space is $S_c^{*}$-closed function.

**Proof:** Let $f$ be $S_c^{**}$-continuous function (where $X$ is $S_c$-compact and $Y$ is locally indiscrete $T_2$-spaces). We will prove that $f$ is $S_c^{*}$-closed function. Let $F$ be $S_c^{*}$-closed set in $X$, thus $F$ is $S_c$-compact set, and since $f$ is $S_c^{**}$-continuous function, then $f(F)$ is $S_c$-compact set in $Y$, since $Y$ is locally indiscrete $T_2$-space thus $f(F)$ is closed set. Therefore $f$ is $S_c^{*}$-closed function.

**Proposition 15:** Let $X$ be topological space and let $Y$ be semi-regular subspace of $X$. If $A$ is $S_c$-compact set in $X$, thus $A$ is $S_c$-compact set in $Y$ (where $A \subseteq Y$).

**Proof:** Let $A$ be $S_c$-compact set in $X$, to prove $A$ is $S_c$-compact set in $Y$. Let $\{U_i : i \in I\}$ be $S_c$-open cover of $A$ in $Y$, thus $\{U_i : i \in I\}$ is $S_c$-open cover of $A$ in $X$ (by proposition 5). Since $A$ is $S_c$-compact set in $X$, then there is $I \subseteq I$, thus $A \subseteq \cup_{i \in I} U_i$. Therefore $A$ is $S_c$-compact set in $Y$.

**Proposition 16:** Let $Y$ be $a$-open subspace of $X$. Then if $A$ is $S_c$-compact set in $Y$ then $A$ is $S_c$-compact set in $X$ (where $A \subseteq Y$).

**Proof:** Let $A$ be $S_c$-compact set in $Y$, to prove $A$ is $S_c$-compact set in $X$. Let $\{U_i : i \in I\}$ be $S_c$-open cover of $A$ in $X$, thus $\{U_i : i \in I\}$ is $S_c$-open cover of $A$ in $Y$ (by proposition 4). Since $A$ is $S_c$-compact set in $Y$, then there is $I \subseteq I$, thus $A \subseteq \cup_{i \in I} U_i$. Therefore $A$ is $S_c$-compact set in $X$.

**Proposition 17:** If $A$ is $S_c$-compact set in $X$ and $F$ is $S_c$-closed set in $X$, then $A \cap F$ is $S_c$-compact set in $X$.

**Proof:** Let $\{U_i : i \in I\}$ be $S_c$-open cover of $A \cap F$, that is; $A \cap F \subseteq \cup_{i \in I} U_i$. Since $F$ is $S_c$-closed set in $X$, then $F^C$ is $S_c$-open set, thus $\{U_i \cap U_i^C\}$ is a $S_c$-open cover of $A$, as well as $A$ is a $S_c$-compact set in $X$, then there is finite subcover $(U_i \cap U_i^C \cap \ldots \cap U_n \cap C^C)$ of $A$, that is; $A \subseteq (\cup_{i = 1}^n U_i)$ and $F^C$. Then $A \cap F \subseteq (\cup_{i = 1}^n U_i)$, thus $A \cap F$ is $S_c$-compact set in $X$.

Now we study the restriction of $S_c$-compact function and the composition of $S_c$-compact set.

**Remark 6:** If $X$ and $Y$ are topological spaces and $f: X \rightarrow Y$ is $S_c$-compact function, then the function $f|_A: A \rightarrow Y$ is not necessary $S_c$-compact function. But if we add a condition the remark is true.

**Proposition 18:** Let $f: X \rightarrow Y$ is $S_c$-compact function and let $A$ be semi-regular $S_c$-closed subset of $X$, then $f|_A: A \rightarrow Y$ is $S_c$-compact function.

**Proof:** Let $K \subseteq Y$ be $S_c$-compact set, to show that $(f|_A)^{-1}(K) = A \cap f^{-1}(K)$ is $S_c$-compact set in $A$. Since $f$ is $S_c$-compact function, thus $f^{-1}(K) \subseteq X$ is $S_c$-compact set in $X$. And since $A$ is $S_c$-closed subset of $X$, thus by above proposition $A \cap f^{-1}(K)$ is $S_c$-compact set in $X$. Now to prove $(f|_A)^{-1}(K) = A \cap f^{-1}(K)$ is $S_c$-compact set in $A$. Since $A$ is Semi-regular thus $A \cap f^{-1}(K)$ is $S_c$-compact set in $A$ (by proposition 5). Therefore $f|_A$ is $S_c$-compact function.

**Theorem 1:** Let $X$ and $Y$ be topological spaces and let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be a functions then

1. If $f$ is $S_c$-compact function and $g$ is $S_c^{**}$-compact function. Then $g \circ f$ is $S_c^{**}$-compact function.
2. If \( f \) is \( S_* \)-compact function and \( g \) is \( S_\text{c} \)-compact function. Then \( g \circ f \) is \( S_* \)-compact function and \( S_*^{**} \)-compact function.

3. If \( f \) is \( S_* \)-compact function and \( g \) is \( S_*^{**} \)-compact function. Then \( g \circ f \) is \( S_* \)-compact function and \( S_*^{**} \)-compact function.

4. If \( f \) is \( S_*^{**} \)-compact function and \( g \) is \( S_*^{**} \)-compact function. Then \( g \circ f \) is \( S_*^{**} \)-compact function and \( S_*^{**} \)-compact function.

5. If \( f \) is \( S_*^{**} \)-compact function and \( g \) is \( S_* \)-compact function. Then \( g \circ f \) is \( S_*^{**} \)-compact function and \( S_*^{**} \)-compact function.

6. If \( f \) is \( S_*^{**} \)-compact function and \( g \) is \( S_*^{**} \)-compact function. Then \( g \circ f \) is \( S_*^{**} \)-compact function.

7. If \( f \) is \( S_*^{**} \)-compact function and \( g \) is \( S_*^{**} \)-compact function. Then \( g \circ f \) is \( S_*^{**} \)-compact function.

**Definition 13:** Let \( X \) be a topological space and \( W \subseteq X \), then \( W \) is said to be \( S_* \)-compactly closed set if \( W \cap K \) is \( S_* \)-compact set in \( X \) for every \( S_* \)-compact set \( K \) in \( X \).

**Example 3:** Any subset of discrete space is \( S_* \)-compactly closed set.

**Remark 7:** Every \( S_* \)-closed set is \( S_* \)-compactly closed set, but the converse is no true in general, as in the following example.

**Example 4:** Let \( (X, T) \) be indiscrete space, then any proper subset of \( X \) is \( S_* \)-compactly closed set, but it is not \( S_* \)-closed set (since only \( S_* \)-closed set in \( X \) is \( \emptyset \) and \( X \)).

**Theorem 2:** Let \( (X, T) \) be a topological space and let \( A \) be semi-regular subset of \( X \), then if \( A \) is \( S_* \)-compactly closed set in \( X \), then the inclusion function \( i: A \rightarrow X \) is \( S_* \)-compact function.

**Proof:** Let \( A \) be \( S_* \)-compactly closed set, to show that \( i: A \rightarrow X \) is \( S_* \)-compact function. Let \( K \subseteq X \) be semi-compact set thus \( A \) is \( S_* \)-compact set, to prove \( i^{-1}(K) \) is \( S_* \)-compact set in \( A \). Since \( A \) is \( S_* \)-compactly closed set in \( X \), thus \( A \cap K \) is \( S_* \)-compact set in \( X \). But \( i^{-1}(K) = A \cap K \) is \( S_* \)-compact set in \( X \). And since \( A \) is semi-regular set in \( X \), then \( i^{-1}(K) \) is \( S_* \)-compact set in \( A \). Therefore \( i: A \rightarrow X \) is \( S_* \)-compact function.

**References**


