On m-light mappings

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ABSTRACT
In this work we introduce m-disconnected and m-totally disconnected spaces, also we study m-totally disconnected, m-light and m-monotone functions, some propositions and remarks about this concept have been given. Finally different examples are taken to consolidate our results.

INTRODUCTION
Throughout this paper by $m_X$-space we mean an m-structure space $(X, m_X)$. Which introduced by Maki-H in [2] the author introduce the concept m-structure (A subfamily $m_X$ of the power set $P(X)$ of a nonempty set X is called a minimal structure (briefly m-structure) if $\phi \in m_X$ and $X \in m_X$. Each member of $m_X$ is said to be $m_X$-open set and the complement of an $m_X$-open is said to be $m_X$-closed set. We denoted $(X, m_X)$ be the m-structure space also the author study its basic properties to this concept. The author in [3] define m-compact spaces these are the spaces in which every $m_X$-open cover has finite sub cover. An m-structure space X is called m-continuum if X is m-compact and m-connected.

Remark 1:
Every topological space is an m-structure space but the converse may be not true in general.

Example 1:
Let $X=\{1,2,3\}$ and $T=\{\phi, X, \{1\}, \{2\}\}$ is m-structure but not topology since $\{1\} \cup \{2\} = \{1,2\} \notin T$.

Definition 1:
Let $(X, m_X)$ be m-structure and let A, B be $m_X$-open nonempty sets of X, then $A \cup B$ is said to be m-disconnection to X if and only if $A \cup B = X$, $A \cap B = \phi$.

Definition 2:
An m-structure space X is said to be m-disconnected if there is m-disconnection $A \cup B$ to X.

So m-structure space X is called m-connected if it is not m-disconnected.

Remarks 2:
1- If the set is both $m_X$-open and $m_X$-closed, then we say that its $m_X$-clopen set.
2- We say that $m_0$ is the discrete m-structure space, $n_{ad}$ is the indiscrete m-structure space and $m_{cof}$ is cofinite m-structure space like $(R, m_D)$, $(R, m_{ind})$ and $(R, m_{cof})$.
3- By ($\Rightarrow$) we mean onto function.

Lemma 1:
An m-structure space X is an m-disconnected if it has nonempty proper subset which is $m_X$-clopen set.

Proof:
Let A be a nonempty proper m-structure subset of X which is $m_X$-clopen.
To prove X is m-disconnected let $B = A^c$, then B is nonempty (since A is proper m-structure subset of X) moreover, $A \cup B = X$ and $A \cap B = \phi$.
Since A is $m_X$-clopen that is, A is $m_X$-closed so B is $m_X$-open X is m-disconnected.

Definition 3:
An m-structure space X is said to be m-totally disconnected if for every pair of distinct points a, b CX X has an m-disconnection $A \cup B$ to X such that $a \in A$ and $b \in B$. 

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Remark 3:
Every discrete space is m-totally disconnected.

Example 2:
Let R be the set of all real number with discrete space, then for every two distinct points p and q we have \{p\} and R-{p} are two m-open sets containing p and q respectively.

Remark 4:
Every m-totally disconnected is m-disconnected but the converse may not be true in general.

Example 3:
Consider the m-structure space \(m_X=\{f, \{a\}, \{b, c\}, X\}\) on \(X=\{a, b, c\}\) then the space \((X, m_X)\) is m-disconnected space.
Since \(\{a\}\) is a nonempty proper subset of X which is m-open and m-closed.

Definition 4:
Let \((X, m_X)\) be m-structure space and \(Y\) be a subset of \(X\) then \(m_Y=\{U \subseteq Y: U=\bigcup_{x \in m_X} Y=\bigcup_{U \subseteq X} X \subseteq m_X\}\) so \(m_Y\) relative m-structure and the pair \((Y, m_Y)\) is said to be m-structure subspace.

Example 4:
Let \(X=\{1, 2, 3\}\) and let \(m_X=\{\phi, X, \{1\}, \{2\}\}\)
\(Y_1=\{2, 3\}\), \(m_{Y_1}=\{\phi, Y, \{2\}\}\)
\(Y_2=\{1, 2\}\), \(m_{Y_2}=\{\phi, Y_2, \{1\}, \{2\}\}\).

In a topological space the property of space being m-totally disconnected in a hereditary property .now by the same context we can show it in m-structure space.

Proposition 1:
Let \(X\) be m-structure space and let \(Y \subseteq X\) if \(X\) is m-totally disconnected then \(Y\) is also m-totally disconnected space.

Proof:
Let \(a, b\) be different points in \(Y\) but \(Y \subseteq X\), then \(a, b \in X\), which is m-totally disconnected, then there exists disconnection \(A \cup B\) to \(X\) such that \(a \in A\) and \(b \in B\), so \(a \in A \cap Y\) and \(b \in B \cap Y\).
But \((A \cap Y) \cup (B \cap Y) = (A \cup B) \cap Y = X \cap Y = Y\)
\((A \cap Y) \cap (B \cap Y) = (A \cap B) \cap Y = \phi \cap Y = \phi\).
Then there exists m-disconnection \(A \cup B\) to \(Y\), such that \(A \cap Y \neq \phi\) and \(B \cap Y \neq \phi\) are \(m_X\)-open sets in \(Y\).
Then \(Y\) is also m-totally disconnected subspace.

Definition 5:
Let \((X, m_X)\) and \((Y, m_Y)\) be two m-structure and let \(f:X \to Y\) be a function, so \(X\) is an m-structure space and \(Y\) is an m-structure space, then \(f\) is m-continuous if and only if the inverse image under \(f\) of every \(m_Y\)-open set in \(Y\) is \(m_X\)-open set in \(X\) [4].

Remark 5:
If a function \(f\) from an m-structure space \(X\) into an m-structure space \(Y\) is m-continuous, then we say that \(f\) is an m-mapping for example:

Example 5:
Let a function \(f:(X, m_X) \to (Y, m_Y)\) such that \(f(x)=x\) for each \(x \in X\), where \(X \neq \phi\).

Definition 6:
An m-mapping \(f: X \to Y\) is said to be m-totally disconnected mapping if and only if for every m-totally disconnected \(U \subseteq X\), \(f(U)\) is m-totally disconnected in \(Y\).

Remark 6:
An m- continuous image of m-totally disconnected is not necessary m-totally disconnected.

Example 6:
let \(I:(X, m_D) \to (X, m_{tot})\) where \(X\) has more than one point such that \(I(x)=x\) clearly, \(I\) is m-continuous , then \((X, m_D)\) is m-totally disconnected but its image \((X, m_{tot})\) is not m-totally disconnected.

Definition 7:
The m- structure spaces \((X, m_X)\) and \((Y, m_Y)\) are called m-homeomorphic if there exists a function \(f:X \to Y\) for which \(f\) is an m-open mapping .In this case \(f\) is called an m-homeomorphism [1].

Proposition 2:
Let \(X\) and \(Y\) are m-structure spaces and let \(f:X \to Y\) be m-homeomorphism .
If \(X\) is m-totally disconnected, then \(Y\) is also m-totally disconnected.

Proof:
Let \(y_1\) and \(y_2 \in Y\) with \(y_1 \neq y_2\), but \(f\) is onto m- mapping, then there exist only two points \(x_1, x_2 \in X\) such that \(f(x_1)=y_1\) and \(f(x_2)=y_2\), also \(X\) is m-totally disconnected.
Then there exists m-disconnection \(A \cup B\) to X such that \(f(x_1)=y_1\) and \(f(x_2)=y_2\), also \(X\) is m-totally disconnected.
Then there exists m-disconnection \(A \cup B\) to X such that \(x_1 \in A\) and \(x_2 \in B\), but \(A\) and \(B\) are \(m_X\)-open sets in \(X\) and \(f\) is m-homeomorphism.
Then \(f(A)\) and \(f(B)\) are \(m_Y\)-open sets in \(Y\) and \(f(A) \cup f(B) = f(A \cup B) = f(X) = Y\), but \(f\) is onto function, \(f(A) \cap f(B) = f(A \cap B) = f(\phi) = \phi\) such that \(y_1 \in f(A)\), \(y_2 \in f(B)\).
Then \(f(A) \cup f(B)\) is m-disconnection to \(Y\).
Therefore \(Y\) is m-totally disconnected.

Definition 8:
An m-mapping \(f:X \to Y\) is said to be m-inversely totally disconnected space if and only if for every m-totally disconnected set \(U \subseteq Y\), \(f^{-1}(U)\) is m-totally disconnected set in \(X\).
Definition 9:  
Let X and Y be two m-structure spaces. An m-mapping \( f: X \rightarrow Y \) is said to be m-light mapping if \( f^{-1}(y) \) is m-totally disconnected for each \( y \in Y \) as the following example:

Example 7:  
Let \( f:(X, m_X) \rightarrow (Y, m_Y) \) where \( m_Y \) is any m-structure space, such that \( f(x)=c \) for each \( x \in X \) and \( c \) is constant.  
\[
\begin{align*}
  f^{-1}(Y) &= \begin{cases} 
  \emptyset & \text{if } y \neq c \\
  X & \text{if } y = c
  \end{cases} \\
\end{align*}
\]
But \( \emptyset \) and \( X \) are m-totally disconnected then \( f^{-1}(y) \) is m-totally disconnected for each \( y \in Y \).
So \( f \) is m-light mapping.

Remark 7:  
Every m-homeomorphism is m-light mapping but the converse may be not true in general, as in the following example:

Example 8:  
Let \( X = \{a, b, c\} \) and \( Y = \{d, e\} \). Let \( f:(X, m_X) \rightarrow (Y, m_Y) \) be m-mapping define by the following \( f(a)=f(b)=e, f(c)=d \).  
Then \( f \) is m-light mapping, but \( f \) is not m-homeomorphism since \( f \) is not bijective m-mapping, since \( f(a)=f(b) \) but \( a \neq b \).

Proposition 3:  
Let \( f:X \rightarrow Y \) is m-inversely totally disconnected mapping then \( f(U) \) is m-totally disconnected in \( Y \) whenever \( U \) is m-totally disconnected in \( X \).

Proof:  
Let \( U \) be m-totally disconnected to prove \( f(U) \) is m-totally disconnected.  
Let \( y_1, y_2 \in f(U) \), there exist m-disconnection to \( f(U) \). But \( f \) is onto \( \Rightarrow \) there exists \( x_1, x_2 \in U \) such that \( f(x_1)=y_1 \) and \( f(x_2)=y_2 \).  
But \( U \) is m-totally disconnected, there exists m-disconnection to \( U \) such that \( \sim U=U \cap B=\emptyset \).  
\( x_1 \in A \) and \( x_2 \in B, y_1=f(x_1)=f(A) \) and \( y_2=f(x_2)=f(B) \).  
\( \sim A \cap B=\emptyset \Rightarrow f(U)=f(A \cup B)=f(A) \cup f(B) \) (since \( f \) is onto)  
\( \sim A \cap B=\emptyset \Rightarrow f(A \cap B)=f(\emptyset)=f(A) \cap f(B)=\emptyset \).  
Then \( f \) is m-totally disconnected.

Proposition 4:  
Let \( f:X \rightarrow Y \) and \( g:Y \rightarrow K \) be m-mappings if \( f \) is m-inversely totally disconnected and \( g \) is m-light mapping, then an m-mapping \( h:X \rightarrow K \) such that \( h=g \circ f \) is an m-light mapping.

Proof:  
To prove \( h \) is m-light mapping.  
Let \( k \in K \) but \( g \) is m-light mapping, then \( g^{-1}(k) \) is m-totally disconnected set in \( Y \). Also \( f \) is m-inversely totally disconnected, then \( f^{-1}(g^{-1}(k)) \) is m-totally disconnected. But \( \sim g^{-1}(k) \Rightarrow (g \circ f)^{-1}(k)=h^{-1}(k) \), so \( h \) is m-light mapping.

Remark 8:  
If \( h=g \circ f \) is m-light mapping it is not necessary \( f \) and \( g \) are both m-light mappings in general.

Example 9:  
Let \( X=\{1,2,3\} \) and \( y=\{1,2\} \), let \( f:(X, m_X) \rightarrow (X, m_{mod}) \) and \( g:(Y, m_Y) \rightarrow (Y, m_{mod}) \) such that \( g(1)=1, g(2)=g(3)=2 \) where \( m_{mod} \) is any m-structure space.  
Then \( f \) is identity mapping, then \( g \circ f \) is m-light mapping, but \( g \) is not m-light mapping, since \( g \) is not onto m-light mapping.

Proposition 5:  
Let \( h:(X, m_X) \rightarrow (Y, m_Y) \) be m-mapping, \( h=g \circ f \) such that each of \( f:(X, m_X) \rightarrow (Z, m_Z) \) and \( g:(Z, m_Z) \rightarrow (Y, m_Y) \) are m-mappings where \( X, Y \), and \( Z \) are m-structure spaces then:  
1. If \( g \) onto m-mapping and \( f \) is m-light mapping, then \( h \) is m-light mapping.
2. If \( h \) is m-light mapping and \( g \) is one to one m-mapping, then \( f \) is m-light mapping.
3. If \( h \) is m-light mapping and \( f \) is onto m-totally disconnected then \( g \) is m-light mapping again.

Proof:  
1.-let \( y \in Y \)  
Since \( g \) is onto m-mapping then there exists one and only one point \( z \in Z \) such that \( g(z)=y \).  
But \( h^{-1}(y)=(g \circ f)^{-1}(y)=f^{-1}(g^{-1}(y))=f^{-1}(g^{-1}(g(z)))=f^{-1}(z) \).  
But \( f \) is m-light mapping, then \( f^{-1}(z) \) is m-totally disconnected in \( X \).  
Also \( h^{-1}(y)=f^{-1}(z) \), then \( h^{-1}(y) \) is m-totally disconnected.  
So we get \( h \) is m-light mapping.
2.-let \( z \in Z \), then \( g(z) \in Y \) but \( h \) is m-light mapping, then \( h^{-1}(g(z)) \) is m-totally disconnected in \( X \).  
Let \( h^{-1}(g(z))=(g \circ f)^{-1}(g(z))=f^{-1}(g^{-1}(g(z)))=f^{-1}(z) \) (since \( g \) is an onto m-mapping).  
So \( f^{-1}(z) \) is m-totally disconnected in \( X \), then \( f \) is m-light mapping.
3.-let \( y \in Y \)  
Since \( h \) is m-light mapping, so \( h^{-1}(y) \) is m-totally disconnected in \( X \).  
Also \( f \) is m-totally disconnected, then \( f(h^{-1}(y)) \) is m-totally disconnected set in \( Z \).  
But \( f(h^{-1}(y))=f(g \circ f)^{-1}(y)=f(g^{-1}(y))=g^{-1}(y) \) (since \( f \) is an onto m-mapping).  
So \( g^{-1}(y) \) is m-totally disconnected in \( Z \).  
Then \( g \) is m-light mapping.

Proposition 6:  
Let \( f_1:X_1 \rightarrow Y_1 \) and \( f_2:X_2 \rightarrow Y_2 \) be m-mappings so a mapping \( f_1 \times f_2:X_1 \times X_2 \rightarrow Y_1 \times Y_2 \) is m-light mapping if \( f_1 \) is m-homeomorphism and \( f_2 \) is m-light mapping.

Proof:  
Let \( (y_1, y_2) \in Y_1 \times Y_2 \)  
So \( (f_1 \times f_2)^{-1}(y_1, y_2)=(f_1^{-1} \times f_2^{-1})(y_1, y_2)=(f_1^{-1}(y_1)) \times f_2^{-1}(y_2) \)  
But \( f_1 \) is m-homeomorphism, so there exist \( x_1 \in X_1 \) such that \( f_1^{-1}(y_1)=f_1^{-1}(f_1(x_1))=x_1 \) \( \Rightarrow \)  
I mean \( (f_1 \times f_2)^{-1}(y_1, y_2)=x_1 \times f_2^{-1}(y_2) \)}
Proposition 7:

Let \( f: (X, m_X) \rightarrow (Y, m_Y) \) be m-continuous function, then if \( X \) is m-connected, then \( Y \) is also m-connected.

Proof:

Let \( (X, m_X) \) be an m-connected space and \( (Y, m_Y) \) be any m\(_Y\) space.

Let \( f: (X, m_X) \rightarrow (Y, m_Y) \) is m-continuous and onto.

To prove \( (Y, m_Y) \) is m-connected, suppose \( (Y, m_Y) \) is m-disconnected.

There exist \( U, V \) are m\(_Y\)-open in \( Y \) such that \( U \cap V = \emptyset \) and \( U \cup V = Y \).

So \( f^{-1}(U) \) and \( f^{-1}(V) \) are m\(_X\)-open sets in \( X \) (since \( f \) is m-continuous).

\[ f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset \]

\[ f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = f^{-1}(Y) = X \]

But \( U \cap V = \emptyset \) (since \( f \) is onto)

Then \( f^{-1}(a) \cap f^{-1}(b) = f^{-1}(U) \) so \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \) (since \( f \) is onto).

So \( (X, m_X) \) is m-disconnected (contradiction)

Then \( (Y, m_Y) \) is m-connected.

Remark 9:

Every m-continuous mapping is m-connected mapping but the converse may be not true in general for example:

Example 12:

Let \( I_R: (R, m_{ind}) \rightarrow (R, m_{ad}) \) is m-connected mapping.

Since every m-connected set \( U \) in \( (R, m_{ind}) \), \( U \cap V = \emptyset \) is also m-connected set in \( (R, m_{ad}) \) because \( I(U) \) cannot be separated by two nonempty disjoint m\(_X\)-open sets.

To prove \( I_R \) is not m-continuous mapping.

Let \( R \) be \( y \) and \( R \) be \( x \) in \( (R, m_{ad}) \).

Since \( I_R \) is the identity function, so \( I_R \) is m-connected but not m-continuous.

REFERENCES


