Some Types of Lindelof in Bitopological Spaces

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Abstract:
In this paper, we define another types of Lindelof on bitopological space, namely N-Lindelof , S-Lindelof and pair-wise Lindelof spaces, and we introduce some properties about these types.

Key words : Lindelof , Bitopological Spaces

Introduction:
In 1963, the term of bitopological space was used for the first time by Kelly [1]. A set equipped with two topologies is called a bitopological space and denoted by (X,τ,τ′), where (X,τ) and (X,τ′) are two topological spaces. In 2010 N.A.Jabbar and A.I.Nasir introduced N-open set [2]. A subset A of a bitopological space (X,τ,τ′) is called an N-open set if and only if it is open in the space (X,τ ∨ τ′), where τ ∨ τ′ is the supremum topology on X contains τ and τ′. In this paper, we introduce the concept of N-Lindelof space. A bitopological space (X,τ,τ′) is said to be an N-Lindelof space if and only if every N-open cover of X has a countable subcover, we study some properties of this kind of Lindelof space. Also we introduce the concept of S-Lindelof and pair-wise Lindelof. We also study the relationships among the three kinds of Lindelof spaces.

1. N-Lindelof Space
In this section, we give the definitions of N-open set, N-Lindelof space in bitopological spaces.

open cover of X which is an N-Lindelof space .Therefore, there exists a countable number of \( \{ U_{\alpha} : \alpha \in \Delta \} \) \( X - A \) \( \cup \{ U_{\alpha} : i \in \Delta \cap \mathbb{N} \} \) is a countable subcover of X . Since A \( \subseteq X \) and X - A covers no part of A , then \( \{ U_{\alpha} : i \in \Delta \cap \mathbb{N} \} \) is a countable subcover of A . So A is N-Lindelof set .

1.1 Definition [1]
Let X be a non-empty set, let \( \tau, \tau' \) be any two topologies on X , then (X,τ,τ′) is called bitopological space.

1.2 Definition [2]
A subset A of a bitopological space (X,τ,τ′) is called an N-open set if and only if it is open in the space (X,τ ∨ τ′), where τ ∨ τ′ is the supremum topology on X contains τ and τ′.

1.3 Definition [2]
The complement of an N-open set in a bitopological space (X,τ,τ′) is called N-closed set.

1.4 Remark
Let (X,τ,τ′) be a bitopological space, then:
Every open set in (X,τ) or in (X,τ′) is an N-open set in (X,τ,τ′).
Every closed set in (X,τ) or in (X,τ′) is an N-closed set in (X,τ,τ′).

1.5 Note:
The opposite direction of remark 1.4 may be untrue as the following example shows:
1.11 Definition [2]
A function \( f : (X,\tau,\tau') \rightarrow (Y,T,T') \) is said to be an N-continuous function if and only if the inverse image of each N-open subset of Y is an N-open subset of X .
1.12 Theorem
The N-continuous image of an N-Lindelof space is an N-Lindelof space .

Proof:
Let (X,τ,τ′) be an N-Lindelof space, and let \( f : (X,\tau,\tau') \rightarrow (Y,T,T') \) be an N- continuous, onto function. To show that (Y,T,T') is an N-Lindelof space ,let \( \{ U_{\alpha} : \alpha \in \Delta \} \) be an N-open
cover of \(Y\), then \(\{f^{-1}(U_{a}) : \alpha \in \Delta\} \) is an N-open cover of \(X\), which is N-Lindelof space. So there exists a countable number of \(\{f^{-1}(U_{a}) : \alpha \in \Delta\} \) such that the family \(\{f^{-1}(U_{a}) : i \in \Delta \subset N\} \) covers \(X\) and since \(f\) is onto, then \(\{U_{a} : i \in \Delta \subset N\} \) is a countable subcover of \(Y\). Hence \(Y\) is N-Lindelof space.

2. S-Lindelof Space

In this section, we give the definition of S-Lindelof space, then we study pair-wise Lindelof space in bitopological spaces and relations between them and the N-Lindelof.

2.1 Definition [3]

A subset \(A\) of a bitopological space \((X, \tau, \tau')\) is said to be S-open set if it is \(\tau\)-open or \(\tau'\)-open. The complement of the S-open set is called S-closed set.

Example

Let \(X = \{1, 2, 3\}\), \(\tau = \{\phi, \{1\}, X\}\) and \(\tau' = \{\phi, \{2\}, X\}\) then \(\tau \cup \tau' = \{\phi, \{1\}, \{2\}, \{1, 2\}, X\}\) is the family of all N-open subsets of \((X, \tau, \tau')\). \([1, 2]\) is an N-open set in \((X, \tau, \tau')\) but it is not open in both \((X, \tau)\) and \((X, \tau')\). So \([3]\) is an N-closed set in \((X, \tau, \tau)\) which is not closed in both \((X, \tau)\) and \((X, \tau')\).

1.6 Definition [2]

Let \((X, \tau, \tau')\) be a bitopological space, let \(A\) be a subset of \(X\). A subcollection of the family \(\{U_{a} : \alpha \in \Delta\} \) \(\cap \{X - A\} \cup \{U_{a} : i \in \Delta \subset N\}\) is a countable subcover of \(X\). Since \(A \subset X\) and \(X - A\) covers no part of \(A\), then \(\{U_{a} : i \in \Delta \subset N\}\) is a countable subcover of \(A\). So \(A\) is S-Lindelof set.

2.8 Definition [4]

Let \(f : (X, \tau, \tau') \to (Y, T, T')\) be a function. Then \(f\) is said to be a bicontinuous function if and only if \(f^{-1}(U) \in \tau\) for each \(U \in T\), and \(f^{-1}(V) \in \tau'\) for each \(V \in T'\).

2.9 Example

Let \(X = \{1, 2, 3\}\), \(\tau = \{\phi, X, \{1\}, \{2\}, \{1, 2\}\}\) and \(\tau' = \tau_{D}\)

Let \(Y = \{a, b, c\}\), \(T = \{\phi, Y, \{a\}\}\) and \(T' = \tau_{I}\). Define \(\tau \cup \tau'\) is called an N-open cover of \(A\) if the union of members of this collection contains \(A\).

1.7 Definition

A bitopological space \((X, \tau, \tau')\) is said to be an N-Lindelof space if and only if every N-open cover of \(X\) has a countable subcover.

1.8 Corollary

If \((X, \tau, \tau')\) is an N-Lindelof space. Then both \((X, \tau)\) and \((X, \tau')\) are Lindelof spaces.

Proof:

Follows from remark 1.4.

1.9 Corollary

If \(\tau\) is a subfamily of \(\tau'\). Then \((X, \tau, \tau')\) is an N-Lindelof space if and only if \((X, \tau)\) and \((X, \tau')\) are Lindelof spaces.

Proof:

Let \((X, \tau, \tau')\) be an N-Lindelof space and let \(A\) be an N-closed subset of \(X\). To show that \(A\) is an N-Lindelof set. Let \(\{U_{a} : \alpha \in \Delta\}\) be an N-open cover of \(A\). Since \(A\) is N-closed subset of \(X\), then \(X - A\) is an N-open subset of \(X\), so \(\{X - A\} \cup \{U_{a} : \alpha \in \Delta\}\) is an N-open cover of \(X\). Then \(f(1) = a, f(2) = b, f(3) = c\). Then \(f\) is bicontinuous function. Where \(\tau_{D}\) and \(\tau_{I}\) are the discrete and indiscrete topologies on \(X\) and \(Y\) respectively.

2.10 Lemma

A bicontinuous image of an S-Lindelof space is an S-Lindelof space.

Proof:

Let \(f : (X, \tau, \tau') \to (Y, T, T')\) be a bicontinuous, onto function and let \((X, \tau, \tau')\) be an S-Lindelof space. To show that \((Y, T, T')\) is an S-Lindelof, let \(\{U_{a} : \alpha \in \Delta\}\) be an S-open cover of \(Y\), then \(\{f^{-1}(U_{a}) : \alpha \in \Delta\}\) is an S-open cover of \(X\), which is an S-Lindelof space. Therefore there exists a countable...
number of \( \{ f^{-1}(U_{\alpha}) : \alpha \in \Delta \} \) such that the family
\( \{ f^{-1}(U_{\alpha}) : i \in \Delta \subset N \} \) covers \( X \) and since \( f \) is onto.
then \( \{ U_{\alpha} : i \in \Delta \subset N \} \) is a countable subcover of \( Y \). Hence \( Y \) is S-Lindelof space.

2.11 Corollary
Every N-Lindelof space is an S-Lindelof space.
Proof:
Follows from remark 2.2.

2.2 Remark
Every S-open (S-closed) set in bitopological space \((X, \tau, \tau')\) is an N-open (N-closed) set.
The converse of the remark 2.2 need not be true, see the
equation of note (1.5), where the set \( \{1,2\} \) is N-open set
which is not S-open. So \( \{3\} \) is N-closed set which is not S-
closed set.

2.3 Definition [3]
Let \( (X, \tau, \tau') \) be a bitopological space, let \( A \) be a subset
of \( X \). A subcollection of the family \( \tau \cup \tau' \) is called an
S-open cover of \( A \) if the union of members of this
collection contains \( A \).

2.4 Definition
A bitopological space \((X, \tau, \tau')\) is called an S-Lindelof
if and only if every S-open cover of \( X \) has a countable
subcover.

2.5 Corollary
If \( (X, \tau, \tau') \) is an S-Lindelof space. Then both \( (X, \tau) \)
and \( (X, \tau') \) are Lindelof spaces.
Proof:
Clear.

2.6 Corollary
If \( \tau \) is a subfamily of \( \tau' \). Then \( (X, \tau, \tau') \) is an S-
Lindelof space if and only if \( (X, \tau) \) and \( (X, \tau') \) are
Lindelof spaces.
Proof:
Clear.

2.7 Lemma
The S-closed subset of an S-Lindelof space is S-Lindelof.
Proof:
Let \( (X, \tau, \tau') \) be an S-Lindelof space and let \( A \) be an
S-closed subset of \( X \). To show that \( A \) is an S-Lindelof
set. Let \( \{ U_{\alpha} : \alpha \in \Delta \} \) be an S-open cover of \( A \).
Since \( A \) is S-closed subset of \( X \), then \( X - A \) is an S-
open subset of \( X \), so \( \{ X - A \} \cup \{ U_{\alpha} : \alpha \in \Delta \} \) is
an S-open cover of \( X \) which is an S-Lindelof space.
Therefore, if \( \tau \) is a subfamily of \( \tau' \) and \( (X, \tau') \) is a
Lindelof space, then \( (X, \tau, \tau') \) is a pair-wise Lindelof space.

Proof:
Suppose that \( \tau \) is a subfamily of \( \tau' \) and \( (X, \tau', \tau') \) be a
Lindelof space. Then by corollary (1.9) \( (X, \tau, \tau') \) is an
N-Lindelof and by corollary (2.11) \( (X, \tau, \tau') \) is S-
Lindelof. Therefore \( (X, \tau, \tau') \) is a pair-wise Lindelof.

2.19 Lemma
If \( (X, \tau) \) and \( (X, \tau') \) are Lindelof spaces. Then
\( (X, \tau, \tau') \) is S-Lindelof if and only if it is a pair-wise
Lindelof.
Proof:
Necessities, follows from corollary (2.17).

Sufficiency, suppose \( (X, \tau, \tau') \) is a pair-wise Lindelof
space, to prove it is an S-Lindelof space, let
\( \{ U_{\alpha} : \alpha \in \Delta \} \) be an S-open cover of \( X \), then there are
three probabilities:
If \( \{ U_{\alpha} : \alpha \in \Delta \} \) is a \( \tau \)-open cover, since \( (X, \tau) \) is
Lindelof space, then \( \{ U_{\alpha} : \alpha \in \Delta \} \) has a countable
subcover of \( X \), so \( X \) is an S-Lindelof.
If \( \{ U_{\alpha} : \alpha \in \Delta \} \) is a \( \tau' \)-open cover, since \( (X, \tau') \) is
Lindelof space, then \( \{ U_{\alpha} : \alpha \in \Delta \} \) has a countable
subcover of \( X \), so \( X \) is an S-Lindelof.
If \( \{ U_{\alpha} : \alpha \in \Delta \} \) is a pair-wise open cover, since
\( (X, \tau, \tau') \) is a pair-wise Lindelof space, then
\( \{ U_{\alpha} : \alpha \in \Delta \} \) has a countable subcover, so \( X \) is an S-
Lindelof.

From the above three probabilities we have \( X \) is an S-
Lindelof.

2.12 Corollary
Let \( (X, \tau, \tau') \) be a bitopological space. If \( \tau \) is a
subfamily of \( \tau' \). Then the concepts of S-Lindelof and N-
Lindelof are coincident.
Proof:
Follows from corollary (1.9) and corollary (2.6).

2.13 Definition [3]
Let \( (X, \tau, \tau') \) be a bitopological space and let \( A \subseteq X \). An S-open cover of \( A \) is called a pair-wise
open cover if it contains at least one non-empty element
from \( \tau \) and at least one non-empty element from \( \tau' \).

2.14 Example
Let \( X = \{1,2,3\} \) and \( \tau = \{\emptyset, X, \{1\}\} \).
Let \( \tau' = \{\emptyset, X, \{2\}, \{3\}, \{2,3\}\} \).
Then the cover \( C = \{\{1\}, \{2\}, \{3\}\} \) is a pair-wise open
cover of \( X \).

2.15 Remark
It follows from the definition (2.13) that every pair-wise
open cover of the bitopological space \((X, \tau, \tau')\) is an S-open cover.
The converse of the remark 2.15 need not be true, for example [2]:

Let \(X = \{1, 2, 3\}\), \(\tau = \{\phi, X, \{1\}\}\).

Let \(\tau' = \{\phi, \{2\}, \{3\}, \{2, 3\}, \{1, 2\}, X\}\).

Then the cover \(C = \{\{1, 2\}, \{3\}\}\) is an S-open cover of \(X\), but it is not pair-wise open cover.

2.16 Definition

A bitopological space \((X, \tau, \tau')\) is called a pair-wise Lindelof space if every pair-wise open cover of \(X\) has a countable subcover.

2.17 Corollary

Every S-Lindelof space is a pair-wise Lindelof space.

Proof:

Follows from remark (2.15).

References


بعض أنواع فضاءات ليندلوف على الفضاءات الثنائية

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الخلاصة:

في هذا البحث ، قمنا بتعريف أنواع أخرى من فضاءات ليندلوف على الفضاءات الثنائية أسسناها فضاءات ليندلوف - \(\alpha\) وفضاءات ليندلوف - \(S\) وفضاءات ليندلوف - \(N\) ودراسة بعض خواص هذه الفضاءات والعلاقة بينها.