Polynomials Over Splitting Fields

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Abstract : - In this paper we study some results concerning the existence of splitting fields which are generated by roots of polynomials. Also we study the roots of cubic polynomials.

Key words : Polynomials , Over Splitting Fields

Introduction and preliminaries
These results are basic to Galois theory consider the polynomial ring $K[X]$ over field K .Let $f(x)$ belong to $K[X]$ in the quotient ring $K[X]/f(x)$. We let $g(x)$ denotes the coset $(g(x)+f(x))$. Thus if $g(x)=\sum_{i=0}^{n}K_{i}x^{i}$, then by the definition of addition and multiplication of cosets we have that $g(x)=\sum_{i=0}^{n}K_{i}x^{i}$, we considered a field K contains in a complex numbers $C$ and a cubic polynomial $f(x)=x^{3}+px+q\in K[X]$ . Also, we obtained explicit expression involving extraction of square and cubic roots for the three roots $\alpha_{1},\alpha_{2}$ and $\alpha_{3}$ of $f(x)$ in C and we were beginning to study the splitting field extension $E=K(\alpha_{1},\alpha_{2},\alpha_{3})$ . If $f(x)$ factors in $K[X]$ either all the roots are in K or exactly one of them (say $\alpha_{3}$) is in K and the other two roots of irreducible quadratic polynomial in $K[X]$ . In this case $E=K(\alpha_{1})$ is a field extension of dimension 2 over K . Therefore if $\alpha_{1}$ denotes one of the roots, we know that $K(\alpha_{1})=K(\alpha_{1})/f(x)$ is a field extension of dimension 3 is irreducible in $R[X]$ . Now, $R[X]=\{a+bx | a,b \in R\}$ is a field where $x=x+(x^{2}+1)$ . Since $x^{2}=-1$ , we may call C the field of the complex numbers.

Definition . [2]
A polynomial $f(x)$ belong to $K[X]$ is said to split over a field S contains K,if $f(x)$ can be write it factor as product of linear a factors in $S[X]$,such that K is a field.

Remark .[1]
$\delta=(\alpha_{1}-\alpha_{2})(\alpha_{2}-\alpha_{3})\in E$ ,since $\delta^{2}=-4p^{3}-27q^{2}\in K$ ,either K or $K(\delta)$ is an extension field of dimension 2 over K, since $K \subseteq K(\delta) \subseteq E$ it follows that 2 also divides dim$_{K}(E)$.

$\delta \in K$ and dim$_{K}(E)=3$ or $\delta \notin K$ and dim$_{K}(E)=6$.

Proposition .[ 4 ]
Let K be a field . If $f(x)$ is a non-constant polynomial in $K[X]$, then there exists a field extension $F/K$ such that $F$ contains a root of $f(x)$ .

Now by the following we can show that $C$ is the field of complex numbers $\{x^{2}+1\}$ is irreducible in $R[X]$ . Now, $R[X]=\{a+bx | a,b \in R\}$ is a field where $x=x+(x^{2}+1)$ . Since $x^{2}=-1$ , we may call C the field of the complex numbers.

Definition . [5]
Let K be a field . A polynomial $f(x)\in K[X]$ is said to split over a field $S \supseteq K$ if $f(x)$ can be factored as a product of line a factors in $S[x]$ .

A field S containing K is said to be a splitting field for $f(x)$ over K if $f(x)$ splits over S but over no proper intermediate field of $S/K$ . For example The field of complex numbers C is a splitting field for the polynomial $x^{2}+1$ over R .this
follows, since \( x^2 + 1 = (x + i)(x - i) \) in \( \mathbb{C}[x] \), and \( \mathbb{C}/\mathbb{R} \) has no proper intermediate field because \( [\mathbb{C}:\mathbb{R}] = 2 \). Now if \( C \supseteq L \supseteq R \) where \( L \) is an intermediate field of \( \mathbb{C}/\mathbb{R} \), then \( 2 = [C:R] = [L:R] \) and so either \( [C:L] = 1 \) or \( [L:R] = 1 \). Then either \( C=L \) or \( C=R \) and note that \( C \) is the splitting field of \( x^2 + 1 \) over \( \mathbb{Q} \) since \( x^2 + 1 \) splits over \( \mathbb{Q} \) (L).

**Proposition [5]**

Let \( K \) be a field and \( f(x) \) be a polynomial in \( K[x] \) of degree \( n \). Let \( F/K \) be a field extension. If \( f(x) = (x-c_1)(x-c_2)\cdots(x-c_n) \) in \( F(x) \), then \( c_1 \) is a splitting field for \( f(x) \) over \( K \).

Also, if we have \( K \) a finite field. Then cardinality of \( K \) is \( p^n \) for some prime \( p \) and some positive integer \( n \). Every \( k \) belong to \( K \) is a root of the polynomial \( X \) and \( K \) is the splitting field of this polynomial over prime subfield \( \mathbb{Z}_p \).

Therefore, if the roots are known as \( a_1 \) and \( a_2 \) then The field \( Q(\lambda_1, \lambda_2) \) for the last example is a splitting field for \( x^3 - 3x^2 + 1 \) over \( \mathbb{Q} \).

Now we can say that if \( K \) be a field and \( f(x) \) be a constant polynomial over \( K \). Then there is a splitting field for \( f(x) \) over \( K \), and if \( E/K \) is a field extension and \( f(x) \) is an irreducible polynomial in \( K[x] \). If \( a, b \in E \) are roots of \( f(x) \) then \( K(a) \cong K(b) \).

Also, we can use other concept to obtain splitting field by normal extension such that (if a finite extension \( E \) to \( K \) is normal, then it is a splitting field over \( K \) and \( f(x) \) belong to \( K[X] \)).

Therefore, if \( E/L \) and \( L/K \) be a finite extensions and if \( E/K \) is normal then \( E/L \) is normal (\( E/L \) is splitting). Now we can give the following fact about two splitting fields [Let \( f(x) \in K[x] \)]. Any two splitting fields for \( f(x) \) over \( K \) are isomorphic.

**Examples**

1. The field \( Q(\sqrt{2}) = \{a+b\sqrt{2} : a, b \in \mathbb{Q}\} \) is a splitting field of \( x^2 - 2 \in \mathbb{Q}[x] \) over \( \mathbb{Q} \).

2. A splitting field of \( x^2 + 1 \in \mathbb{R}[x] \) over \( \mathbb{R} \) is the field \( \mathbb{C} \).

**Proposition [2]**

If \( K \) is field and \( f \in K[x] \), then:

There exists splitting field of polynomial; \( f \) on \( K \).

Any two splitting fields of \( f \) on \( K \) are two isomorphism fields on \( K \).

Splitting fields are unique up to isomorphism over \( K \).

**Proposition [3]**

Let \( K \) be a subfield of \( C \) let \( f(x) = x^3 + px^2 + q \in K[x] \) an irreducible cubic polynomial and let \( E \) denotes the splitting field of \( f(x) \) in \( C \). Let \( \delta = (a_1-a_2)(a_1-a_3)(a_2-a_3) \) where \( a_i \) are the roots of \( f(x) \). If \( \delta \in K \), then \( \dim_{K} (E)=6 \).

**Proposition [1]**

Suppose \( K \subseteq L \) is any field extension \( f(x) \in K[x] \) and \( \beta \) is the root of \( f(x) \) in \( L \). If \( \delta \) is an automorphism of \( L \) leaves \( F \) fixed pointwise, then \( \delta(\beta) \) is also a root of \( f(x) \). Proof:

If \( f(x) = \sum f_{i}x^{i} \) and since \( \beta \) is one of the roots that is mean \( f(\beta) = 0 \) then \( \sum f_{i}\delta(\beta)^{i} = \delta(0) = 0 \).

**Example**

Let \( f(x) = x^3 - 2 \), which is irreducible over \( \mathbb{Q} \).

The three roots of \( f \) in \( C \) are \( \sqrt[3]{2} \), \( \omega \sqrt[3]{2} \) and \( \omega^{2} \sqrt[3]{2} \), where \( \omega = \frac{1}{2} + \frac{\sqrt{-3}}{2} \) is a primitive cube root of 1.

Finally, to show that the splitting fields always exist[for if \( g \) is any irreducible factor of \( f \), then \( K[x]/(g) = K(\alpha) \) is an extension of \( K \) for which \( g(\alpha) = 0 \), where \( \alpha \) denotes the image of \( X \). Then \( g \) and \( f \) are splits off a linear factor, induction implies that exists a splitting field \( L \) for \( f \).

**Conclusions**

We got that a polynomial \( f(x) \in K[x] \) always has a splitting field, namely the field generated by its roots in a given algebraic closure \( \overline{K} \) of \( K \). Also we can apply these roots of any non-constant polynomials by Galois theory. We obtained a new result (every normal extension is splitting field, and splitting fields are unique, let \( K \) be a field by a root of polynomials \( f(x) \in K[x] \) we mean an element \( \alpha \) in an over field of \( K \) such that \( f(\alpha) = 0 \). It is easy to see that a non-zero polynomial in \( K[x] \) of degree \( n \) has most \( n \) roots.

**References**


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الخلاصة
قدنا في هذا البحث بدراسة بعض النتائج المتعلقة بوجود الحقل المنفصل الذي يتولد عن طريق جذور متحدثات الحدود. كذلك قدنا بدراسة نوع واحد من هذه الجذور وهي الجذور التكعيبية.