About Asymmetric Noisy Chaotic Models

Prof. PhD. Salah Hamza Abid
Hassan Mazin Hassan

Al-Mustansiriya University, College of Education, Department of Mathematics

In this paper we will introduce three families of asymmetric maps, and discuss some dynamical properties for these families in the deterministic case, and noisy case. New mixed noisy chaotic map will suggest and then study with some details.

I : Basic concepts

Let \((\Omega, \mathcal{F}, \mathbb{P})\) denoted a probability space, and let \((\vartheta^t)_{t \in \mathbb{Z}^+: \Omega \to \Omega}\) be a map such that \(\vartheta^t\) is measurable, \(\vartheta \mathbb{P} = \mathbb{P}\), \(\vartheta = \text{id}_\Omega\), \(\vartheta^{n+m} = \vartheta^n \circ \vartheta^m\), and all \(B \in \mathcal{F}\) with \(\vartheta^{-1}(B) = B\) then \(\mathbb{P}(B) \in \{0, 1\}\), we will define the
ergodic process $\xi_n(\omega) = \xi(\delta^n(\omega))$, $\omega \in \Omega$, $n \in \mathbb{Z}^+$, where $\xi: \Omega \to \mathbb{R}$ is measurable function.

**Definition 1.1**: [1]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denoted a probability space then the **random dynamical system** is define as a random difference equation $x_{t+1}(\omega) = \varphi(t, x_t, \xi_t(\omega))$, where $t \in \mathbb{Z}^+, \varphi: \mathbb{Z}^+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ and $\xi_t(\omega)$ defined above. such that

i) $\varphi(\cdot, x, \cdot, \xi_.(\omega)) = \varphi(\cdot, \varphi(x, \cdot, \xi_.(\omega)))$

ii) $\varphi(n + m, x, \cdot, \xi_.(\omega)) = \varphi(n, \varphi(m, x, \cdot, \xi_.(\omega)))$

**Definition 1.2**: [6].

A **random fixed point** $p(\omega)$ of a random dynamical system $\varphi$ on $\mathbb{R}$ is a random variable $p(\omega): \Omega \to \mathbb{R}$ such that

$$p(\omega) = \varphi(\cdot, p(\omega), \xi_.(\omega))$$

almost surely (a.s.)

The random fixed point $p(\omega)$ is attracting with probability $P$ if $P(\lim_{t \to \infty} |x_t(\omega) - p(\omega)| = 0) = P$, for all $x_.(\omega) \in (p(\omega) - \varepsilon, p(\omega) + \varepsilon)$ for some $\varepsilon > 0$.

If $P = 1$ (i.e. $\lim_{t \to \infty} |x_t(\omega) - p(\omega)| = 0$, a.s.) then $p(\omega)$ is called **globally attracting** [6].

**Lemma 1.1**: let $p(\omega)$ be a random fixed point to the random dynamical system $\varphi$ on $\mathbb{R}$, and $\varphi$ partial differentiable at $p(\omega)$ then

$$P(\text{p(\omega) is attracting}) = P\left(\left|\frac{\partial \varphi(t, p, \xi_t(\omega))}{\partial x}\right| < 1\right)$$
II: Some dynamical properties in deterministic case

II.1: The family of asymmetric tent map

The dynamical system for the asymmetric tent map can be defined as follows:

\[ x_{t+1} = T_{a,\mu}(x_t) = \begin{cases} \frac{x_t}{\mu} & \text{if } 0 \leq x_t \leq a \\ \frac{a - x_t}{\mu} & \text{if } a < x_t \leq 1 \end{cases} \quad \text{...(1)} \]

Where \( T_{a,\mu} : [\cdot, 1] \to [\cdot, 1] \) and \( 0 \leq \mu \leq 1, \ 0 < a < 1 \).

**Theorem 2.1:** For a dynamical system in (1).

1) If \( \mu < a \) then the system has one attracting fixed point \( p = \cdot \).
2) If \( \mu = a \) then all points of the interval \([\cdot, a]\) represent fixed points to the system and all points from the interval \((a, 1]\) represent eventually fixed points and it is orbit have only two points.
3) If \( a < \mu < 1 - a \) then the system has two fixed points \( p_1 = \cdot \) 
replying and \( p_r = \frac{\mu}{\mu + (1-a)} \) attracting.

4) If \( \mu > a \) and \( \mu > 1 - a \) then there are no any attracting fixed or 
periodic point and the system becomes chaotic, see Lyapunov exponent figure (1).

**Proof:** Clear.

The satisfaction of the above properties can be seen from the 
bifurcation diagrams figure (2).

![Figure 1: Lyapunov exponent of asymmetric tent map](image)

- (a) \( a = 0.5 \)
- (b) \( a = 0.6 \)
- (c) \( a = 0.7 \)
- (d) \( a = 0.8 \)
The dynamical system for asymmetric logistic map can be defined as follows:

$$x_{t+1} = G_{a, \mu}(x_t) = \begin{cases} \frac{x_t}{a}(1 - \frac{x_t}{a}) & \text{if } \cdot \leq x_t \leq a \\ \frac{x_t + \gamma - \gamma a}{\mu(\frac{x_t + \gamma - \gamma a}{1 - \gamma a} - \frac{1 - x_t}{1 - \gamma a})} & \text{if } a < x_t \leq \gamma a \end{cases}$$

Where $G_{a, \mu} : [\cdot, \gamma a] \rightarrow [\cdot, \gamma a]$ and $0 < a < k$ $\cdot \leq \mu \leq \gamma$.

**Theorem 2.2**: For a dynamical system in (2)

1) If $\mu < \gamma a$ then the system has an attracting fixed point $p_1 = \cdot$.
2) If $\gamma a < \mu \leq \xi a$ then the system has two fixed points $p_1 = \cdot$ is repelling and $p_2 = \frac{\gamma a \mu - \xi a \gamma}{\mu}$ is attracting and $p_1, p_2 \in [\cdot, a]$.
3) If $\gamma a < \mu < \gamma a + \sqrt{\xi a \gamma + (\gamma - \gamma a) \sqrt{\gamma}}$ then the second fixed point
\[ p_r = \frac{\frac{\gamma \mu}{(\gamma - \mu)^2} - \sqrt{\left(\frac{\gamma \mu}{(\gamma - \mu)^2}\right)^r + \left(\frac{\mu^2 - \gamma \mu}{(\gamma - \mu)^2}\right)^r}}{\frac{\gamma \mu}{(\gamma - \mu)^2}} \] is attracting and \( p_r \in [\cdot, \cdot] \).

**proof**: Clear.
Figure (4) : bifurcation diagram of asymmetric logistic map family, x-axes represent the values of μ and y-axes represent the values of $x_{t+1}$ with $x_{t+1} \in [0,1]$.  
(a) $\alpha = \cdot$, (b) $\alpha = \cdot$,  
(c) $\alpha = \cdot$, (d) $\alpha = \cdot$,  
(e) $\alpha = \cdot$, (f) $\alpha = \cdot$,  
(g) $\alpha = \cdot$, (h) $\alpha = \cdot$.  

II.3 : The family of mixed logistic-tent map

The dynamical system for the mixed logistic-tent map can be defined as follows:

$$x_{t+1} = GT_{a,\mu}(x_t) = \begin{cases} \frac{\mu x_t}{\gamma a} & \text{if } \gamma \leq x_t \leq a \\ \frac{\mu}{\gamma (1-a)} & \text{if } a < x_t \leq 1 \end{cases}$$

Where $GT_{a,\mu} : [\gamma, 1] \to [\gamma, 1]$ and $\gamma < a < 1$, $\gamma \leq a \leq 1$.

**Theorem 2.3** : For a dynamical system in (3)

1. If $\mu < \gamma a$ then the system has one attracting fixed point $p = \gamma$.
2. If $\gamma a < \mu < \gamma (1-a)$ then the system has two fixed points $p_{\gamma} = \gamma$ is repelling and $p_{\mu} = \frac{\gamma a - \mu}{\mu} is attracting$, and $p_{\gamma}, p_{\mu} \in [\gamma, a]$.
3. If $\mu = \gamma a$ then the second fixed point $p_{\gamma} = a$ is attracting.
4. If $\gamma (1-a) < \mu < \gamma (1-a)$ then $p_{\mu} = \frac{\mu}{\mu + \gamma (1-a)}$ is attracting, and $p_{\mu}$ repelling if $\mu > \max \{\gamma a, \gamma (1-a)\}$.

**Proof** :

The proofs of (1), (2) and (4) are clear.

To prove (3), $p_{\mu} = \frac{\gamma a - \mu}{\mu} = \frac{\gamma a - \mu}{\gamma a} = a$

To show that $p_{\mu}$ is attracter we must find open interval $J = (p_{\mu} - \varepsilon, p_{\mu} + \varepsilon)$, $\varepsilon > 0$, such that for any $x_0 \in J$ then the orbit of $x_t \to p_{\mu}$, as $t \to \infty$.

Claim : $J = (\gamma, 1)$
At first let $x_0 \in (\cdot, a)$ since $GT_{a, a}(x_t) \leq GT_{a, a}(a) = a$, for every $x_t \in (\cdot, a)$, then $\{x_t\}$ bounded from above and $a$ represent $\sup(\{x_t\})$, and since $GT_{a, a}(x_t) = \frac{a}{1-a} \left(1 - \frac{x_t}{a}\right) > 0$ for every $x_t \in (\cdot, a)$ then $\{x_t\}$ increasing, that implies $\{x_t\}$ converge to $a$.

To complete the proof we need to show for any $x_0 \in (a, 1)$ then $x_0 \in (\cdot, a)$.

$$x_{t+1} = GT_{a, a}(x_t) = a \left(1 - \frac{x_t}{1-a}\right),$$ since $\cdot < a < x_t < 1$ then $\cdot < 1 - x_t < 1 - a < 1$, and hence $\frac{1-x_t}{1-a} < 1$ which implies to $\cdot < x_{t+1} = a \left(1 - \frac{x_t}{1-a}\right) < a$.

\[\]

![Figure (5): Lyapunov exponent of mixed logistic-tent map](image)

(a) (b) (c) (d) (e) (f) (g) (h) (i) (j) (k) (l) (m) (n) (o) (p) (q) (r) (s) (t) (u) (v) (w) (x) (y) (z)
Table (1) : Some properties of the asymmetric tent, asymmetric logistic and mixed logistic-tent map

<table>
<thead>
<tr>
<th>Nu.</th>
<th>Property</th>
<th>map</th>
<th>value of “µ”</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>1</td>
<td>Value of &quot;µ&quot; which Fixed point becomes unstable .</td>
<td></td>
<td>3.3240</td>
</tr>
<tr>
<td></td>
<td>A. L.</td>
<td></td>
<td>0.9000</td>
</tr>
<tr>
<td>2</td>
<td>Value of &quot;µ&quot; which Chaotic region begins .</td>
<td></td>
<td>3.8510</td>
</tr>
<tr>
<td></td>
<td>A. L.</td>
<td></td>
<td>0.9000</td>
</tr>
<tr>
<td>3</td>
<td>Value of &quot;µ&quot; which cycle with period 3 appears .</td>
<td></td>
<td>3.9905</td>
</tr>
<tr>
<td></td>
<td>A. L.</td>
<td></td>
<td>0.995</td>
</tr>
<tr>
<td></td>
<td>M. L-T</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>Value of &quot;µ&quot; which Chaotic region ends .</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>A. L.</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>A. T.</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>Are there stable cycles in the chaotic region ?</td>
<td></td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>A. L.</td>
<td></td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>A. T.</td>
<td></td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>M. L-T</td>
<td></td>
<td>Yes</td>
</tr>
</tbody>
</table>
III : Some dynamical properties in noisy case.

All models treated with in previous section were defined on closed interval $[0,1]$ , but in the noisy case it is not true generally

III.1 : The family of noisy asymmetric tent map .

The noisy dynamical system for asymmetric tent map with additive noise is given by:

$$x_{t+1}(\omega) = T_{a,\mu}(x_t, \xi_t(\omega)) = \begin{cases} \frac{\mu x_t}{a} + \xi_t(\omega) & \text{if } \cdot \leq x_t \leq a \\ \frac{\mu - x_t}{a -\cdot} + \xi_t(\omega) & \text{if } a < x_t \leq 1 \end{cases} \quad (i)$$

Where $T_{a,\mu}: \mathbb{R} \times \Omega \to \mathbb{R}$, $\{\xi_t(\omega)\}$ is a sequence of independent identically distributed (iid) random variables and $\cdot \leq a \leq 1, \cdot \leq \mu \leq 1$

Theorem 4 : For the dynamical system (4) , where $\xi_t(\omega) \sim Gauss(\cdot,\sigma^2)$,

1) If $\mu < a$ then the system has one random fixed point $p(\omega) \sim Gauss(\cdot,\sigma^2)$, and
\( P_r(p(\omega) \text{ is attracting}) = P_r(\xi(\omega) < a), \text{ if } \mu > 1 - a, \text{ and} \\
P_r(p(\omega) \text{ is attracting}) = 1 \text{ if } \mu < 1 - a \)

2) If \( \mu = a \) then there are infinite number of random fixed points
\( p(\omega) \sim \text{Gauss}(p^*, \sigma^*), p^* \in [\cdot, a]. \)

3) If \( a < \mu < 1 - a \) then the system has two random fixed points
\( p_1(\omega) \sim \text{Gauss}(\cdot, \sigma^*) \text{ and } p_2(\omega) \sim \text{Gauss}(\frac{\mu}{\mu+(1-a)}, \sigma^*), \)
where
\( P_r(p_1(\omega) \text{ is attracting}) = P_r(\xi(\omega) > a) \text{ and} \\
P_r(p_2(\omega) \text{ is attracting}) = P_r(\xi(\omega) > a - \frac{\mu}{\mu+(1-a)}). \)

4) If \( \mu > a \) and \( \mu > 1 - a \) then for any random fixed (periodic) point \( p(\omega), P_r(p(\omega) \text{ is attracting}) = 0, \text{ and the system become noisy chaotic}. \)

**Proof:**

The proofs of (1), (2) and (3) are clear.

To prove (4), since
\[
\frac{\partial \tilde{T}_{a, \mu}(x_t, \xi_t(\omega))}{\partial x_t} = \begin{cases} \\
\frac{\mu}{a} & \text{if } x_t < a \\
\frac{\mu}{1-a} & \text{if } x_t > a \\
\end{cases}
\]

Then by lemma 1 any random fixed (periodic) point \( p(\omega) \) is attracter with probability 0.

To show that the system is noisy chaotic we will calculate stochastic Lyapunov exponent (SLE) as follows. [2]

\[
\lambda_{\omega}(x_\omega) = \lim_{n \to \infty} - \frac{1}{n} \sum_{j=1}^{n-1} \log \left| \frac{\partial \tilde{T}_{a, \mu}(x_j, \xi_j(\omega))}{\partial x_j} \right|
\]
And since \[ \left| \frac{\partial \xi_j(\omega)}{\partial \xi_i} \right| > 1 \] then \( \lambda_s(\chi) > 1 \) and the system is noisy chaotic.

Example 1:

Let \( \xi_j(\omega) \sim Gauss(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot) \), for system (4)

Case 1: let \( \mu = \cdot, a = \cdot \) then by theorem (4 - 1), the system has one random fixed point \( p(\omega) \sim Gauss(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot) \) and

\[ P_r(p(\omega) \text{ is attracting}) = P_r(\xi(\omega) < \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) = P_r(z(\omega) < \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) \cong 1, \]

where \( z(\omega) \sim Gauss(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot), \) and the SLE of the system is about \( \lambda_s = -\cdot,\cdot,\cdot,\cdot,\cdot,\cdot \).

Case 2: let \( \mu = a = \cdot, 7 \) then by theorem (4-2), the system has an infinity number of random fixed points \( p(\omega) \sim Gauss(p^*, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) \), for all points \( p^* \in [\cdot, a] \), and the SLE of the system is about

\[ \lambda_s = -\cdot,\cdot,\cdot,\cdot,\cdot,\cdot \times 1, \cdot,\cdot,\cdot,\cdot,\cdot,\cdot. \]

Case 3: let \( \mu = \cdot, a = \cdot \) then by theorem (4-3), the system has two random fixed points \( p_1(\omega) \sim Gauss(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot), \) \( p_r \sim Gauss(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot) \) and

\[ P_r(p(\omega) \text{ is attracting}) = P_r(\xi(\omega) > \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) = P_r(z(\omega) > \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) \cong 1, \]

\[ P_r(p(\omega) \text{ is attracting}) = P_r(\xi(\omega) < \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) = P_r(z(\omega) > \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) = \cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\...
Note that Stochastic Lyapunov exponent of the system (4) and deterministic Lyapunov exponent of the system (1) are the same, because the partial derivative of system (4) is not dependent on $x_t(\omega)$.

The satisfaction of the above properties can be seen from the bifurcation diagrams figure (7).
III.2: The family of noisy asymmetric logistic map.

The noisy dynamical system for asymmetric logistic map with additive noise is given by:

\[
x_{t+1}(\omega) = \mathcal{G}_{a,\mu}(x_t, \xi_t(\omega)) = \begin{cases} 
\mu \frac{x_t}{1-a} \left(1 - \frac{x_t}{1-a}\right) + \xi_t(\omega) & \text{if } \ast \leq x_t \leq a \\
\mu \left(x_t + \gamma a\right) \left(1 - \frac{x_t + \gamma a}{1-a}\right) + \xi_t(\omega) & \text{if } a < x_t \leq 1
\end{cases}
\]

Where \( \mathcal{G}_{a,\mu}: \mathbb{R} \times \Omega \rightarrow \mathbb{R} \), \( \{\xi_t(\omega)\} \) is a sequence of (iid) random variables and \( \ast \leq a \leq 1 \), \( \ast \leq \mu \leq \gamma \).

Theorem 5: For the dynamical system (5), where \( \xi_t(\omega) \sim \text{Gauss}(\cdot, \sigma) \).

1) If \( \mu < \gamma a \), then the system has one random fixed point

\[
P_r(p_1(\omega) \text{ is attracting}) = P_r(\xi(\omega) < a) \ast P_r \left( a - \frac{\gamma a}{\mu} < p_1(\omega) < a + \frac{\gamma a}{\mu} \right) + P_r(\xi(\omega) > a) \ast P_r \left( a - \frac{(1-\gamma a)}{\gamma} < p_1(\omega) < a + \frac{(1-\gamma a)}{\gamma} \right)
\]

\( p_1(\omega) \sim \text{Gauss}(\cdot, \sigma) \), and...
2) If \( a < \mu < \xi a \) then the system has another random fixed point

\[ p_\gamma (\omega) \sim Gauss \left( \frac{\gamma a \mu - \xi a}{\mu}, \sigma^r \right) \]

where

\[ P_r (p_\gamma (\omega) \text{ is attracting}) = P_r \left( \xi (\omega) < a - \frac{\gamma a \mu - \xi a}{\mu} \right) \cdot P_r \left( \frac{\gamma a}{\mu} < a + \frac{\gamma a}{\mu} \right) + P_r \left( \xi (\omega) > a - \frac{\gamma a \mu - \xi a}{\mu} \right) \cdot P_r \left( \frac{\gamma a}{\mu} < a + \frac{\gamma a}{\mu} \right) \]

3) If \( \xi a < \mu \) then the second random fixed point

\[ p_\gamma (\omega) \sim Gauss \left( \left( \frac{\gamma a \mu}{(\gamma - \theta)^{-1}} \right) - \sqrt{\left( \frac{\gamma a \mu}{(\gamma - \theta)^{-1}} \right)^2 + \left( \frac{\mu - \gamma a \mu}{(\gamma - \theta)^{-1}} \right)^2}, \sigma^r \right) \]

is attracting with probability

\[ P_r (p_\gamma (\omega) \text{ is attracting}) = P_r \left( \frac{\gamma a}{(\gamma - \theta)^{-1}} - \sqrt{\left( \frac{\gamma a \mu}{(\gamma - \theta)^{-1}} \right)^2 + \left( \frac{\mu - \gamma a \mu}{(\gamma - \theta)^{-1}} \right)^2} \right) \cdot P_r \left( \frac{\gamma a}{\mu} < a + \frac{\gamma a}{\mu} \right) + P_r \left( \frac{\gamma a}{(\gamma - \theta)^{-1}} - \sqrt{\left( \frac{\gamma a \mu}{(\gamma - \theta)^{-1}} \right)^2 + \left( \frac{\mu - \gamma a \mu}{(\gamma - \theta)^{-1}} \right)^2} \right) \cdot P_r \left( \frac{\gamma a}{\mu} < a + \frac{\gamma a}{\mu} \right) \]

**Proof:**

Clear, by definition 2 and lemma 1
Example 2:

Let $\xi_{\omega}(\omega) \sim Gauss(\cdot, \cdot, \cdots, \cdot, \omega)$, for system (4)

Case 1: let $\mu = 1, a = 0.3$, by theorem (5-1), the system has one random fixed point $p_{1}(\omega) \sim Gauss(\cdot, \cdot, \cdots, \cdot, \omega)$, and it is attracting with probability

$$P_{r}(p(\omega) \text{is attracting}) = P_{r}(\xi(\omega) < 0.7) * P_{r}(\xi < p_{1}(\omega) < 1.3) +$$

$$P_{r}(\xi(\omega) > 0.7) * P_{r}(\xi = p_{1}(\omega) < 1.3) = P_{r}(z(\omega) < 1.3) * P_{r}(\xi < z(\omega) < 0.7) + P_{r}(z(\omega) > 1.3) * P_{r}(\xi = z(\omega) < 0.7) \cong \cdots$$

, where $z(\omega) \sim Gauss(\cdot, 1)$, and the SLE of system is about $\lambda_{z} = -1.1^{\omega}$. 

Case 2: let $\mu = 0.7, a = 0.3$, by theorem (5-2), the system has two random fixed point $p_{1}(\omega) \sim Gauss(\cdot, \cdot, \cdots, \cdot, \omega)$, and

$$P_{r}(p_{1}(\omega) \text{is attracting}) = P_{r}(\xi(\omega) < 0.7) * P_{r}(\xi < p_{1}(\omega) < 1.3) +$$

$$P_{r}(\xi(\omega) > 0.7) * P_{r}(\xi = p_{1}(\omega) < 1.3) = P_{r}(z(\omega) < 1.3) * P_{r}(\xi < z(\omega) < 0.7) + P_{r}(z(\omega) > 1.3) * P_{r}(\xi = z(\omega) < 0.7) \cong \cdots$$

, where $z(\omega) \sim Gauss(\cdot, 1)$ and the SLE of the system is about $\lambda_{z} = -1.1^{\omega}$. 

Case 3: let $\mu = 0.7, a = 0.3$, by theorem (4.5-3), the second fixed point $p_{1}(\omega) \sim Gauss(\cdot, \cdot, \cdots, \cdot, \omega)$, and

$$P_{r}(p_{1}(\omega) \text{is attracting}) = P_{r}(\xi(\omega) < 0.7) * P_{r}(\xi < p_{1}(\omega) < 1.3) +$$

$$P_{r}(\xi(\omega) > 0.7) * P_{r}(\xi = p_{1}(\omega) < 1.3) = P_{r}(z(\omega) < 1.3) * P_{r}(\xi < z(\omega) < 0.7) + P_{r}(z(\omega) > 1.3) * P_{r}(\xi = z(\omega) < 0.7) \cong \cdots$$

, where $z(\omega) \sim Gauss(\cdot, 1)$ and the SLE of the system is about $\lambda_{z} = -1.1^{\omega}$.
In this example the approximate value of $\mu_{nc} = 3.23$.  

Figure (8) : S.L.E. for noisy asymmetric logistic map with additive Gauss (0 , 0.0025) noise , x_ axes represent the values of $\mu$ and y_ axes represent the values of S.L.E. , (a) $\mu = \Upsilon$ , (b) $\alpha = \Upsilon, i$ , (c) $\alpha = \Upsilon, \Lambda$ , (d) $\alpha = \Upsilon, \lambda$ .

The satisfaction of the above properties can be seen from the bifurcation diagrams figure (9).
Figure (9) : bifurcation diagram of noisy asymmetric logistic map family with additive noise Gauss (0, 0.0025), x-axes represent the values of $\mu$ and y-axes represent the values of $x_{t+1}$.
III.3 : The family of noisy mixed logistic-tent map.

The noisy dynamical system for mixed logistic-tent map with additive noise is given by:

\[ x_{t+1}(\omega) = \hat{G}_{a,\mu}^{\xi}(x_t, \xi_t(\omega)) = \begin{cases} \mu \frac{x_t}{a} \left( 1 - \frac{x_t}{a} \right) + \xi_t(\omega) & \text{if } \cdot \leq x_t \leq a \\ \mu \left( 1 - \frac{x_t}{\xi(t-a)} \right) + \xi_t(\omega) & \text{if } a < x_t \leq 1 \end{cases} \]

Where \( \hat{G}_{a,\mu}^{\xi} : \mathbb{R} \times \Omega \rightarrow \mathbb{R} \), \( \{\xi_t(\omega)\} \) sequence of (iid) random variables and \( \cdot \leq a \leq 1, \cdot \leq \mu \leq \xi \).

**Theorem 4.6** : For the dynamical system (6), if \( \xi_t(\omega) \sim Gauss(\cdot, \sigma^\gamma) \).

1) If \( \mu < \gamma a \), then the system has one random fixed point

\( p_\gamma(\omega) \sim Gauss(\cdot, \sigma^\gamma) \), and

\[ P_\gamma(p_\gamma(\omega) \text{ is attracting}) = P_\gamma(\xi(\omega) < a) \times P_\gamma\left( a - \frac{\gamma a^\gamma}{\mu} < p_\gamma(\omega) < a + \frac{\gamma a^\gamma}{\mu} \right) + P_\gamma(\xi(\omega) > a) \]

if \( \mu < \xi(1 - a) \), and if \( \mu > \xi(1 - a) \) then

\[ P_\gamma(p_\gamma(\omega) \text{ is attracting}) = P_\gamma(\xi(\omega) < a) \times P_\gamma\left( a - \frac{\gamma a^\gamma}{\mu} < p_\gamma(\omega) < a + \frac{\gamma a^\gamma}{\mu} \right) \]

2) If \( \gamma a < \mu \leq \xi a \) then the system has another random fixed points

\( p_\gamma(\omega) \sim Gauss\left( \frac{\gamma a^\gamma - \gamma a}{\mu}, \sigma^\gamma \right) \), and

\[ P_\gamma(p_\gamma(\omega) \text{ is attracting}) = P_\gamma\left( \xi(\omega) < a - \frac{\gamma a^\gamma - \gamma a}{\mu} \right) \times P_\gamma\left( a - \frac{\gamma a^\gamma}{\mu} < p_\gamma(\omega) < a + \frac{\gamma a^\gamma}{\mu} \right) + P_\gamma(\xi(\omega) > a - \frac{\gamma a^\gamma - \gamma a}{\mu}) \]

if \( \mu < \xi(1 - a) \) and if \( \mu > \xi(1 - a) \), then
3) If \( \zeta a < \mu < \zeta (1 - a) \) then the second random fixed point
\[
p_r(\omega) \sim \text{Gauss}\left(\frac{\mu}{\mu + i(\zeta - a)}, \sigma^\prime\right)
\]
is attracting with probability
\[
P_r(p_r(\omega) \text{ is attracting }) = P_r\left(\xi(\omega) < a - \frac{\tau a \mu - i a^\prime}{\mu}\right) \times P_r\left(a - \frac{\tau a}{\mu} < p_r(\omega) < a + \frac{\tau a}{\mu}\right)
\]
and
\[
P_r(p_r(\omega) \text{ is attracting }) = P_r\left(\xi(\omega) < a - \frac{\mu}{\mu + i(\zeta - a)}\right) \times P_r\left(a - \frac{\tau a}{\mu} < p_r(\omega) < a + \frac{\tau a}{\mu}\right)
\]
if \( \mu > \max\{\zeta a, \zeta (1 - a)\} \).

**Proof:**

The proofs of (1) and (3) is in the similar way for (2)

To prove (2), let
\[
p_r(\omega) = \tilde{\sigma} T_{a, \mu}\left(p, \xi(\omega)\right) = \mu \frac{p}{\tau a} \left(1 - \frac{p}{\tau a}\right) + \xi(\omega) = \frac{\tau a \mu - i a^\prime}{\mu} + \xi(\omega), \text{ since } \tau a < \mu \leq \zeta \alpha
\]

Implies \( p_r(\omega) \sim \text{Gauss}\left(\frac{\tau a \mu - i a^\prime}{\mu}, \sigma^\prime\right) \)

Now to find probability of \( p_r(\omega) \) is attracter

\[
P_r(p_r(\omega) \text{ is attracting })
\]
\[
= P_r(p_r(\omega) < a) \times P_r(p_r(\omega) \text{ is attracting } p_r(\omega) < a) + P_r(p_r(\omega) > a) \times P_r(p_r(\omega) \text{ is attracting } p_r(\omega) > a)
\]
We know that if \( \mu < \xi (1 - a) \) then 
\[ P_r(p_r(\omega) \text{ is attracting } \ p_r(\omega) > a) = 1 \]

Then if \( \mu < \xi (1 - a) \)

\[ P_r(p_r(\omega) \text{ is attracting }) = \]
\[ P_r\left(\xi(\omega) < a - \frac{r_{a\mu-1}a^\gamma}{\mu}\right) \ast P_r\left(a - \frac{r_{a\mu-1}a^\gamma}{\mu} < p_r(\omega) < a + \frac{r_{a\mu-1}a^\gamma}{\mu}\right) + P_r\left(\xi(\omega) > a - \frac{r_{a\mu-1}a^\gamma}{\mu}\right) \]

, by lemma 1

And if \( \mu > \xi (1 - a) P_r(p_r(\omega) \text{ is attracting } \ p_r(\omega) > a) = \ast \) and hence 

\[ P_r(p_r(\omega) \text{ is attracting }) = \]
\[ P_r\left(\xi(\omega) < a - \frac{r_{a\mu-1}a^\gamma}{\mu}\right) \ast P_r\left(a - \frac{r_{a\mu-1}a^\gamma}{\mu} < p_r(\omega) < a + \frac{r_{a\mu-1}a^\gamma}{\mu}\right) \]

Example 3:

Let \( \xi_t(\omega) \sim Gauss(\cdot, \cdot, \cdot, \cdot, \cdot) \), for system (6)

Case 1: let \( \mu = 1, a = \cdot, \cdot, \cdot \), by theorem (6-1), the system has on random fixed point \( p(\omega) \sim Gauss(\cdot, \cdot, \cdot, \cdot, \cdot) \), and it is attracting with probability 
\[ P_r(p(\omega) \text{is attracting}) = \]
\[ P_r(\xi(\omega) < \cdot, \cdot, \cdot) \ast P_r(\cdot, \cdot, \cdot, \cdot < \cdot, \cdot, \cdot) \ast P_r(\xi(\omega) > a) = P_r(z(\omega) < \cdot, \cdot, \cdot, \cdot) \ast P_r(\cdot, \cdot, \cdot, \cdot < \cdot, \cdot, \cdot) \ast P_r(\xi(\omega) > a) \]

, where \( z(\omega) \sim Gauss(\cdot, \cdot) \) and the SLE of system is about \( \lambda_s = -\cdot, \cdot, \cdot, \cdot \)

Case 2: let \( \mu = \cdot, a = \cdot, \cdot \) by theorem (6-2), the system has another random fixed point \( p_r(\omega) \sim Gauss(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot) \), and
\[ P_r(p_r(\omega) \text{ is attracting}) = P_r(\xi(\omega) < .12) \times P_r(.34 < p_r(\omega) < .91) = \]
\[ P_r(z(\omega) < .04) \times P_r(-.85 < z(\omega) < .91) \cong .9918 \]
where \( z(\omega) \sim Gauss(.1, .1) \) and the SLE of the system is about \( \lambda_s = -1.2144 \).

Case 3: let \( \mu = .5, a = .7, \) by theorem (6-3) in above the second random fixed point \( p_r(\omega) \sim Gauss(.59, .59, \ldots, .59) \), and
\[ P_r(p_r(\omega) \text{ is attracting}) = P_r(\xi(\omega) < .91) \times P_r(.31 < p_r(\omega) < .818) = \]
\[ P_r(z(\omega) < .19) \times P_r(-.956 < z(\omega) < .674) = .477690 \]
and the SLE of the system is about \( \lambda_s = -1.9315 \).
In this example the approximate value of \( \mu_{nc} = \gamma_{1}\gamma_{2}\gamma_{3} \).

The satisfaction of the above properties can be seen from the bifurcation diagrams figure (11).
Figure (11): Bifurcation diagram of noisy mixed logistic-tent map family with additive noise Gauss (0, 0.0025). x-axes represent the values of μ and y-axes represent the values of $x_n(\omega)$. (a) $a = 0.7$, (b) $a = 0.9$, (c) $a = 1.0$, (d) $a = 1.1$, (e) $a = 1.3$, (f) $a = 1.5$, (g) $a = 1.7$, (h) $a = 1.9$. 
References:


