إنشاء الأقواس الكاملة 
في الفضاء الإسقاطي ثلاثي الأبعاد

حوت حقل كالوا (4)

فاطمة فيصل كريم
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المستخلص

في هذا البحث، قمنا بانتشار الأقواس الإسقاطية المختلفة في الفضاء الإسقاطي (3,4) حول حقل كالوا (4) عندما 5 ≤ n ≤ 21. ووجدنا الأقواس الكاملة عندما k ≥ 5. كذلك برهنا k أن بعض القوس (4) (85,21) ينتمي إلى هذه القوس (4). 

هندسياً ان أعظم قوس كامل في (3,4) هو القوس PG(3,4) من النقاط لا يوجد منها على استقامة واحدة. القوس الكامل (k,n) هو القوس الغير محتوى في القوس (k+1,n).
The Construction of Complete (k,n)-arcs in 3-Dimensional Projective Space Over Galois Field GF(4)

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Abstract
In this work, we construct the projectively distinct (k,n)-arcs in PG(3,4) over Galois field GF(4), where k \geq 5, and we found that the complete (k,n)-arcs, where 3 \leq n \leq 21, moreover we prove geometrically that the maximum complete (k,n)-arc in PG(3,4) is (85,21)-arc. A(k,n)-arcs is a set of k points no n+1 of which are collinear. A(k,n)-arcs is complete if it is not contained in a (k+1,n)-arcs.

1. Introduction
Ahmed A.M., Al-Mukhtar A.Sh., Kadhum S.J., (2002), [1], gives the maximum arcs in the projective plane PG(2,7) over Galois field GF(7), Ismael N.A., (2005), [2] give the complete (k,n)-arcs in the projective plane PG(2,13) and Al-Mukhtar A.Sh. in (2008), [4] proved that the completeness of (k,n)-arcs in PG(3,q), where q = 2,3,5 and 3 \leq n \leq q^2 + q + 1, this paper divided into three sections, section one consists of the basic theorems and definitions of a projective 3-space PG(3,q). In section two the addition's and multiplication operations of GF(4). The construction of complete (k,n)-arcs, for 3 \leq n \leq 21 explained in section three.

1. Basic Concepts
1.1 Definition: "PG(3,q)" , [3]
A projective 3-space PG(3,q) over Galois field GF(q), where q = p^m for some prime number p and some integer m is a three-dimensional projective space which consists of points, planes and lines with incidence relation between them. PG(3,q) is satisfying the following axioms:

a. Any two distinct points are contained in a unique line.
b. Any three distinct non-collinear points, also any line and point not on it are contained in a unique plane.
c. Any two distinct coplanar lines intersect in a unique point.
d. Any line not on a given plane intersects the plane in a unique point.
e. Any two distinct planes intersect in a unique line.

Any point in PG(3, q) has the form of a quadrable \((x_1, x_2, x_3, x_4)\), where \(x_1, x_2, x_3, x_4\) are elements in GF(q) with the exception of the quadrable consisting of four zero elements.

Two quadrables \((x_1, x_2, x_3, x_4)\) and \((y_1, y_2, y_3, y_4)\) represent the same point if there exists \(\lambda\) in GF(q) \(\setminus\{0\}\) such that \((x_1, x_2, x_3, x_4) = \lambda (y_1, y_2, y_3, y_4)\). Similarly, any plane in PG(3, q) has the form of a quadrable \([x_1, x_2, x_3, x_4]\), where \(x_1, x_2, x_3, x_4\) are elements in GF(q) with the exception of the quadrable consisting of four zero elements.

Two quadrables \([x_1, x_2, x_3, x_4]\) and \([y_1, y_2, y_3, y_4]\) represent the same plane if there exists \(\lambda\) in GF(q) \(\setminus\{0\}\) such that \([x_1, x_2, x_3, x_4] = \lambda [y_1, y_2, y_3, y_4]\).

A point \(p(x_1, x_2, x_3, x_4)\) is incident with the plane \(\pi[a_1, a_2, a_3, a_4]\) iff \(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0\).

**1.2 Definition: "Plane \(\pi\", [3]\)**

A plane \(\pi\) in PG(3, q) is the set of all points \(p(x_1, x_2, x_3, x_4)\) satisfying a linear equation \(u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0\). This plane is denoted by \(\pi[u_1, u_2, u_3, u_4]\), where \(u_1, u_2, u_3, u_4\) are elements in GF(q) with the exception of the quadrable consisting of four zero elements.

**1.3 Theorem: [4]**

The points of PG(3, q) have unique forms which are \((1,0,0,0)\), \((x,1,0,0)\), \((x,y,1,0)\) and \((x,y,z,1)\) for all \(x, y, z\) in GF(q) which are \((1,0,0,0)\) is one point, \((x,1,0,0)\) are q points, \((x,y,1,0)\) are \(q^2\) points, and \((x,y,z,1)\) are \(q^3\) points, for all \(x,y,z\) in PG(q).

**1.4 Theorem: [4]**

The planes of PG(3, q) have unique forms which are \([1,0,0,0]\), \([x,1,0,0]\), \([x,y,1,0]\), \([x,y,z,1]\) for all \(x, y, z\) in GF(q). which are \([1,0,0,0]\) is one plane, \([x,1,0,0]\) are q planes, \([x,y,1,0]\) are \(q^2\) planes, and \([x,y,z,1]\) are \(q^3\) planes, for all \(x,y,z\) in PG(q).
1.5 **Corollary:** [4] 
There exists \( q^3 + q^2 + q + 1 \) of points in \( PG(3,q) \) and there exist \( q^3 + q^2 + q + 1 \) of planes.

1.6 **Theorem:** [4] 
Every plane in \( PG(3,q) \) contains exactly \( q^2 + q + 1 \) points (lines) and every point is on exactly \( q^2 + q + 1 \) planes.

1.7 **Theorem:** [4] 
Every line in \( PG(3,q) \) contains exactly \( q + 1 \) points and every point is on exactly \( q + 1 \) lines.

1.8 **Corollary:** [4] 
Any two planes in \( PG(3,q) \) intersect in exactly \( q + 1 \) points, and any two points are on exactly \( q + 1 \) planes. Also any line is on exactly \( q + 1 \) planes.

1.9 **Definition:** "(k,n)-arcs", [1] 
A \((k,n)\)-arc \( A \) in \( PG(3,q) \) is a set of \( k \) points such that at most \( n \) points of which lie in any plane, \( n \geq 3 \). \( n \) is called degree of the \((k,n)\)-arc.

1.10 **Definition:** [1] 
In \( PG(3,q) \), if \( k \) is any \( k \)-set, then an \( n \)-secant of \( k \) is a line (a plane) \( \ell \) such that \( |\ell \cap k| = n \). A 0-secant is called an external line (plane) of \( k \), a 1-secant is called a unisecant line (plane), a 2-secant is called a bisecant line and 3-secant is called a trisecant line.

1.11 **Definition:** [1] 
A point \( N \) not on a \((k,n)\)-arc has index \( i \) if there are exactly \( i \) (\( n \)-secant) of \( K \) through \( N \), one can denoted the number of point \( N \) of index \( i \) by \( C_i \).

1.12 **Remark:** [2] 
A \((k,n)\)-arc \( A \) is complete iff \( C_0 = 0 \). Thus the \( k \)-set is complete iff every point of \( PG(3,q) \) lies on some \( n \)-secant of the \((k,n)\)-set.
1.13 Definition: [2]
Let $T_i$ be the total number of the $i$-secant of a $(k,n)$-arc $A$, then the type of $A$ w.r.t. its planes denoted by $(T_n, T_{n-1}, T_{n-2}, \ldots, T_0)$. One can also say that $A$ is of type $m$ where $m = m_i$; that is $m$ is the smallest integer $i$ for which $T_i \neq 0$.

1.14 Definition: [4]
Let $(k_1,n)$-arc $A$ is of type $(T_n, T_{n-1}, \ldots, T_0)$ and $(k_2,n)$-arc $B$ of type $(S_n, S_{n-1}, \ldots, S_0)$, then $A$ and $B$ have the same type iff $T_i = S_i$, for all $i$, in this case they are projectively equivalent.

1.15 Theorem: [4]
Let $T_i$ represents the number of $i$-secants (planes) for the arc $A$ in $\text{PG}(3,q)$, that is $T_2$ is the number of bisecants, $T_1$ is the number of unisecants, and $T_0$ is the number of external line $t = p + 2 - k$, then:

1. $T_1 = k \cdot t$
2. $T_2 = \frac{k(k-1)}{2}$
3. $T_3 = \frac{k(k-1)(k-2)}{3!}$
4. $T_n = \frac{k(k-1) \cdots (k-n+1)}{n!}$
5. $T_0 = q^3 + q^2 + q + 1 - k \cdot t - \frac{k(k-1)}{2} - \frac{k(k-1)(k-2)}{3!} - \cdots - \frac{k(k-1)(k-2) \cdots (k-n+1)}{n!}$

1.16 Theorem: [4]
Let $C_i$ be the number of points of index $i$ in $\text{PG}(3,q)$ which are not on a $(k,n)$-arc $A$, then the constants $C_i$ of $A$ satisfy the following equations:

(1) $\sum_{a}^{e} C_i = q^3 + q^2 + q + 1 - k$
(2) \[ \sum_{i=\alpha}^{\beta} C_i = \frac{k(k-1)\cdots(k-n+1)}{n!} \left( q^2 + q + 1 - n \right) \]

where \( \alpha \) is the smallest \( i \) for which \( C_i \neq 0 \), \( \beta \) be the largest \( i \) for which \( C_i \neq 0 \).

1.17 Theorem: [1]

A \((k,n)\)-arc \( A \) is maximum if and only if every line in \( PG(3,q) \) is a 0-secant or \( n \)-secant.

2. The Addition's and Multiplication's Operations of \( GF(4) \): [5]

To find the addition and multiplication tables in \( GF(4) \), we have the order pairs \((x_1,x_2)\) such that \( x_1, x_2 \) in \( GF(2) \), as follows:
\[ 0 \equiv (0,0), 1 \equiv (1,0), 2 \equiv (0,1), 3 \equiv (1,1). \]
Put these points in one orbit, \((1,0)\) at the first point and by the principle of \((1,0)A \), \( i=0,1,2,3 \) and \( A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \).

\( (1,0)A = (0,1) \) and \((1,0)A^2 = (1,1)\), so \((1,0) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

Now, in the left of the following table, \( m \) is the operation of multiplication and in the right \( n \) is the operation of addition, in multiplication side we write the numeration of points as last, and the addition side takes the normal sequence.

<table>
<thead>
<tr>
<th>( m(*) )</th>
<th>((+n = f(m))</th>
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<tr>
<td>1</td>
<td>(1,0)</td>
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<td>2</td>
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In addition table, we have the following relation:
\((x_1,x_2) + (y_1,y_2) = (z_1,z_2)\) where \( z_i = (y_i + x_i) \mod (2) \), for \( i = 1, 2 \).

In multiplication table, we have the following relation
\[ m_1 \cdot m_2 = m_3 \Leftrightarrow ((1,0)A^{f(m_1)}) A^{f(m_2)} = (0,1)A^{(f(m_1)+f(m_2)) \mod 3} = (x_1, x_2) \]

For example: \(2 \cdot 3 = 1 \Leftrightarrow ((1,0)A^1) A^2 = (1,0)A^0 = (1,0)\)

where \((1,0)\) equal to 1 in multiplication side.

Now we have addition and multiplication tables:

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### 3. The Construction of Complete (k,n)-arcs in PG(3,4):

#### 3.1 A Projective Space PG(3,4):

A projective space PG(3,4) contains 85 points and 85 planes such that each point is on 21 planes and every plane contains 21 points, any line contains 5 points and it is the intersection of 5 planes. All the points, planes and lines of PG(3,4) are given in tables 1 and 2.

#### 3.2 The Construction of Complete (k,3)-arcs in PG(3,4):

Let \(A=\{1,2,6,22,43\}\) be a set of five points no four of them are on a plane, where \(1(1,0,0,0), 2(0,1,0,0), 6(0,0,1,0), 22(0,0,0,1), 43(1,1,1,1)\), which are called the reference and unit points of PG(3,4), (table 1). A is a (5,3)-arc, since A intersects any plane in at most three points. A is complete (5,3)-arc, since every point of PG(3,4) not in A is on trisecant (plane), that is there are no points of index zero for A.
The Construction of Complete (k,4)-arcs in PG(3,4):

The distinct (k,4)-arcs can be constructed by adding to A in each time one point from the remaining points of PG(3,4) as follows:

\[ A_1 = A \cup \{3\}, \quad A_2 = A \cup \{4\}, \quad A_3 = A \cup \{5\}, \quad A_4 = A \cup \{7\}, \quad A_5 = A \cup \{8\}, \quad A_6 = A \cup \{9\}, \quad A_7 = A \cup \{10\}, \quad A_8 = A \cup \{11\}, \ldots, \quad A_{22} = A \cup \{23\}, \ldots, \quad A_{43} = A \cup \{44\}, \ldots, \quad A_{84} = A \cup \{85\}. \]

By the definition (1.14), there are only four projectively distinct (6,4)-arcs, since \( A_1, A_7, A_8, A_{19}, A_{22}, A_{34}, A_{35}, A_{38} \) are projectively equivalent, for \( T_0 = 14, T_1 = 32, T_2 = 31, T_3 = 8, T_4 = 4 \), the arcs \( A_2, A_3, A_4, A_5, A_6, A_{11}, A_{15}, A_{20}, A_{23}, A_{26}, A_{30}, A_{39}, A_{40}, A_{42}, A_{46}, A_{49}, A_{54}, A_{65} \) are projectively equivalent, for \( T_0 = 14, T_1 = 32, T_2 = 27, T_3 = 9, T_4 = 3 \) the arcs \( A_9, A_{10}, A_{12}, A_{13}, A_{16}, A_{18}, A_{21}, A_{24}, A_{25}, A_{27}, A_{28}, A_{31}, A_{33}, A_{36}, A_{37}, A_{41}, A_{43}, A_{45}, A_{48}, A_{50}, A_{53}, A_{55}, A_{57}, A_{58}, A_{59}, A_{66}, A_{68}, A_{69}, A_{70}, A_{72}, A_{77}, A_{78}, A_{80} \) projectively equivalent, for \( T_0 = 15, T_1 = 29, T_2 = 28, T_3 = 1, T_4 = 2 \) and the arcs \( A_{14}, A_{17}, A_{29}, A_{32}, A_{44}, A_{47}, A_{51}, A_{52}, A_{56}, A_{60}, A_{61}, A_{62}, A_{63}, A_{64}, A_{67}, A_{71}, A_{73}, A_{74}, A_{75}, A_{76}, A_{79} \) are projectively equivalent, for \( T_0 = 15, T_1 = 31, T_2 = 24, T_3 = 14, T_4 = 1 \), hence we have four distinct (6,4)-arcs: \( A_1 = \{1,2,3,6,22,43\}, \quad A_2 = \{1,2,4,6,22,43\}, \quad A_3 = \{1,2,6,12,22,43\}, \quad A_4 = \{1,2,6,17,22,43\} \). We try to show the completeness of these arcs:

1. \( A_1 \) is not complete, since there exists some points of index zero which are:
   \[ 32,33,35,36,37,46,48,49,50,52,53,62,63,65,66,67,69,78,79,80,82,83,84. \]
   We add the points 31, 50 to \( A_1 \), then \( B_1 = A_1 \cup \{31,50\} = \{1,2,3,6,22,31,43,50\} \) is a complete (8,4)-arc, since every point of the space not in \( B \) lies on a 4-secant of \( B_1 \).

2. \( A_2 \) is not complete, since there exists some points of index zero which are:
   \[ 30,31,32,33,34,35,36,37,46,47,48,49,50,51,52,53,62,63,64,65,66,67, \]
   \[ 68,69,78,79,80,81,82,83,84,85. \]
   We add the points 30,35 to \( A_2 \), then \( B_2 = A_2 \cup \{30,35\} = \{1,2,4,6,22,30,35,43\} \) is a complete (8,4)-arc.

3. \( A_3 \) is not complete, since there exists some points of index zero which are:
   \[ 26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,46,47,48,49,50,51, \]
   \[ 52,53,54,55,56,57,58,59,60,61,66,67,68,69,70,71,72,73,74,75,76,77, \]
   \[ 78,79,80,81. \]
   We add the points 26, 41 to \( A_3 \), then \( B_3 = A_3 \cup \{26,41\} = \{1,2,6,12,22,26,41,43\} \) is a complete (8,4)-arc.
(4) A₄ is not complete, since there exists some points of index zero which are: 23, 24, ..., 85. We add the points 23, 46 to A₄, then B₄ = A₂ ∪ {23, 46} = {1, 2, 6, 17, 22, 23, 43, 46} is a complete (8,4)-arc.

3.4 The Construction of Complete (k,5)-arcs in PG(3,4):
There are four complete arcs of degree 4, B₁, B₂, B₃ and B₄. By taking the union of two of these arcs, say B₁ and B₃, then C₁ = B₁ ∪ B₃ = {1, 2, 3, 6, 12, 22, 26, 31, 41, 43, 50} is a (k,5)-arc, since no six points of it are coplanar. C₁ is not complete, since there exists some points of index zero which are 57, 63, 72, 83. By adding the point 57 to C₁, then C₂ = C₁ ∪ {57} = {1, 2, 3, 6, 12, 22, 26, 31, 41, 43, 50, 57} is a complete (12,5)-arc.

3.5 The Construction of Complete (k,6)-arcs in PG(3,4):
In this section, one can tries to construct a complete (k,6)-arc as follows: we take the union of two complete (k,4)-arcs, say B₂ and B₃, then C₃ = B₂ ∪ B₃ = {1, 2, 4, 6, 12, 22, 26, 30, 35, 41, 43, 50} is a (11,6)-arc, since there are no seven points are coplanar. C₃ is not complete, since there exists some points of index zero which are 3, 5, 10, 11, 13, 14, 15, 16, 18, 20, 21, 38, 39, 42, 45, 46, 47, 49, 50, 51, 53, 54, 55, 56, 58, 59, 60, 63, 64, 66, 67, 68, 70, 72, 73, 74, 76, 77, 78, 80, 81, 82, 84, 85. By adding the points 3, 46, 53 to C₃ then C₄ = C₃ ∪ {3, 46, 53} = {1, 2, 3, 4, 6, 12, 22, 26, 30, 35, 41, 43, 46, 53} is a complete (14,6)-arc.

3.6 The Construction of Complete (k,7)-arcs in PG(3,4):
In this section, one can tries to construct a complete (k,7)-arc as follows: by taking the union of arcs B₁, B₂ and B₃, denoted by C₅ = B₁ ∪ B₂ ∪ B₃ = {1, 2, 3, 4, 6, 12, 22, 26, 30, 31, 35, 41, 43, 50}, C₅ is a (14,7)-arc and C₅ is not complete, since there exists some points of index zero which are 5, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 38, 39, 40, 42, 44, 45, 46, 47, 49, 50, 51, 52, ..., 85. By adding the point 5 to C₅ then C₆ = C₅ ∪ {5} = {1, 2, 3, 4, 5, 6, 12, 22, 26, 30, 31, 35, 41, 43, 50} is a complete (15,7)-arc.
3.7 The Construction of Complete (k,8)-arcs in PG(3,4):
By taking the union of arcs B₁, B₂, B₃ and B₄, denoted by C₇={1,2,3,4,6,12,17,22,23, 26,30,31,35,41,43,46,50}, C₇ is not complete (17,8)-arc, since there exists some points of index zero which are 5,10,11,13,14,15,16,18,20,21,38,39,40,42,44,47,48,49, 51,52,54,55,57,58,59,61,62,63,65,66,67,69,71,72,73,75,76,77,79,80, 81,84,85. By adding the point 5,42,47,51 to C₇ then C₈={1,2,3,4,5,6,12,17,22,23,26,30,31,35,41,42,43,46,47,50,51} is a complete (21,8)-arc.

3.8 The Construction of Complete (k,9)-arcs in PG(3,4):
A complete (k,9)-arc can be construction from the complete (21,8)-arc C₈ by adding some points of index zero which are 7,,54,60,63 to C₈, a complete(25,9)-arcC₉ is obtained, C₉={1,2,3,4,5,6,7,12,17,22,23,26,30,31,35,41,42,43,46,47,50,51,54,60,63}.

3.9 The Construction of Complete (k,10)-arcs in PG(3,4):
A complete (k,10)-arc can be construction from the complete (25,9)-arc C₉ by adding some points of index zero which are 8,55,58,64 to C₉, a complete (29,10)-arc C₁₀ is obtained, C₁₀={1,2,3,4,5,6,7,8,12,17,22,23,26,30,31,35,41,42,43,46,47,50,51,54,55,58,60,63,64}.

3.10 The Construction of Complete (k,11)-arcs in PG(3,4):
A complete (k,11)-arc can be construction from the complete (29,10)-arc C₁₀ by adding some points of index zero which are 9,70,75,80,84 to C₁₀, a complete (34,11)-arc C₁₁ is obtained, C₁₁={1,2,3,4,5,6,7,8,9,12,17,22,23,26,30,31,35,41,42,43,46,47,50,51,54,55,58,60,63,64,70,75,80,84}.
3.11 The Construction of Complete \((k,12)\)-arcs in \(PG(3,4)\):

A complete \((k,12)\)-arc can be construction from the complete \((34,11)\)-arc \(C_{11}\) by adding some points of index zero which are 10,32,52,61 to \(C_{11}\), we obtain a complete \((38,12)\)-arc \(C_{12}\),

\[
C_{12} = \{1,2,3,4,5,6,7,8,9,10,12,17,22,23,26,30,31,32,35,41,42,43,46,47,50,51,52,54,55,58,60,61,63,64,70,75,80,84\}.
\]

3.12 The Construction of Complete \((k,13)\)-arcs in \(PG(3,4)\):

A complete \((k,13)\)-arc can be construction from the complete \((38,12)\)-arc \(C_{12}\) by adding some points of index zero which are 11,24,38,48 to \(C_{12}\), we obtain a complete \((42,13)\)-arc \(C_{13}\),

\[
C_{13} = \{1,2,3,4,5,6,7,8,9,10,11,12,17,22,23,24,26,30,31,32,35,38,41,42,43,46,47,48,50,51,52,54,55,58,60,61,63,64,70,75,80,84\}.
\]

3.13 The Construction of Complete \((k,14)\)-arcs in \(PG(3,4)\):

A complete \((k,14)\)-arc can be construction from the complete \((42,13)\)-arc \(C_{13}\) by adding some points of index zero which are 13,27,39,62 to \(C_{13}\), we obtain a complete \((46,14)\)-arc \(C_{14}\),

\[
C_{14} = \{1,2,3,4,5,6,7,8,9,10,11,12,13,17,22,23,24,26,27,30,31,32,35,38,39,41,42,43,46,47,48,50,51,52,54,55,58,60,61,62,63,64,70,75,80,84\}.
\]

3.14 The Construction of Complete \((k,15)\)-arcs in \(PG(3,4)\):

A complete \((k,15)\)-arc can be construction from the complete \((46,14)\)-arc \(C_{14}\) by adding some points of index zero which are 14,25,40,56,65 to \(C_{14}\), we obtain a complete \((51,15)\)-arc \(C_{15}\),

\[
C_{15} = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,17,22,23,24,25,26,27,30,31,32,35,38,39,40,41,42,43,46,47,48,50,51,52,54,55,56,58,60,61,62,63,64,65,70,75,80,84\}.
\]
3.15 The Construction of Complete \((k,16)\)-arcs in \(PG(3,4)\):

A complete \((k,16)\)-arc can be constructed from the complete \((51,15)\)-arc \(C_{15}\) by adding some points of index zero which are \(15,28,44,57,78\) to \(C_{15}\), we obtain a complete \((56,16)\)-arc \(C_{16}\),

\[C_{16} = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,17,22,23,24,25,26,27,28,30,31,32,35,38,39,40,41,42,43,44,46,47,48,50,51,52,54,55,56,57,58,60,61,62,63,64,70,75,78,80,84\}\).

3.16 The Construction of Complete \((k,17)\)-arcs in \(PG(3,4)\):

A complete \((k,17)\)-arc can be constructed from the complete \((56,16)\)-arc \(C_{16}\) by adding some points of index zero which are \(16,29,45,59,71\) to \(C_{16}\), we obtain a complete \((61,17)\)-arc \(C_{17}\),

\[C_{17} = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,22,23,24,25,26,27,28,30,31,32,35,38,39,40,41,42,43,44,45,46,47,48,50,51,52,54,55,56,57,58,59,60,61,62,63,64,65,70,75,78,80,84\}\).

3.17 The Construction of Complete \((k,18)\)-arcs in \(PG(3,4)\):

A complete \((k,18)\)-arc can be constructed from the complete \((61,17)\)-arc \(C_{17}\) by adding some points of index zero which are \(18,33,49,53,68,72,74\) to \(C_{17}\), we obtain a complete \((68,18)\)-arc \(C_{18}\),

\[C_{18} = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,22,23,24,25,26,27,28,30,31,32,35,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,68,70,71,72,74,75,78,80,84\}\).

3.18 The Construction of Complete \((k,19)\)-arcs in \(PG(3,4)\):
A complete (k,19)-arc can be construction from the complete (68,18)-arc C_{18} by adding some points of index zero which are 19,34,66,77 to C_{18}, we obtain a complete (72,19)-arc C_{19} which is, C_{19} = \{1,2,3,\ldots,18,22,\ldots,33,34,35,38,\ldots,65,66,68,70,71,72,74,75,78,80,84\}.

3.19 The Construction of Complete (k,20)-arcs in PG(3,4):

A complete (k,20)-arc is construction from the complete (72,19)-arc C_{19} by adding some points of index zero which are 20,36,67,69 to C_{19}, we obtain a complete (76,20)-arc C_{20}, C_{20}=\{1,2,\ldots,18,20,22,\ldots,36,38,\ldots,72,74,75,78,80,84\}.

3.20 The Construction of Complete (k,21)-arcs in PG(3,4):

A complete (k,21)-arc is construction from the complete (76,20)-arc C_{20} by adding all points of index zero to C_{20} which are 21,37,73,76,79,81,82,83,85, we obtain the complete (85,21)-arc C_{21}, C_{21}=\{1,2,\ldots,85\}=PG(3,4). C_{21} is a maximum complete arc, since C_{21} is the whole space PG(3,4).

Conclusions:

Form the above results, the distinct complete (k,n)-arcs in PG(3,4), 3 \leq n \leq 21, as follows:

- (k,3)-arc, where k=5, is a complete.
- (k,4)-arc, where k=8, is a complete.
- (k,5)-arc, where k=12, is a complete.
- (k,6)-arc, where k=14, is a complete.
- (k,7)-arc, where k=15, is a complete.
- (k,8)-arc, where k=21, is a complete.
- (k,9)-arc, where k=25, is a complete.
- (k,10)-arc, where k=29, is a complete.
- (k,11)-arc, where k=34, is a complete.
- (k,12)-arc, where k=38, is a complete.
(k,13)-arc, where k=42, is a complete.
(k,14)-arc, where k=46, is a complete.
(k,15)-arc, where k=51, is a complete.
(k,16)-arc, where k=56, is a complete.
(k,17)-arc, where k=61, is a complete.
(k,18)-arc, where k=68, is a complete.
(k,19)-arc, where k=72, is a complete.
(k,20)-arc, where k=76, is a complete.
(k,21)-arc, where k=85, is a complete.

References: