ON NONLINEAR DIFFERENTIAL OPERATORS THAT COMMUTES WITH ANY FUNCTION

Inaam A.Malloki
Amal Khalaf Haydar
Sajda Kadhum Mohammed

Abstract: A natural differential operator series is one that commutes with every function .This paper discusses natural nonlinear "normally ordered" differential operators series .The operators provides a wide range of higher order derivative identities , these identities specialize to a large variety of identities among binomial coefficient and the orthogonal polynomials ,a number of which are new.

1.*Introduction*:

An operator Ω is called natural if it commutes with arbitrary function , i.e. ,

$$\phi(\Omega u) = \Omega\phi(u), \qquad (1)$$

for all scalar function ϕ[5]. A formal power series f in k variables $t_1, t_2, ..., t_k$ over a field C is a formal expression of the following type

$$f = f(t) = f(t_1, t_2, ..., t_k) = \sum_{\mu \geq 0} a_\mu t^\mu = \sum a_{\mu_1,\mu_2....,\mu_k} t_1^{\mu_1} t_2^{\mu_2} ....t_k^{\mu_k},$$

where $a_\mu = a_{\mu_1,\mu_2....,\mu_k}$,the coefficients of f .

In this paper we will take $u(t)$ to be a formal power series in the variable t, and Ω to be a formal series of differential operators. A simple example of a natural operator in this context is the exponential operator e^{zD},where $D = d/dt.$

Olver [5] show that the translation operators $e^{\alpha z D}$ are essentially the only linear natural differential series. The main result of this paper is the nonlinear differential operator series

$$D^{-1}:e^{zD\psi(u,u',...,u^{(n)})}:D=1+\sum_{n=1}^{\infty}\frac{z^n}{n!}D^{n-1}.\psi(u,u',...,u^{(n)})^n.D,$$

(2)

is natural ,i.e., for any analytic function $\phi(u)$ and any formal power series $\psi(u,u',...,u^{(n)})$

$$D^{-1}:e^{zD\psi(u,u',...,u^{(n)})}:D\phi(u)=\phi(D^{-1}:e^{zD\psi(u,u',...,u^{(n)})}:Du),$$

(3)

via Lagrange inversion formula [6] which is given by

$$f(z)=f(a)+\sum_{n=1}^{\infty}\frac{x^n}{n!}\frac{d^{n-1}[\phi^n(a)f'(a)]}{da^{n-1}},$$

where $f(z)=f(z(x))$ is an analytic function.

In (2) the colons mean that the operator is normally ordered "meaning that all the multiplication terms appear after all the differentiations.

In our work ,and for the purpose of computations, we suppose that $\psi=u^r$, r is a real number, i.e.,

$$D^{-1}:e^{zD\psi(u,u',...,u^{(n)})}:D=D^{-1}:e^{zDu^r}:D$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{n!}D^{n-1}.u^{rn}.D,$$

(4)

which is a generalization of the nonlinear differential operator series

$$D^{-1}:e^{zDu}:D=1+\sum_{n=1}^{\infty}\frac{z^n}{n!}D^{n-1}.u^n.D,$$

which introduced by Olver [5],consequently , we obtain derivative identities reduce to those in [5] if $r=1$.

2.Natural operators

Certain formal series differential operators play a distinguished role , in that commute with functional evaluation.

Definition:A series differential operator Ω is called natural, [5] if it commutes with all functions, i.e.,

$\phi(\Omega u) = \Omega\phi(u),$ for all scalar functions ϕ and all formal series u.

The main result of this paper are the following examples of a nonlinear natural differential operators.

Theorem:Let $u(t)$ be a formal power series and let $D = d/dt$,then the series differential operator

$$D^{-1} : e^{zD\psi(u,u',...,u^{(n)})} : D = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} D^{n-1}.\psi(u,u',...,u^{(n)})^n.D$$

(5)

is natural, i.e., for any analytic functions $\phi(u),\psi(u,u',...,u^{(n)})$

$$D^{-1} : e^{zD\psi(u,u',...,u^{(n)})} : D\phi(u) = \phi(D^{-1} : e^{zD\psi(u,u',...,u^{(n)})} : Du).$$

(6)

Proof: This result follows as a direct consequence of the famous Lagrange inversion formula when $z = x$, $\mathrm{a} = \mathrm{t}$, $\phi(\mathrm{a}) = \psi(\mathrm{u(t)},\mathrm{u}'(\mathrm{t}),....,\mathrm{u}^{(n)}(t))$.If $u(t)$ is any analytic function (or formal power series),and we define $x = \xi(z,t)$ implicitly by the formula

$$x = t + z\psi(u(x),u'(x),...,u^{(n)}(x)),$$

(7)

then, for any analytic function $f(t)$, we have the classical Lagrange inversion formula

$$f(x) = f(t) + \sum_{n=1}^{\infty} \frac{z^n}{n!} \{D^{n-1}.\psi(u(t),u'(t),...,u^{(n)}(t))^n.D\} f(t)$$
$$= D^{-1} : e^{zD\psi(u(t),u'(t),....,u^{(n)}(t))} : Df(t). \quad (8)$$

Now set $f(t)=\phi(u(t))$, so that (8) becomes

$$\phi(u(x))=D^{-1}:e^{zD\psi}:D\phi(u(t)).$$

(9)

On the other hand , according to the formula at the bottom of page 144 of [6] for any analytic function $g(x)$, evaluated at (7) if $\psi=u$.

$$\frac{\partial^n}{\partial z^n}g(\xi(z,t))\Big|_{z=0}=D^{n-1}(u(t)^n Dg(t)),\ \text{n}\geq 1.$$

It is clear that ,for any analytic function g(x), $x=t+z\psi(u(x),u'(x),...,u^{(n)}(x))$, can be expanded at x ,by verifing

$$\frac{\partial^n}{\partial z^n}g(\xi(z,t))\Big|_{z=0}=D^{n-1}(\psi(u(x),u'(x),...,u^{(n)}(x)\ )^n Dg(t)),\ \text{n}\geq 1.$$

Therefore, taking $g=u$ in the last formula ,we find the expansion

$$\begin{aligned}u(x)&=\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{\partial^n}{\partial z^n}\psi(u(\xi(z,t)),u'(\xi(z,t)),...,u^{(n)}(\xi(z,t))\Big|_{z=0}\\&=\sum_{n=0}^{\infty}\frac{z^n}{n!}D^{n-1}\psi(u(t),u'(t),...,u^{(n)}(t))^n Du(t)\\&=D^{-1}:e^{zD\psi(u(t),u'(t),...,u^{(n)}(t))}:Du(t).\end{aligned}$$

Substituting this into (9) completes the proof of the theorem .

We will suppose that $\psi=u^r$, r is a real number ,so we introduce the following results:

3 .Derivative identities:

Any natural differential operator leads to a large class of derivative identities, obtained by considering different functions ϕ in the basic condition (6) . Here we present some of the more elementary derivative identities to be found as consequence of the main theorem . We , first compute the basic formula

$$\zeta(\mathrm{u}) = \mathrm{D}^{-1} : e^{zDu^r} : Du = \sum_{\mathrm{n}=0}^{\infty} \frac{\mathrm{z}^{\mathrm{n}}}{n!} D^{n-1}(u^{rn}\mathrm{u}')$$

$$= \sum_{\mathrm{n}=0}^{\infty} \frac{\mathrm{z}^{\mathrm{n}}}{(rn+1)n!} D^n(u^{rn+1}) . \qquad (10)$$

More generally , we find that , for $\phi(u) = u^k$,

$$D^{-1} : e^{zDu^r} : Du^k = \sum_{n=0}^{\infty} \frac{k}{(rn+k)} \frac{z^n}{n!} D^n(u^{rn+k}) . \qquad (11)$$

As long as k is not a negative integer , (11) is valid as it stands . It also , remains correct when $k = -j$ is a negative integer, provided we interpret that term corresponding to $n = j$ in the summation according to the general "rule "

$$\lim_{m\to 0} \frac{1}{m} D^n u^m = \lim_{m\to 0} D^{n-1}(u^{m-1}u') = D^n \log u , \quad \mathrm{n} \ge 1.$$

(12)

Now , according to the main theorem , the series (11) is the kth power of the series (10) , this implies certain identities among higher-order derivatives of powers of u .For instance , taking the case $k = 2$, the series identity

$$\sum_{n=0}^{\infty} \frac{2}{rn+2} \frac{z^n}{n!} D^n(u^{rn+2}) = (\sum_{n=0}^{\infty} \frac{1}{rn+1} \frac{1}{n!} D^n(u)^{rn+1})^2 ,$$

implies the following derivative identities :

$$D^n(u^{rn+2}) = \sum_{i=0}^{n} \frac{rn+2}{2(ri+1)(r(n-i)+1)} \binom{n}{i} D^i(u^{ri+1}) D^{n-i}(u^{r(n-i)+1}).$$

More generally , if we apply the identities corresponding to $\phi(u)$ being u^{k+l}, u^k and u^l , then the series identity

$$\sum_{n=0}^{\infty} \frac{k+l}{rn+k+l} \frac{z^n}{n!} D^n(u^{rn+k+l}) = (\sum_{n=0}^{\infty} \frac{k}{rn+k} \frac{z^n}{n!} D^n(u^{rn+k})).(\sum_{n=0}^{\infty} \frac{l}{rn+l} \frac{z^n}{n!} D^n(u^{rn+l})),$$

implies the additional derivative identities

$$\frac{k+l}{rn+k+l}D^n(u^{rn+k+l})=\sum_{i=0}^{n}\frac{kl}{(ri+k)(r(n-i)+l)}\binom{n}{i}D^i(u^{ri+k})D^{n-i}(u^{r(n-i)+l}).$$

(13)

These identities are valid for arbitrary (positive and negative) values of k,l provided we use the rule (12) if either $rn+k+l=0$,or any of the summation terms $ri+k=0$ or $r(n-i)+l=0$.

3.1 Inverse operators

To obtain the inverse series for the natural differential operators (4), we have two cases by taking $k=-1,l=1$ in (13)

Case 1.If r is an integer number differs from 1,we find the series

$$\eta(u)=\sum_{n=0}^{\infty}\frac{1}{1-rn}\frac{z^n}{n!}D^n(u^{rn-1}),$$

is the inverse series for (4) ,$r\in Z-\{1\}$,where Z is the set of integer numbers ,and hence we have the series identity

$$1=(\sum_{n=0}^{\infty}\frac{1}{rn+1}\frac{z^n}{n!}D^n(u^{rn+1}))(\sum_{n=0}^{\infty}\frac{1}{1-rn}\frac{z^n}{n!}D^n(u^{rn-1})).$$

Rearranging the terms of degree n in z in this formula results the new derivative identities.

$$\sum_{i=0}^{n}\frac{1}{(1-ri)i!(r(n-i)+1)(n-1)!}D^i(u^{ri-1}).D^{n-i}(u^{r(n-i)+1})=0,$$

(14)

$n\geq 1,r\in Z-\{1\}$,which does not appear in literatures.

Case 2. If $r=\frac{1}{s},s\in Z-\{0\}$,we find the series

$$\eta(u)=-z^sD^s(\log u)+\sum_{\substack{n=0\\ n\neq s}}^{\infty}\frac{1}{1-rn}\frac{z^n}{n!}D^n(u^{rn-1}),$$

is the series inverse for (4) , $r=\frac{1}{s}, s\in Z-\{0\}$,and hence we have the series identity

$$1=\Big(\sum_{n=0}^{\infty}\frac{s}{n+s}\frac{z^n}{n!}D^n(u^{\frac{n}{s}+1})\Big).\Big(-z^sD^s(\log u)+\sum_{\substack{n=0\\ n\neq s}}^{\infty}\frac{1}{1-rn}\frac{z^n}{n!}D^n(u^{rn-1})\Big).$$

(15)

Rearranging the terms of degree n in z in this formula results the new derivative identities.

$$D^n(u^{\frac{n}{s}+1})=\frac{(n+s)(n-1)!}{(n-s)!}uD^s(\log u)D^{n-s}(u^{\frac{n}{s}})+\sum_{\substack{i=1\\ i\neq s}}^{n}\frac{(n+s)n!}{i!(ri-1)(n-i+s)(n-i)!}u$$
$$D^i(u^{ri-1}).D^{n-i}(u^{\frac{n-i}{s}+1}), \qquad (16)$$

$r=\frac{1}{s}, s\in Z-\{0\}$,which does not appear in literatures.

It is clear that if $s=1$ then the series identity

$$1=\Big(\sum_{n=0}^{\infty}\frac{z^n}{(n+1)!}D^n(u^{n+1})\Big).\Big(\frac{1}{u}-z\frac{u'}{u}+\sum_{n=2}^{\infty}\frac{1}{(1-n)}\frac{z^n}{n!}D^n(u^{n-1})\Big), \qquad n\geq 1,$$

and the resulting derivative identity

$$D^n(u^{n+1})=(n+1)u'D^{n-1}(u^n)+\sum_{i=2}^{n}\frac{1}{i-1}\binom{n+1}{i}uD^i(u^{i-1}).D^{n-i}(u^{n-i+1}),$$

which introduced by Olver [5] are a special cases of (15) and (16) respectively.

3.2 Binomial and orthogonal polynomial identities :-

We now specialize the above derivative identities for particular function $u(t)$, and find that reduce to a wide range of identities among binomial coefficient and orthogonal polynomials .

1. First consider the case

$u(t) = t^{\alpha}$,so $\frac{1}{n!} D^n u^m = \binom{m\alpha}{n} t^{m\alpha - n}$.

Then(13) reduces to the identity

$$\frac{k+l}{rn+k+l}\binom{(rn+k+l)\alpha}{n} = \sum_{i=0}^{n} \frac{kl}{(ri+k)(r(n-i)+l)}\binom{(ri+k)\alpha}{i}.\binom{(r(n-i)+l)}{n-i}. \quad (17)$$

This is equivalent to the Hagen –Rothe identity [2], which generalizes the classical Vandermonde convolution identity for binomial coefficients,

$$\binom{r+s}{n} = \sum_{i=0}^{n}\binom{n}{i}\binom{s}{n-i}. \quad (18)$$

As another example, the formula (14) in this case reduces to the identity

$$\sum_{i=0}^{n} \frac{-1}{(ri-1)[r(n-i)+1]}\binom{(ri-1)\alpha}{i}\binom{[r(n-i)+1]}{n-i} = 0, \quad n \geq 1. \quad (19)$$

2. Let $u = e^{\alpha t},$ so $D^n u^m = m^n \alpha^n e^{\alpha mt}$.

Then (13) reduces to the identity

$$(rn+k+l)^{n-1} = \sum_{i=0}^{n} \frac{kl}{k+l}\binom{n}{i}(ri+k)^{i-1}(r(n-i)+l)^{n-i-1}. \quad (20)$$

If we set $k = -x/z, l = -rn - y/z$,we deduce

$$(x+y)^{n-1} = \sum_{i=0}^{n} \frac{x(y+rnz)}{x+y+rnz}\binom{n}{i}(x-riz)^{i-1}(y+riz)^{n-i-1}, \quad (21)$$

which is very similar to the Abel identity [4] ,

$$(x+y)^{n} = \sum_{i=0}^{n}\binom{n}{i}x(x-iz)^{i-1}(y+iz)^{n-i}. \quad (22)$$

3. Let $u = e^{-t^2}$, $\quad D^n u^m = (-1)^n m^{n/2} H_n(\sqrt{m}\, t)e^{-mt^2}$,

where H_n denotes the usual Hermite polynomial [3], [1].In this case ,(13) reduces to the identity

$$(rn+k+l)^{\frac{n}{2}-1} H_n(\sqrt{rn+k+l}t) = \sum_{i=0}^{n} \frac{kl}{k+l}\binom{n}{i}(ri+k)^{\frac{i}{2}-1}(r(n-i)+l)^{\frac{n-i}{2}-1}.$$

$$H_i(\sqrt{ri+kt})H_{n-i}(\sqrt{r(n-i)+lt}) \; , \quad (23)$$

which we can interpret as an Abel-type identity for Hermite polynomials.It is not the same as the usual addition formula ,since the arguments of the Hermite polynomials appearing in the summation depend on the summation index i. If either $rn+k+l=0$, $ri+k=0$,or $r(n-i)+l=0$, then we view the corresponding term in (23) according to the rule (12)

$$\lim_{m\to 0}(-1)^n m^{(n/2)-1} H_n(\sqrt{m}\ t) = \begin{cases} -2t & n=1, \\ -2 & n=2, \\ 0 & n\ge 3. \end{cases}$$

4. Let $u = t^{\alpha}e^{-t}$, so $\frac{1}{n!}D^n u^m = t^{m\alpha-n}e^{-mt}L_n^{m\alpha-n}(mt)$,

where L_n^{α} are generalized Laguerre polynomials [1], [3] .Again (13) reduces to an Abel-type identity

$$\frac{k+l}{rn+k+l}L_n^{(rn+k+l)\alpha-n}((rn+k+l)t) = \sum_{i=0}^{n}\frac{kl}{(ri+k)(r(n-i)+l)}L_i^{(ri+k)\alpha-i}((ri+k)t).$$
$$L_{n-i}^{(r(n-i)+l)\alpha-n+i}((r(n-i)+l)t)\ , \qquad (24)$$

for Laguerre polynomials .As in the previous example , we make the convention according to the rule (12)

$$\lim_{m\to 0}\frac{1}{m}L_n^{m\alpha-n}(mt) = \begin{cases} \alpha - t & n=1, \\ [(-1)^{n-1}/n]\alpha, & n\ge 2, \end{cases}$$

5. Let the case $u=(1-t)^{\alpha}(1+t)^{\beta}$, so

$$D^n u^m = n!(-2)^n(1-t)^{m\alpha-n}(1+t)^{m\beta-n}\,p_n^{(m\alpha-n,m\beta-n)}(t),$$

where $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials [3] .In this case (13) reduces to the Hagen – Rothe type formula

$$\frac{k+l}{rn+k+l}P_n^{((rn+k+l)\alpha-n,(rn+k+l)\beta-n)}(t) =$$
$$\sum_{i=0}^{n}\frac{kl}{(ri+k)(r(n-i)+l)}p_i^{((ri+k)\alpha-i,(ri+k)\beta-i)}(t)\,p_{n-i}^{((r(n-i)+l)\alpha-(n-i),(r(n-i)+l)\beta-(n-i))}(t)\,. \qquad (25)$$

Again , from (12) we have

$$\lim_{m\to 0}\frac{1}{m}p_n^{(m\alpha-n,m\beta-n)}(t) = \frac{[\alpha(t+1)^n-\beta(t-1)^n]}{[n(-2)^n]}, \qquad n\ge 1.$$

Finally ,the identities (13),(17),(21),(23),(24),(25) are reduced to those in [5] if $r=1$.

REFERENCES

[1] E.KAYSZIG,Introduction Functional Analysis ,1978.

[2] H.W,GOULD,Combinatorial Identities, Morgantown,WV,1972.

[3] J.RIORDAN ,An Introduction to Combinatorial Analysis,Princeton University Press, Princeton ,NJ,1980

[4] L.COMTET,Advanced Combinatorics:The Art of Finite and Infinite Expansions,D.Reidel,Boston,MA,1974.

[5] P.J.OLVER,A Nonlinear Differential Operator series that commutes with any Function ,Siam J.Math.Anal.Vol. 23,No.1,1992

[6] Z.A.MELZAK, Companion to Concrete Mathematics,Wiley-Interscience, New York,1973.

الخلاصه

المؤثر التفاضلي الطبيعي المتسلسل هو المؤثر الذي يتبادل مع كل داله.هذا البحث يناقش مؤثرات تفاضليه طبيعيه غير خطيه متسلسله, والتي زودتنا بعدد واسع ومتنوع من المتطابقات التفاضليه من رتب عليا, من بينها متعددات الحدود المتعامده ومعاملات ذي الحدين. بعض من المتطابقات التي اوجدناها جديده.