بناء نموذج احتمالي موزون مضاعف موزون مع تطبيق عملي

أ.د. ضوية سلمان حسن | الباحث. مهدي وهاب نعمة
الكلية التقنية الادارية /بغداد | قسم الاحصاء / جامعة كربلاء

Constructing a New Weighted Exponential Pareto Distribution with Estimation

Dhwyia S. Hassan1,Mahdi W. Nasrallah2

1Department of Business Information Technology, College of Business Administration of Informatics, University of Information Technology &Communications , Iraq

2Department of Statistics, College of Administration and Economic, University of Karbala

Dhwyia.salman@yahoo.com, Mahdi_na2002@yahoo.com

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. **Dhwyia S. Hassan[1],Mahdi W. Nasrallah[2]**

ABSTRACT

This paper deals with constructing a new probability distribution used for length biased data, which can be employed in development of proper model that use for data come for population which are size biased distribution, here we use the method to adjust the original distribution function from real data, and expectation of those data. The constructed distribution is called weighted exponential – Pareto (WEPD). The researcher work on construction the ($p.d.f$), C.D.F of this distribution, also deriving the general form of non-central and central moment (r^{th} moments about origin, and about μ), also the derivation of C.D.F need integration by incomplete Gamma formula. The r^{th} central moment formula is necessary for finding coefficient of Skewness and Kurtosis, which are used as a tool of some numerical method applied maximum likelihood estimators, and proposed estimators based on Cran's estimators. The last method is L – moment estimators, the comparison has been done through simulation using different values of sample size ($n = 40,80,100,150$), the estimators were compared using MSE, MAPE ,all the results are explained in tables .

Keywords: biased sized distribution, WEPD, incomplete Gamma, maximum likelihood, maximum entropy, L – moment, Cran's estimator, MSE, MAPE.

This paper represent a sub – research taken from Ph.D thesis submitted to the department of statistics / college of administration and economic / University of Baghdad, Iraq.

1. Introduction

Statistical models and methods for life time data analysis, have been studied very well and extensively applied in many fields, including biomedical sciences, engineering and management. The researcher work in developing proper models for life time data when the original data are skewed or have (size biased) in sampling with applications to wildlife populations and human families. The problem of determining proper model for interested information are one thing that important for data analysis, as proposed by (Rao [1965], Rao & Patil [1978]) to use weighted distribution that better fit, the length biased population (Olvyede[2006, Das & Roy [2011]) as well as many researcher applying the concept of length biased distribution. Continuing with previous research we introduce this contribution to construct a new probability weighted family (WEPD) called weighted exponential Pareto distribution which defined as follows;

Definition; The Exponential – Pareto Distribution

Let $x \ be \ r.v \sim exponential \ pareto \ with \ CDF$ (1);

$$F(x;p,\lambda,\theta) = 1 - e^{-\lambda\left(\frac{x}{p}\right)^{\theta}} \quad (1)$$

This is the CDF of (EPD), while the $p.d.f$ is;

$$f(x;p,\lambda,\theta) = F'(x;p,\lambda,\theta) = \left(\frac{\lambda\theta}{p}\right)\left(\frac{x}{p}\right)^{\theta-1} e^{-\lambda\left(\frac{x}{p}\right)^{\theta}} \quad x > 0 \quad (2)$$

The r^{th} moment for $p.d.f$ in equation (2) is;

$$E(x^r) = \int_0^\infty x^r \ f(x;p,\lambda,\theta)\,dx = \left(\frac{p}{\sqrt[\theta]{\lambda}}\right)^r \Gamma\left(\frac{r}{\theta}+1\right) \quad r = 1,2,3,\ldots \quad (3)$$

Also the r^{th} central moments about mean;

$$E(x-\mu)^r = \int_0^\infty (x-\mu)^r \left(\frac{\lambda\theta}{p}\right)\left(\frac{x}{p}\right)^{\theta-1} e^{-\lambda\left(\frac{x}{p}\right)^{\theta}} \quad (4)$$

Let;

$$u = \lambda\left(\frac{x}{p}\right)^{\theta} \qquad du = \left(\frac{\lambda\theta}{p}\right)\left(\frac{x}{p}\right)^{\theta-1} dx \qquad x = p\sqrt[\theta]{\frac{u}{\lambda}}$$

$$E(x-\mu)^r = \int_0^\infty \left(p\sqrt[\theta]{\frac{u}{\lambda}} - \mu\right)^r e^{-u}\ du$$

$$= \int_0^\infty \left(\frac{p}{\sqrt[\theta]{\lambda}}\, u^{\frac{1}{\theta}} - \mu\right)^r e^{-u}\,du$$

$$= \int_0^\infty \sum_{j=0}^{r} C_j^r \left(\frac{p}{\sqrt[\theta]{\lambda}}\, u^{\frac{1}{\theta}}\right)^j (-\mu)^{r-j} e^{-u}\,du$$

$$E(x-\mu)^r = \sum_{j=0}^{r} C_j^r \left(\frac{p}{\sqrt[\theta]{\lambda}}\right)^j (-\mu)^{r-j}\,\Gamma\left(\frac{j}{\theta}+1\right) \quad (5)$$

When applying the definition of size biased distribution [$g_1(w)$];

$$g_1(w) = \frac{x^c f(x)}{E(x^c)} \quad (6)$$

On the distribution given in equation (2), we have the weighted exponential Pareto distribution as follows;

$$g_1(w) = \frac{\left(\frac{\theta}{p}\right)\left(\lambda^{1+\frac{c}{\theta}}\right)\left(\frac{x}{p}\right)^{\theta+c-1} e^{-\lambda\left(\frac{x}{p}\right)^{\theta}}}{\Gamma\left(\frac{c}{\theta}+1\right)} \quad x > 0, \lambda, p > 0, \theta > 2, c \geq 1 \quad (7)$$

$\lambda, p \ scale \ parameters, \theta, c \ shape \ parameters$

And its integrate to one.

The cumulative distribution function for [$g_1(w)$] is;

$$G_1(w) = \int_0^x g_1(u)\,du \quad (8)$$

Therefore the cumulative distribution function for the sized – biased weighted exponential – Pareto (SBWEP) is given by;

$$F(x) = \left[\lambda\left(\frac{x}{p}\right)^{\theta}\right]^{\frac{c}{\theta}+1} e^{-\lambda\left(\frac{x}{p}\right)^{\theta}} \sum_{k=0}^{\infty} \frac{\left[\lambda\left(\frac{x}{p}\right)^{\theta}\right]^k}{\Gamma\left(\frac{c}{\theta}+k+2\right)} \quad (9)$$

Which is used for generating observation (x_i) according to incomplete – Gamma and given values of (c, λ, θ, p).

2. Estimation of Parameters

For the four parameters of (SBWEP) defined in equation (7), which are two shape parameters (θ, c) and two scale parameters (λ, p), these four parameters are estimated using different methods like maximum likelihood method, as well as the formula of linear moments is derived to obtain L – moment, also research work to derive a formula for estimating by applying maximum entropy like transformation also we apply numerical method on the statistical measures like mean, variance, skewnwss, kurtosis, obtained from applying the (r^{th}) moment for length biased distribution which is obtained from applying $[E_L(x^r)]$ on equation (6), i.e;

$$E_L(x^r) = \frac{E(x^{r+1})}{E(x)} \qquad r = 1,2,3, \dots$$

Then for given values of four parameters (c, λ, θ, p), the comparison has been done numerically and the set of parameters which gives smallest (skewness & kurtosis) is the best.

3. Methods of Estimation

In this section we introduce four methods of estimation for the four parameters (c, λ, θ, p), these methods are maximum likelihood method for the generated weighted exponential – Pareto, the second method is proposed estimators depend on Cran's method, and the third is the L- moment estimators. After explaining the estimators briefly, the results of simulation procedure also explained.

3.1 Maximum Likelihood Method

The estimators by this method obtained from maximizing log L which is defined by;

$$h_1(x) = \left(\theta\lambda^{\frac{c}{\theta}+1}\right)\left(\frac{x^{c+\theta-1}}{p^{c+\theta}}\right)\left(\frac{e^{-\lambda\left(\frac{x}{p}\right)^{\theta}}}{\Gamma\left(\frac{c}{\theta}+1\right)}\right)$$

$$L = \lambda^{n\left(\frac{c}{\theta}+1\right)}(\theta^n)\left(\frac{\prod x_i^{(c+\theta-1)}}{p^{n(c+\theta)}}\right)\left(\frac{e^{-\lambda\sum\left(\frac{x_i}{p}\right)^{\theta}}}{\Gamma^n\left(\frac{c}{\theta}+1\right)}\right)$$

$$\log L = n\left(\frac{c}{\theta}+1\right)\log\lambda + n\log\theta - n(c+\theta)\log p + (c+\theta-1)\sum\log x_i$$
$$-\lambda\sum\left(\frac{x}{p}\right)^{\theta} - n\log\Gamma\left(\frac{c}{\theta}+1\right) \dots (10)$$

$$from\ \frac{\partial\log L}{\partial\theta}, \frac{\partial\log L}{\partial\lambda}, \frac{\partial\log L}{\partial p}, \frac{\partial\log L}{\partial c} = 0$$

$$\hat{\theta}_{ML} = \sqrt{\frac{nc\log\lambda - nc - \frac{nc^2(2-\theta)}{\theta^2\Gamma^2(\frac{c}{\theta}+1)}}{\sum_{i=1}^{n}\log\left(\frac{x_i}{p}\right) - \lambda\sum_{i=1}^{n}\left(\frac{x_i}{p}\right)^{\theta}\log\left(\frac{x_i}{p}\right)}} \quad \dots(10)$$

$$\hat{\lambda}_{ML} = \frac{n\left(\frac{\hat{c}}{\hat{\theta}}+1\right)}{\sum\left(\frac{x_i}{\hat{p}}\right)^{\hat{\theta}}} \quad \dots(11)$$

$$\hat{p}_{ML} = \left[\frac{\hat{\lambda}\,\hat{\theta}\sum x_i^{\hat{\theta}}}{n(c+\theta)}\right]^{\frac{1}{\theta}} \quad \dots(12)$$

$$\hat{C}_{ML}=\frac{\theta^4\Gamma^2(\frac{c}{\theta}+1)\left[n\log p-\sum_{i=1}^{n}log(x_i)-\frac{n}{\theta}\log\lambda\right]}{n(2-\theta)}\quad ...(13)$$

3.2 Proposed Method using Cran's Method

The researcher employing the ideas presented by the researcher (Cranes 1988) when estimating three parameters of Weibull distribution. For estimating the parameters of proposed model (four parameter), we find m_k' and equate it with population moments of $(\mu_k^{'})$ as follows :

$$m_k'=\int_0^{\infty}\{1-G_w(x)\}^k\,dt \qquad (14)$$

$$=\sum_{r=0}^{n}\left(1-\frac{r}{n}\right)^k\{x_{r+1}-x_r\}\quad,\ x_0=0 \qquad (15)$$

As the $\mathbf{m_1'}=\bar{\mathbf{x}}$ mean sample ,where the $x_1\le x_2\le\cdots\le x_n$ and CDF has been estimated from the following formula:

$$G_w(x)=\begin{cases}0 & x<x_{(1)}\\ \frac{r}{n} & x_{(r)}\le x\le x_{(r+1)}\\ 1 & x_{(n)}\le x\end{cases} \qquad (16)$$

where

$\mu_k^{'}=m_k^{'}$

$\mu_k^{'}=E(x)^r$

Where $E(x)^r$ defined by equation (3) and($m_k^{'}$) is the knowledge of the equation (14)

3.3 L – Moments Estimation

The estimators by this method obtained from solving ($B_r=b_r$), where;

$$B_r=\int_0^{\infty}x\ G_w^r(x)g_w(x)dx\quad ...(17)$$

$$b_r=\frac{1}{nC_r^{n-1}}\sum_{i=1}^{n}C_r^{i-1}x_{(i)}\qquad x_{(1)}\le x_{(2)}\le\ldots\le x_{(n)}\quad ...(18)$$

$$G_w^r(x)=\left[\lambda\left(\frac{x}{p}\right)^{\theta}\right]^{r\left(\frac{c}{\theta}+1\right)}e^{-\lambda r\left(\frac{x}{p}\right)^{\theta}}\sum_{k=0}^{\infty}\frac{\left[\lambda\left(\frac{x}{p}\right)^{\theta}\right]^{rk}}{\Gamma^r\left(\frac{c}{\theta}+k+2\right)}$$

$$g_w(x)=\frac{\left(\frac{\theta}{p}\right)\lambda^{\left(\frac{c}{\theta}+1\right)}}{\Gamma\left(\frac{c}{\theta}+1\right)}\left(\frac{x}{p}\right)^{c+\theta-1}e^{-\left[\lambda\left(\frac{x}{p}\right)^{\theta}\right]}$$

The final formula for (B_r) is;

$$B_r=\frac{\left(\frac{p}{\theta}\right)\lambda^{\left(\frac{c}{\theta}+1\right)(r+1)}}{\Gamma\left(\frac{c}{\theta}+1\right)}\sum_{k=0}^{\infty}\frac{\lambda^{rk}}{\theta}\frac{\left(p^{(r+1)(c+\theta)+\theta rk+1}\right)}{\left[\lambda(r+1)^{\frac{1}{\theta}}\right]^{(r+1)(c+\theta)+\theta rk+1}}\frac{\Gamma\left[\frac{1}{\theta}(r+1)(c+\theta)+\theta rk+1\right]}{\Gamma^r\left(\frac{c}{\theta}+k+2\right)}\quad ...(19)$$

when $B_r=b_r$ r=1,2,3,4

The estimators by L – moments are obtained by solving the set of equations;

$$(B_1=b_1),(B_2=b_2),(B_3=b_3),(B_4=b_4)$$

4. Simulation

To find the estimator's $(MLE, Cran's, L-Moment)$ we perform simulation experiments using Monte Carlo assuming that;

λ	1.5	
θ	2.5	
p	2	2.5
c	2	2.5

Table (1): Estimated parameters at $(\theta = 2.5, \lambda = 1.5, p = 2, c = 2.5)$.

n	Method		θ	λ	p	c
40	L - Mom	Parameter	2.497981	2.498918	1.890605	1.498259
		MSE	0.003405	0.003274	0.003392	0.014869
		MAPE	0.020253	0.019885	0.033947	0.054698
	Crans	Parameter	2.515743	2.247168	0.593754	1.972911
		MSE	0.067517	0.010542	0.226344	1.977705
		MAPE	0.101133	0.030623	0.315274	0.703123
	MLE	Parameter	2.449763	2.45027	1.95088	1.44827
		MSE	0.02331	0.033341	0.043525	0.033228
		MAPE	0.129892	0.120095	0.334487	0.22456
Best			L - Mom	L - Mom	L - Mom	L - Mom
80	L - Mom	Parameter	2.501034	1.942213	1.500526	2.501726
		MSE	0.003388	0.003114	0.006591	0.003444
		MAPE	0.020266	0.031936	0.033084	0.020459
	Crans	Parameter	2.253165	0.600728	1.971168	2.542818
		MSE	0.008542	0.224525	1.958083	0.064551
		MAPE	0.030677	0.314112	0.699636	0.098734
	MLE	Parameter	2.156692	1.794271	1.244416	2.208564
		MSE	0.115043	0.085019	0.055497	0.158719
		MAPE	0.116574	0.170389	0.102865	0.137323
Best			L - Mom	L – Mom	L - Mom	L - Mom
100	L - Mom	Parameter	2.501034	1.942213	1.500526	2.501726
		MSE	0.003302	0.003476	0.003487	0.00334
		MAPE	0.01977	0.034362	0.025569	0.020104
	Crans	Parameter	2.253165	0.600728	1.971168	2.542818
		MSE	0.008157	0.225387	1.955246	0.064265
		MAPE	0.029769	0.314618	0.699134	0.098669
	MLE	Parameter	2.144188	1.798633	1.24251	2.203177
		MSE	0.117392	0.087884	0.053495	0.16713
		MAPE	0.118729	0.17166	0.100684	0.142325
Best			L - Mom	L – Mom	L - Mom	L - Mom
150	L - Mom	Parameter	2.456175	1.949285	1.453535	2.446147
		MSE	0.003758	0.002913	0.003422	0.002743
		MAPE	0.021541	0.030976	0.025358	0.01753
	Crans	Parameter	2.218991	0.597103	1.918876	2.487756
		MSE	0.002351	0.176139	1.968168	0.079829
		MAPE	0.015468	0.279251	0.701449	0.112404
	MLE	Parameter	2.451749	1.950136	1.452575	2.450979
		MSE	0.003169	0.003015	0.003434	0.003111
		MAPE	0.019608	0.031617	0.024932	0.0193
Best			L - Mom	L – Mom	L - Mom	L - Mom

Table (2): Estimated parameters at ($\theta = 2.5, \lambda = 1.5, p = 2, c = 2$).

n	Method		θ	λ	p	c
40	L - Mom	Parameter	2.000078	1.878727	1.502755	2.500455
		MSE	0.003323	0.003432	0.017751	0.003342
		MAPE	0.019956	0.033887	0.060636	0.024931
	Crans	Parameter	1.824357	0.59144	1.977045	2.114487
		MSE	0.446917	0.230717	1.986884	0.038953
		MAPE	0.16755	0.31803	0.70428	0.089725
	MLE	Parameter	1.652263	1.813043	1.231459	2.150969
		MSE	0.145721	0.093356	0.047873	0.164314
		MAPE	0.139612	0.179027	0.093478	0.173869
Best			L - Mom	L - Mom	L - Mom	L - Mom
80	L - Mom	Parameter	1.998624	1.938582	1.499049	2.468658
		MSE	0.003345	0.003337	0.006955	0.0034
		MAPE	0.020084	0.033493	0.033978	0.025432
	Crans	Parameter	1.784601	0.616761	1.970109	3.20613
		MSE	0.090032	0.223867	1.9139	0.050266
		MAPE	0.070044	0.313406	0.69162	0.107756
	MLE	Parameter	1.6561	1.794851	1.249932	2.188962
		MSE	0.125537	0.081299	0.055723	0.158464
		MAPE	0.124415	0.166712	0.102574	0.17195
Best			L - Mom	L – Mom	L - Mom	L - Mom
100	L - Mom	Parameter	2.000872	2.002297	1.502588	2.49872
		MSE	0.003399	0.003347	0.003244	0.003428
		MAPE	0.020178	0.03356	0.024566	0.025507
	Crans	Parameter	1.780758	0.620005	1.973404	2.409435
		MSE	0.060382	0.226783	1.904706	0.051694
		MAPE	0.058168	0.315603	0.689998	0.109683
	MLE	Parameter	1.654174	1.793046	1.245395	2.180868
		MSE	0.132892	0.086635	0.056061	0.16202
		MAPE	0.127653	0.169737	0.103477	0.172913
Best			L - Mom	L – Mom	L - Mom	L - Mom
150	L - Mom	Parameter	1.952364	1.953021	1.444914	2.450834
		MSE	0.003313	0.003773	0.003058	0.00311
		MAPE	0.019666	0.036724	0.02349	0.023818
	Crans	Parameter	1.736688	0.61867	1.918127	2.408783
		MSE	0.019708	0.175636	1.908136	0.070247
		MAPE	0.041891	0.278751	0.690665	0.131656
	MLE	Parameter	1.950054	1.949279	1.452201	2.452372
		MSE	0.003218	0.003182	0.003383	0.003392
		MAPE	0.019051	0.031866	0.025361	0.024973
Best			L - Mom	L – Mom	MLE	MLE

Table (3): Estimated parameters at ($\theta = 2.5, \lambda = 1.5, p = 2.5, c = 2$).

n	Method		θ	λ	p	c
40	L - Mom	Parameter	1.998401	2.34828	1.500306	2.500104
		MSE	0.003364	0.003394	0.026095	0.003415
		MAPE	0.020129	0.033715	0.060688	0.025038
	Crans	Parameter	1.624955	0.692517	2.200387	2.559939
		MSE	0.037568	0.493176	3.267353	0.144787
		MAPE	0.047819	0.466924	0.722993	0.187522
	MLE	Parameter	1.651013	2.303822	1.237733	2.190816
		MSE	0.123726	0.089245	0.05205	0.158333
		MAPE	0.123674	0.174845	0.078471	0.174493
Best			L - Mom	L - Mom	L - Mom	L - Mom
80	L - Mom	Parameter	1.99923	2.424581	1.500855	2.499216
		MSE	0.003403	0.003255	0.008811	0.003173
		MAPE	0.020252	0.032778	0.03117	0.024211
	Crans	Parameter	1.621475	0.70113	2.201387	2.58443
		MSE	0.01951	0.494302	3.23609	0.14672
		MAPE	0.04207	0.467591	0.719548	0.189262
	MLE	Parameter	1.686555	2.297971	1.252957	2.192186
		MSE	0.126157	0.082492	0.054722	0.136124
		MAPE	0.123125	0.164695	0.080831	0.156817
Best			L - Mom	L – Mom	L - Mom	L - Mom
100	L - Mom	Parameter	2.001434	2.498847	1.501833	2.499903
		MSE	0.003401	0.003423	0.00337	0.003223
		MAPE	0.020077	0.033901	0.019943	0.02457
	Crans	Parameter	1.623353	0.702408	2.200454	2.590675
		MSE	0.016094	0.493045	3.23146	0.145283
		MAPE	0.042127	0.46697	0.719037	0.188324
	MLE	Parameter	1.669885	2.304756	1.245405	2.202768
		MSE	0.118637	0.087092	0.051576	0.150419
		MAPE	0.118893	0.16973	0.078098	0.165057
Best			L - Mom	L – Mom	L - Mom	L - Mom
150	L - Mom	Parameter	1.945555	2.450336	1.450626	2.448173
		MSE	0.003479	0.003289	0.003353	0.00379
		MAPE	0.020731	0.032916	0.019866	0.027223
	Crans	Parameter	1.597412	0.700352	2.151489	2.535092
		MSE	0.003728	0.425082	3.238792	0.16279
		MAPE	0.019492	0.434326	0.719859	0.201294
	MLE	Parameter	1.948981	2.450881	1.451209	2.452904
		MSE	0.002946	0.003396	0.003306	0.003529
		MAPE	0.018838	0.032528	0.019647	0.025618
Best			MLE	MLE	L - Mom	MLE

Conclusion & Recommendation

1. The estimators which have smallest MSE are obtained due to using L-moment estimator for shape parameter and mixing this with proposed estimator.
2. The best methods used is; L – moment for sample size (40, 80, 100), while for large sample size ($n = 150$), the MLE is the best.
4. The proposed distribution contains four parameters (θ, λ, p, c), all are estimators by different methods, we hope to extend the concept of this research by including the parameter of truncation (A), which is necessary for truncated data.

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