Approximation of Burger's Equation Using Sextic B-Spline Galerkin Scheme with Quintic Weight Function

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Abstract

In this study We produce new numerical scheme which rely on sextic B-spline Galerkin method takes with quintic B-spline as a weight function, for solving the Burger's equation, contrasted with exact solution can be done and then we find out a linear stability analysis which is erect on a Fourier (Von Neumann) method.

1- Preface

The Burger's equation first appeared in 1915 [2], where he applied this equation as a sample for the motion of a viscous fluid when the viscosity approaches zero. Later, Burger investigated various aspects of turbulence and developed a mathematical model illustrating the theory as well as statistical and spectral aspects of the equation and related system [4],[5],[6]. Because of comprehensive work of Burger, it's known as Burger's equation. It play an necessary part in studying different problem for sciences and engineering like a model in fields as wide as heat conduction [7], gas dynamics [12], longitudinal elastic waves in an isotropic solid [3], number theory [18], and so forth. The Burger's equation is solved analytically and independently for unintentional intinal conditions [11],[7]. In many states, these solutions include infinite series which may converge very slowly for small values of viscosity coefficients ϵ, which correspond to steep wave fronts in the propagation of the dynamic wave forms. Burgers' equation shows a similar features with Navier-Stokes equation due to the form of the nonlinear convection term and the occurrence of the viscosity term. Before concentrating on the numerical solution of the Navier-Stokes equation, it seems reasonable to first study a simple model of the Burgers' equation. Therefore, the Burgers' equation has been used as a model equation to test the numerical methods in terms of accuracy and stability for the Navier-Stokes equation. Many authors have used a variety of numerical techniques for getting the numerical solution of the Burgers' equation. Numerical difficulties have been come across in the numerical solution of the Burgers' equation with a very small viscosity. Various numerical techniques accompanied with spline functions have been set up for computing the solutions of the Burgers' equation. The nonlinear term UU_x makes it more interesting to study as it is one of the few nonlinear partial differential equations that have been solved analytically.

2- Sextic B-Spline Galerkin Scheme with Quintic Weight Function (SBGQWM)

Consider the one dimensional quasi-linear parabolic differential equation known as Burger's equation given by [11].

where $\epsilon > 0$ is the coefficient for the kinematic viscosity and subscripts x and t denote differentiation. "The boundary conditions" are selected form

$$U(a,t) = 0, \quad U(b,t) = 0,$$
$$U_x(a,t) = 0, \quad U_x(b,t) = 0, \quad t \in (0,T], \tag{2}$$
$$U_{xx}(a,t) = 0, \quad U_{xx}(b,t) = 0,$$

With initial condition

$$U(x,0) = g(x), \qquad a \le x \le b$$

g (x) is a prescribed function. If the Galerkin technique is applied to (1) such that W is the weight function yields the following integral equation:

$$\int_a^b W(U_t + UU_x - \epsilon U_{xx})dx = 0, \tag{3}$$

We consider the mish $a = x_0 < x_1 < \cdots < x_N = b$ is a uniform partition of the solution domain $a \le x \le b$ by the knots x_m and $h = x_m - x_{m-1}$, $m = 1, \ldots, N$, throughout this paper.

The sextic B-spline $F_m(x), (m = -3(1)N + 2)$, which form basis over the solution domain $[a,b]$ at the knots x_m, is defined as [14]

$$F_m(x) = \frac{1}{h^6}\begin{cases} g_1 = (x - x_{m-3})^6, & \text{if } x \in [x_{m-3}, x_{m-2}], \\ g_2 = g_1 - 7(x - x_{m-2})^6, & \text{if } x \in [x_{m-2}, x_{m-1}], \\ g_3 = g_2 + 21(x - x_{m-1})^6, & \text{if } x \in [x_{m-1}, x_m], \\ g_4 = g_3 - 35(x - x_m)^6, & \text{if } x \in [x_m, x_{m+1}], \\ (x - x_{m+4})^6 - 7(x - x_{m+3})^6 + 21(x - x_{m+2})^6, & \text{if } x \in [x_{m+1}, x_{m+2}], \\ (x - x_{m+4})^6 - 7(x - x_{m+3})^6, & \text{if } x \in [x_{m+2}, x_{m+3}], \\ (x - x_{m+4})^6, & \text{if } x \in [x_{m+3}, x_{m+4}], \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

The set of splines $(F_{-3}(x), F_{-2}(x), \ldots, F_{N+1}(x), F_{N+2}(x))$ represent forms a basis for functions defined over $[a,b]$. The approximate solution $U_N(x,t)$ to the exact solution $U(x,t)$ is

$$U_N(x,t) = \sum_{i=-3}^{N+2} F_i(x)\tau_i(t), \tag{5}$$

Such that τ_i are unknown (time-dependent) parameters that will be determined from the weighted residual and boundary conditions. Since each cubic B-spline covers seven elements such that every element $[x_m, x_{m+1}]$ is covered by seven splines. . In each element, we using the following local coordinate transformation [18]

$$h\eta = x - x_m, \quad 0 \le \eta \le 1, \tag{6}$$

The shape functions in expressions η over the element $[x_m, x_{m+1}]$ to the sextic B-spline (4) can be given by:-

$$F_{m-3} = (1-\eta)^6,$$
$$F_{m-2} = (2-\eta)^6 - 7(1-\eta)^6,$$

$$F_{m-1} = (3-\eta)^6 - 7(2-\eta)^6 + 21(1-\eta)^6,$$
$$F_m = (4-\eta)^6 - 7(3-\eta)^6 + 21(2-\eta)^6 - 35(1-\eta)^6,$$
$$F_{m+1} = (-\eta-2)^6 - 7(-\eta-1)^6 + 21(-\eta)^6,$$
$$F_{m+2} = (-\eta-1)^6 - 7(-\eta)^6,$$
$$F_{m+3} = (-\eta)^6, \qquad (7)$$

All splines apart from $F_{m-3}, F_{m-2}, F_{m-1}, F_m, F_{m+1}, F_{m+2}$ and F_{m+3} over the element $[x_m, x_{m+1}]$ equal zero. Variation of the function $U(\eta,t)$ over element $[x_m, x_{m+1}]$ is approximated by

$$U_N(\eta,t) = \sum_{i=m-3}^{m+3} F_i(\eta)\tau_i(t), \qquad (8)$$

Where B-splines $F_{m-3}(\eta), F_{m-2}(\eta), F_{m-1}(\eta), F_m(\eta), F_{m+1}(\eta), F_{m+2}(\eta)$ and $F_{m+3}(\eta)$ behave as element shape functions and $\tau_{m-3}(t), \tau_{m-2}(t), \tau_{m-1}(t), \tau_m(t), \tau_{m+1}, \tau_{m+2}(t)$ and $\tau_{m+3}(t)$ as element parameters. The spline $F_m(x)$ vanishes outside the interval $[x_{m-3}, x_{m+4}]$.

Using trial function (5) and sextic B-spline (4) to determine the value of U with its first and second derivatives U', U'' respectively at the knots (x_m) in terms of element parameters τ_m as follows:

$$U_m = U(x_m) = \tau_{m-3} + 57\tau_{m-2} + 302\tau_{m-1} + 302\tau_m + 57\tau_{m+1} + \tau_{m+2},$$
$$U'_m = U'(x_m) = \frac{6}{h}(\tau_{m-3} + 25\tau_{m-2} + 40\tau_{m-1} - 40\tau_m - 25\tau_{m+1} - \tau_{m+2}),$$
$$U''_m = U''(x_m) = \frac{30}{h^2}(\tau_{m-3} + 9\tau_{m-2} - 102\tau_{m-1} - 102\tau_m + 9\tau_{m+1} + \tau_{m+2}). \quad (9)$$

Weight function $W(x)$ is taken as a quintic B-spline $E_m(x), (m = -2(1)N+2)$, at the knots x_m which form basis over the solution domain [a, b], is defined as [14]

$$E_m(x) = \frac{1}{h^5}\begin{cases} e_1 = (x - x_{m-3})^5, & \text{if } x \in [x_{m-3}, x_{m-2}], \\ e_2 = e_1 - 6(x - x_{m-2})^5, & \text{if } x \in [x_{m-2}, x_{m-1}], \\ e_3 = e_2 + 15(x - x_{m-1})^5, & \text{if } x \in [x_{m-1}, x_m], \\ e_4 = e_3 - 20(x - x_m)^5, & \text{if } x \in [x_m, x_{m+1}], \\ e_5 = e_4 + 15(x - x_{m+1})^5, & \text{if } x \in [x_{m+1}, x_{m+2}], \\ e_6 = e_5 - 6(x - x_{m+2})^5, & \text{if } x \in [x_{m+2}, x_{m+3}], \\ 0, & \text{otherwise.} \end{cases} \qquad (10)$$

In each element, using (6), a shape functions of quintic B-spline in terms of η over the element $[x_m, x_{m+1}]$ is given by

$$E_{m-2} = (1-\eta)^5,$$
$$E_{m-1} = (2-\eta)^5 - 6(1-\eta)^5,$$
$$E_m = (3-\eta)^5 - 6(2-\eta)^5 + 15(1-\eta)^5,$$
$$E_{m+1} = (4-\eta)^5 - 6(3-\eta)^5 + 15(2-\eta)^5 - 20(1-\eta)^5,$$
$$E_{m+2} = (5-\eta)^5 - 6(4-\eta)^5 + 15(3-\eta)^5 - 20(2-\eta)^5 + 15(1-\eta)^5,$$
$$E_{m+2} = \eta^5, \qquad (11)$$

When the Petrov-Galerkin approach is applied to Eq.(1), Using transformation (6), equation(3) for the typical element $[x_m, x_{m+1}]$ becomes ([9],[10])

$$\int_0^1 W\left(U_t + \frac{\hat{U}}{h}U_\eta - \frac{\varepsilon}{h^2}U_{\eta\eta}\right)d\eta = 0, \qquad (12)$$

Where $\hat{U}$ was taken to be a constant over an element to simplify the integral [8],[17]

$$\int_0^1 (W\,U_t + \lambda W U_\eta + \beta W_\eta U_\eta)d\eta = \beta W U_\eta\,|_0^1, \qquad (13)$$

Such that,

$\lambda = \hat{U}/h$ and $\beta = \varepsilon/h^2$,

If we take the weight function with quintic B-spline shape functions given by (11) and reparation approximation (8) in equation (13), get

$$\sum_{i=m-3}^{m+3} [\int_0^1 E_j F_i d\eta]\tau_i^{\cdot e} + \sum_{i=m-3}^{m+3} [\int_0^1 \left(\lambda E_j F_i' + \beta E_j' F_i'\right) d\eta - \beta E_j F_i' \mid_0^1] \tau_i^e = 0 \tag{14}$$

Simplifying, we can write (14) in matrix form as follows:

$$\left(X_{ij}^e\right)\tau^{\cdot e} + [\beta(Y_{ij}^e - R_{ij}^e) + \lambda Q_{ij}^e]\tau^e = 0. \tag{15}$$

Such that $\tau^e = (\tau_{m-3}, \tau_{m-2}, \tau_{m-1}, \tau_m, \tau_{m+1,}\tau_{m+2}, \tau_{m+3})^T$ represent the element parameters. The element matrices $X_{ij}^e, Y_{ij}^e, R_{ij}^e$ and Q_{ij}^e are rectangular (6×7) given by the following integrals :

$$X_{ij}^e = \int_0^1 E_i F_j \, d\eta = \frac{1}{5544}\begin{bmatrix} 462 & 36959 & 244205 & 304250 & 76900 & 2503 & 1 \\ 16171 & 1537535 & 11886590 & 17975130 & 6128395 & 375559 & 1580 \\ 51014 & 5748218 & 52521800 & 96528940 & 42334750 & 3704026 & 25812 \\ 25812 & 3704026 & 42334750 & 96528940 & 52521800 & 5748218 & 51014 \\ 1580 & 375559 & 6128395 & 17975130 & 11886590 & 1537535 & 16171 \\ 1 & 2503 & 76900 & 304250 & 244204 & 36959 & 461 \end{bmatrix},$$

$$Y_{ij}^e = \int_0^1 E_i' F_j' \, d\eta = \frac{1}{42}\begin{bmatrix} 126 & 4621 & 11215 & -7190 & -8140 & -631 & -1 \\ 1805 & 83245 & 274990 & -87150 & -237055 & -35455 & -380 \\ 670 & 65170 & 397840 & 94340 & -438850 & -116950 & -2220 \\ -2220 & -116950 & -438850 & 94340 & 397840 & 65170 & 670 \\ -380 & -35455 & -237055 & -87150 & 274990 & 83245 & 1805 \\ -1 & -630 & -8140 & -7190 & 11215 & 4621 & 127 \end{bmatrix},$$

$$R_{ij}^e = (E_i F_j') \mid_0^1 \; = \begin{bmatrix} 6 & 150 & 240 & -240 & -150 & -6 & 0 \\ 156 & 3894 & 6090 & -6480 & -3660 & -6 & 6 \\ 396 & 9744 & 11940 & -22080 & -3660 & 3504 & 156 \\ 156 & 3504 & -3660 & -22080 & 11940 & 9744 & 396 \\ 6 & -6 & -3660 & -6480 & 6090 & 3894 & 156 \\ 0 & -6 & -151 & -240 & 240 & 150 & 6 \end{bmatrix},$$

$$Q_{ij}^e = \int_0^1 E_j F_j' \, d\eta = \frac{1}{462}\begin{bmatrix} -252 & -8861 & -20445 & 14060 & 14480 & 1017 & 1 \\ -9113 & -388303 & -1161290 & 486520 & 950545 & 120623 & 1018 \\ -29558 & -1529148 & -5905750 & 861980 & 5530290 & 1056688 & 15498 \\ -15498 & -1056688 & -5530290 & -861980 & 5905750 & 1529148 & 29558 \\ -1018 & -120623 & -950545 & -486520 & 1161290 & 388303 & 9113 \\ -1 & -1017 & -14480 & -14060 & 20445 & 8861 & 252 \end{bmatrix}$$

Such that suffices j takes just the values 1(1)6 and i takes values $(m-3)(1)(m+3$ for the typical element $[x_m, x_{m+1}]$. A lumped value is defined as

$\lambda = \frac{3}{4h}(\tau_{m-3} + 57\tau_{m-2} + 302\tau_{m-1} + 302\tau_m + 57\tau_{m+1} + \tau_{m+2})^2$,by gathering all contributions from all element we get,

$$\left(X_{ij}\right)\tau^{\cdot} + \left[\beta\left(Y_{ij} - R_{ij}\right) + \lambda Q_{ij}\right]\tau = 0 \tag{16}$$

where $\tau = (\tau_{-3}, \tau_{-2}, \tau_{-1}, \dots, \tau_N, \tau_{N+1}, \tau_{N+2})^T$ is a global element parameter. The matrices $X, Y$and λQ are rectangular, 13-diagonal and row of each has the following form :

$X = \frac{1}{5544}(1, 4083, 478271, 10187685, 66318474, 162512286, 162512286, 66318474, 10187685, 478271, 4083, 1, 0)$,

$Y = \frac{1}{42}(-1, -1011, -45815, -360525, -447810, 855162, 855162, -447810, -360525, -45815, -1011, -1, 0\,)$,

$\lambda Q = \frac{1}{462}(-\lambda_1, -1017\lambda_1 - 1018\lambda_2, -14480\lambda_1 - 120623\lambda_2 - 15498\lambda_3, -14060\lambda_1 - 950545\lambda_2 - 1056688\lambda_3 - 29558\lambda_4, 20445\lambda_1 - 486520\lambda_2 - 5530290\lambda_3 - 1529148\lambda_4 - 9113\lambda_5,\ 8861\lambda_1 + 1161290\lambda_2 - 861980\lambda_3 - 5905750\lambda_4 - 388303\lambda_5 - 252\lambda_6, 252\lambda_1 + 388303\lambda_2 + 5905750\lambda_3 + 861980\lambda_4 - 1161290\lambda_5 - 8861\lambda_6, 9113\lambda_2 + 1529148\lambda_3 + 5530290\lambda_4 + 486520\lambda_5 - 20445\lambda_6, 29558\lambda_3 + 1056688\lambda_4 + 950545\lambda_5 + 14060\lambda_6, 15498\lambda_4 + 120623\lambda_5 + 14480\lambda_6, 1018\lambda_5 + 1017\lambda_6, \lambda_6, 0)$

where,

$$\lambda_1 = \frac{3}{4h}(\tau_{m-3} + 58\tau_{m-2} + 359\tau_{m-1} + 604\tau_m + 359\tau_{m+1} + 58\tau_{m+2} + \tau_{m+3})^2,$$

$$\lambda_2 = \frac{3}{4h}(\tau_{m-2} + 58\tau_{m-1} + 359\tau_m + 604\tau_{m+1} + 359\tau_{m+2} + 58\tau_{m+3} + \tau_{m+4})^2,$$

$$\lambda_3 = \frac{3}{4h}(\tau_{m-1} + 58\tau_m + 359\tau_{m+1} + 604\tau_{m+2} + 359\tau_{m+3} + 58\tau_{m+4} + \tau_{m+5})^2,$$

$$\lambda_4 = \frac{3}{4h}(\tau_m + 58\tau_{m+1} + 359\tau_{m+2} + 604\tau_{m+3} + 359\tau_{m+4} + 58\tau_{m+5} + \tau_{m+6})^2,$$

$$\lambda_5 = \frac{3}{4h}(\tau_{m+1} + 58\tau_{m+2} + 359\tau_{m+3} + 604\tau_{m+4} + 359\tau_{m+5} + 58\tau_{m+6} + \tau_{m+7})^2,$$

$$\lambda_6 = \frac{3}{4h}(\tau_{m+2} + 58\tau_{m+3} + 359\tau_{m+4} + 604\tau_{m+5} + 359\tau_{m+6} + 58\tau_{m+7} + \tau_{m+8})^2,$$

"replacing the time derivatives of the parameter $\tau^{\cdot}$ by usual forward finite difference approximation and parameter τ by the Crank-Nicholson formulation" [16]

$$\tau^{\cdot} = \frac{\tau^{\,n+1} - \tau^{\,n}}{\Delta t}, \qquad \tau = \frac{1}{2}(\tau^{\,n} + \tau^{\,n+1}),$$

into equation (16), obtain $(N+5)\times(N+6)$ matrix system

$$\left[X_{ij} + \frac{\Delta t}{2}(\beta(Y_{ij} - R_{ij}) - \lambda Q_{ij})\right]\tau^{n+1} = \left[X_{ij} - \frac{\Delta t}{2}(\beta(Y_{ij} - R_{ij}) - \lambda Q_{ij})\right]\tau^{n}, \qquad (17)$$

To make the matrix equation be square we imposing the boundary conditions to (17). The initial vector of parameter $\tau^0 = (\tau_0^0, \tau_1^0, \ldots, \tau_N^0)$ required be determined to iterate system (17). The approximation (5) is rewritten over the interval $[a, b]$ at time $t = 0$ as follows :

$$U_N(x, 0) = \sum_{m=-3}^{N+2} F_m\,\tau_m^0,$$

$U(x, 0)$ must be satisfy the following relations at the mesh points x_m :

$$U_N(x_m, 0) = U(x_m, 0),\ m=0,1,\ldots,N$$
$$U'_N(x_0, 0) = U'(x_N, 0) = 0\,,$$
$$U''_N(x_0, 0) = U''(x_N, 0) = 0.$$

So, the initial vector of parameter τ^0 can be determined as

$$\begin{bmatrix} 30 & 270 & -300 & -300 & 270 & 30 & & & & \\ 6 & 150 & 240 & -240 & -150 & -6 & & & & \\ 1 & 57 & 302 & 302 & 57 & 1 & & & & \\ & & & \ddots & \ddots & & & & & \\ & & & & \ddots & \ddots & & & & \\ & & & 1 & 57 & 302 & 302 & 57 & 1 \\ & & & 6 & 150 & 240 & -240 & -150 & -6 \\ & & & 30 & 270 & -300 & -300 & 270 & 30 \end{bmatrix} \begin{bmatrix} \tau_{-3}^0 \\ \tau_{-2}^0 \\ \tau_{-1}^0 \\ \tau_0^0 \\ \vdots \\ \vdots \\ \tau_N^0 \\ \tau_{N+1}^0 \\ \tau_{N+2}^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ U(x_0) \\ \vdots \\ \vdots \\ \vdots \\ U(x_N) \\ 0 \\ 0 \end{bmatrix}, \qquad (18)$$

To solve this system, first must reduce it to six-diagonal form by eliminating the first three and last pair of equations and then apply Thomas algorithm .[14]

3- Stability

We apply the Von-Neumann stability method for the stability, since this method is applicable to the linear schemes, the nonlinear term UU_x is linearized by taking U as a constant value k [13].

The linearized form of suggested scheme (17) takes the form

$$\wp_1\tau_{m-3}^{n+1} + \wp_2\tau_{m-2}^{n+1} + \wp_3\tau_{m-1}^{n+1} + \wp_4\tau_{m}^{n+1} + \wp_5\tau_{m+1}^{n+1} + \wp_6\tau_{m+2}^{n+1} + \wp_7\tau_{m+3}^{n+1} + \wp_8\tau_{m+4}^{n+1} + \wp_9\tau_{m+5}^{n+1} + \wp_{10}\tau_{m+6}^{n+1} + \wp_{11}\tau_{m+7}^{n+1} + \wp_{12}\tau_{m+8}^{n+1} = \wp_{12}\tau_{m-3}^{n} + \wp_{11}\tau_{m-2}^{n} + \wp_{10}\tau_{m-1}^{n} + \wp_9\tau_{m}^{n} + \wp_8\tau_{m+1}^{n} + \wp_7\tau_{m+2}^{n} + \wp_6\tau_{m+3}^{n} + \wp_5\tau_{m+4}^{n} + \wp_4\tau_{m+5}^{n} + \wp_3\tau_{m+6}^{n} + \wp_2\tau_{m+7}^{n} + \wp_1\tau_{m+8}^{n}.$$

where

$\wp_1 = \frac{1}{5544} - \frac{\Delta t}{2}(\frac{1}{42}\beta + \frac{\lambda}{462})$, $\wp_2 = \frac{4083}{5544} - \frac{\Delta t}{2}(\frac{1011}{42}\beta + \frac{2035\lambda}{462})$,

$\wp_3 = \frac{478271}{5544} - \frac{\Delta t}{2}(\frac{45815}{42}\beta + \frac{150601\lambda}{462})$, $\wp_4 = \frac{10187685}{5544} - \frac{\Delta t}{2}(\frac{360525}{42}\beta + \frac{2050851\lambda}{462})$,

$\wp_5 = \frac{66318474}{5544} - \frac{\Delta t}{2}(\frac{447810}{42}\beta + \frac{7534626\lambda}{462})$, $\wp_6 = \frac{162512286}{5544} + \frac{\Delta t}{2}(\frac{855162}{42}\beta + \frac{5986134\lambda}{462})$,

$\wp_7 = \frac{162512286}{5544} - \frac{\Delta t}{2}(\frac{855162}{42}\beta - \frac{5986134\lambda}{462})$, $\wp_8 = \frac{66318474}{5544} + \frac{\Delta t}{2}(\frac{447810}{42}\beta - \frac{7534626\lambda}{462})$,

$\wp_9 = \frac{10187685}{5544} + \frac{\Delta t}{2}(\frac{-360525}{42}\beta - \frac{2050851\lambda}{462})$, $\wp_{10} = \frac{478271}{5544} + \frac{\Delta t}{2}(\frac{45815}{42}\beta - \frac{150601\lambda}{462})$,

$\wp_{11} = \frac{4083}{5544} + \frac{\Delta t}{2}(\frac{1011}{42}\beta - \frac{2035\lambda}{462})$, $\wp_{12} = \frac{1}{5544} + \frac{\Delta t}{2}(\frac{1}{42}\beta - \frac{\lambda}{462})$,

The error in typical mode of amplitude $\gamma, \tau_m^n = \gamma e^{i\beta mh}$, substituting the above Fourier mode into linearized form gives

$$\gamma^{n+1} = \delta\gamma^n,$$

the growth factor g_{14} has the form :

$$\delta = \frac{b_1e^{6i\beta h}+(b_2+b_{12})(e^{5i\beta h}+e^{-5i\beta h})+(b_3+b_{11})(e^{4i\beta h}+e^{-4i\beta h})+(b_4+b_{10})(e^{3i\beta h}+e^{-3i\beta h})+(b_5+b_9)(e^{2i\beta h}+e^{-2i\beta h})+(b_6+b_8)(e^{i\beta h}+e^{-i\beta h})+b_7}{b_{12}e^{6i\beta h}+(b_{11}+b_1)(e^{5i\beta h}+e^{-5i\beta h})+(b_{10}+b_2)(e^{4i\beta h}+e^{-4i\beta h})+(b_9+b_3)(e^{3i\beta h}+e^{-3i\beta h})+(b_8+b_4)(e^{2i\beta h}+e^{-2i\beta h})+(b_7+b_5)(e^{i\beta h}+e^{-i\beta h})+b_6}$$

So that the magnitude of the growth factor $|\gamma| \leq 1$. The linearized recurrence relation based on the present scheme is unconditionally stable.

4- Numerical example

The test problem is studied so that demonstrate the robustness and numerical accuracy of the suggested

method. Accuracy is measured by using L_2 and L_∞ error norms

$$L_2 = \left\| U - U_N \right\|_2 \cong \sqrt{h \sum_{J=0}^{N} \left| U_J \ - (U_N) \right|^2}, \qquad (19)$$

$$L_\infty = \left\| U - U_N \right\|_\infty \cong \max_J \left| U_J \ - (U_N) \right|, \qquad (20)$$

L_2 and L_∞error norms are used for numerical example and comparison is made with results of the paper [6].

"Burgers' equation has the following form of the analytical solution".

$$U(x,t) = \frac{\frac{x}{t}}{1 + \sqrt{\frac{t}{t_0} exp(\frac{x^2}{4vt})}}, \quad t \geq 1, \quad 0 \leq x \leq 1, \qquad (21)$$

Table (1) Comparison of results at different times for $v = 5 \times 10^{-3}$with $\Delta t = 1 \times 10^{-2}$and $h = 5 \times 10^{-3}$

x	$t = 17 \times 10^{-1}$ SBGM	$t = 17 \times 10^{-1}$ Exact	$t = 24 \times 10^{-1}$ SBGM	$t = 24 \times 10^{-1}$ Exact	$t = 31 \times 10^{-1}$ SBGM	$t = 31 \times 10^{-1}$ Exact
0.1	0.058823	0.058823	0.041666	0.041666	.032258	.032258
0.2	0.117645	0.117645	0.083332	0.083332	.064515	.064515
0.3	0.176458	0.176458	0.124995	0.124995	.096771	.096771
0.4	0.235166	0.235168	0.166640	0.166640	.129021	.129021
0.5	0.291875	0.291904	0.208111	0.208114	.161230	.161231
0.6	0.295812	0.295910	0.247396	0.247417	.193123	.193127
0.7	0.041931	0.041929	0.252093	0.252172	.221847	.221867
0.8	0.000648	0.000646	0.072996	0.073025	.215071	.215135
0.9	0.000005	0.000005	0.003023	0.003023	.070789	.070874
$L_2 \times 10^3$	0.02883		0.02470		0.14833	
$L_\infty \times 10^3$	0.11314		.07877		1.09575	

Where $t_0 = exp(\frac{1}{8v})$ .The propagation of the shock is represented by the equation above. The initial shock which is taken when t =1 in Eq. (21) will be observed as time progresses. To make comparison with earlier study [1], computation is done with parameters $h = 5 \times 10^{-3}$), $v = 5 \times 10^{-3}$) and $\Delta t = 1 \times 10^{-2}$ over the domain [0, 1]. Table (1) show a comparison of the exact solution with numerical values to scheme . Comparisons are presented at time t = 1.7, 2.4 and 3.1 only . The accuracy in the L_2 norm obtained is measured as 2.8 $\times 10^{-5}$ at time t=1.7, 2.4 $\times 10^{-5}$at time t=2.4 and 1.4 $\times 10^{-4}$ at time t = 3.1 for the SBGQWM.

The propagation of the shock is visualized at some times in the Figs. (1), which it is seen that the initial shock becomes steadier as a program runs . At time t = 3.1, the error distribution is drawn over the domain in the Figs.(2), and there appears to be the highest error about the right-hand boundary position.

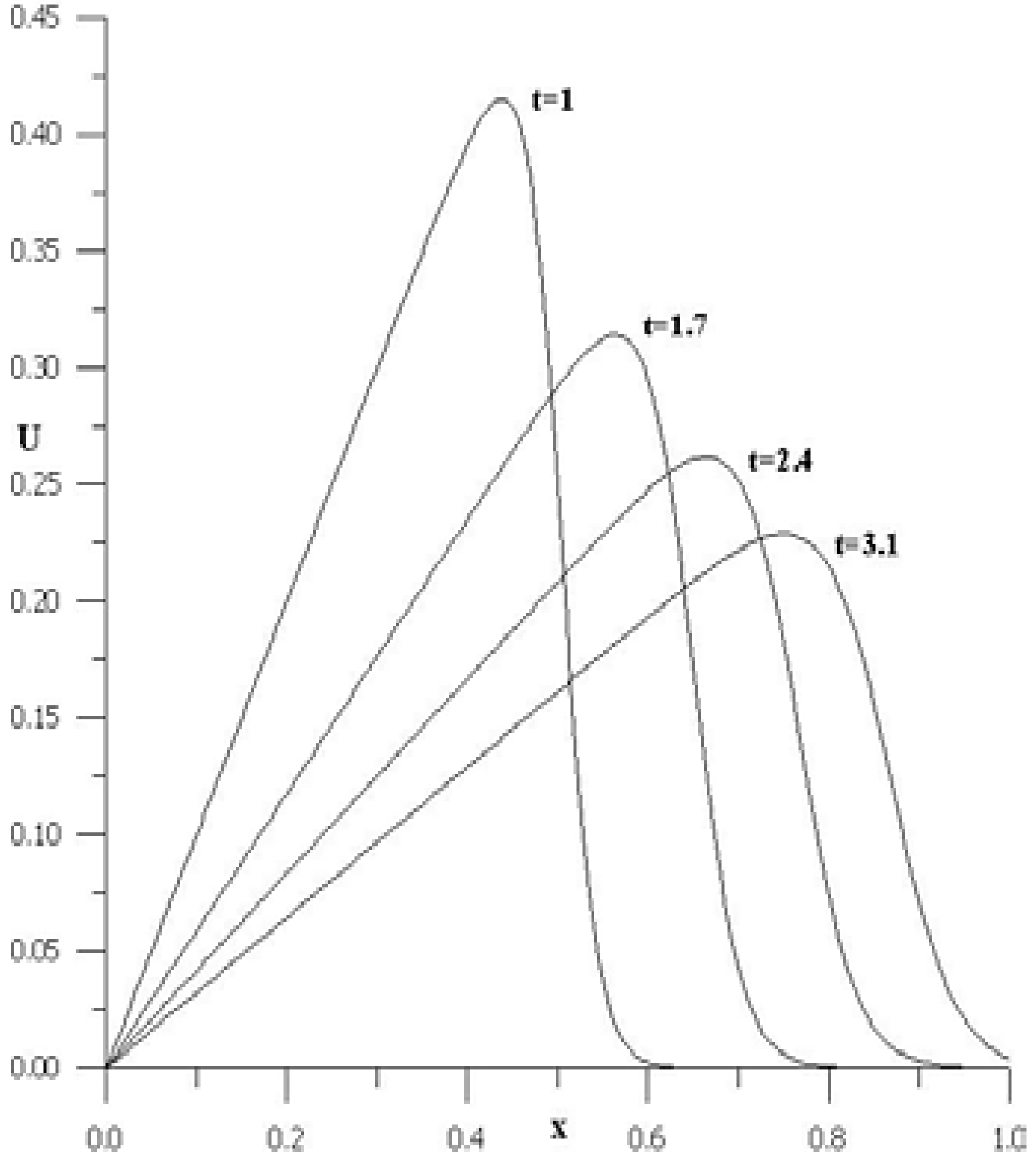

Figure (1) : $\boldsymbol{v = 5 \times 10^{-3}, h = 5 \times 10^{-3}, \Delta t = 1 \times 10^{-2}}$

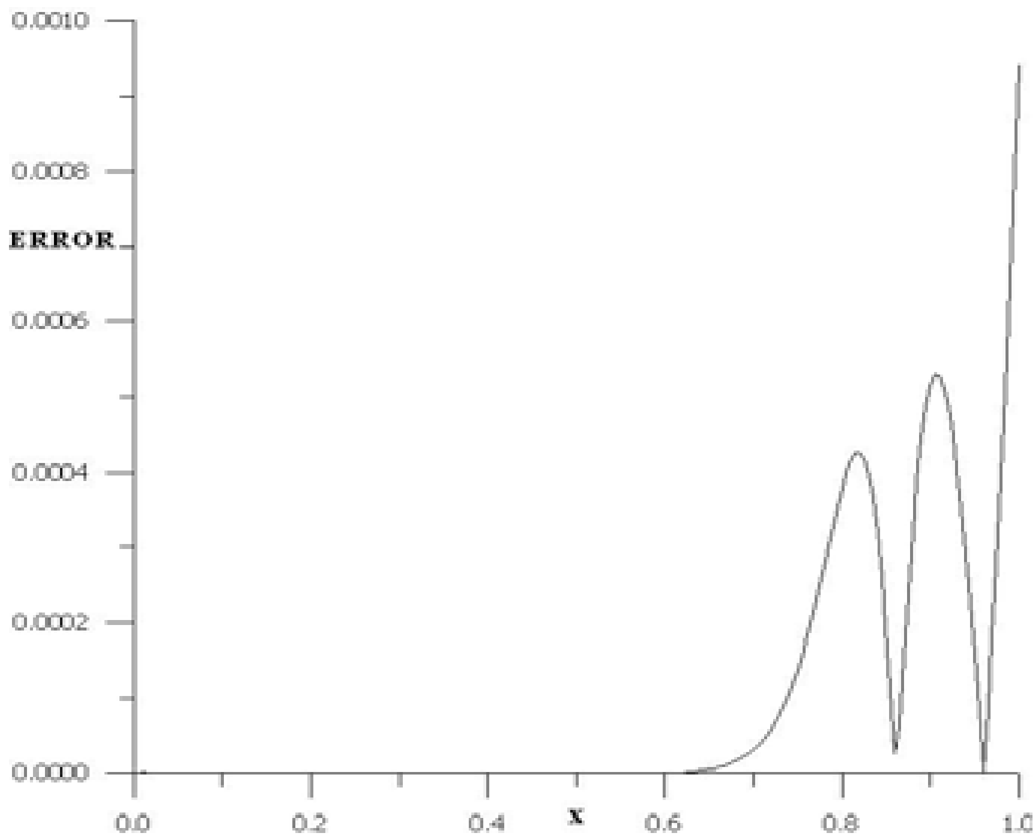

Figure (2): the (|numerical − analytic solution|) Error at time $t = 31 \times 10^{-1}$ with $v = 5 \times 10^{-3}$

5- Conclusions

The numerical algorithm based on Sextic B-Spline as trial function Galerkin method and quintic B-splines as weight function and is constructed of both Burgers' equation. The numerical method appear able to producing numerical solution for high accuracy of the solution to the Burgers' equation. Also we found that there is not frequently effect of the time-splitting for Burgers' equation on obtainment the numerical solution introduction method. The experimental results of the scheme is a lot more acceptable in comparison with the precedent results [1].

Therefore we can be concluded that the introduction method is efficient and credible for getting a numerical solution for the partial differential equations. The simulation process is made by using MATLAB 2011 software package.

CONFLICT OF INTERESTS

There are no conflicts of interest.

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الخلاصة

في هذه الدراسة قمنا باستخدام نظام عددي جديد لحساب الحل التقريبي لمعادلة (بيرغر) وذلك باعتماد دالة وزن تقل بقدار واحد عن درجة دالة الـــ(سبلاين كالركن من النوع B) حيث كانت دالة الـــ (كالركن) من الدرجة السادسة ودالة الوزن من الدرجة الخامسة. وبعد مقارنة النتائج التي حصلنا عليها مع الحل التحليلي لمعادلة (بيرغر) وجدنا ان هذا الأسلوب كان جيداً و غير شرطي الاستقرار من خلال استخدامنا لطريقة فورير(فان نيومن) لدراسة استقرار الاسلوب المقدم.

الكلمات الدالة: طريقة B-spline، طريقة بتروف جاليركين، طريقة Galerkin، طريقة (Fourier Von-Neumann)، معادلة Burger.